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Performance and Turnover in a Stochastic Partnership

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Abstract

This paper characterizes the social-welfare maximizing equilibrium of a "stochastic partnership matching market", in which players paired to play a stochastic game may quit to be costlessly and anonymously re-matched. Patterns of performance and turnover in this equilibrium are consistent with the well-known "survivorship bias" and, if partners form "meaningful first impressions", with the "honeymoon effect". By contrast, maximizing social welfare in standard repeated games with re-matching typically requires that players receive low payoffs at the start of each relationship. Welfare and turnover comparative statics are also provided: higher partnership-states are associated with higher joint payoffs and, in the special case of an exogenous stochastic process, with both higher joint stage-game and joint continuation payoffs as well as longer-lasting relationships.

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1 Introduction

Players in an ongoing interaction often face uncertainty regarding the fundamentals of their relationship. For example, an employer may be unsure about whether his worker will have an incentive in the future to leave for another job. Or, firms engaged in a joint venture may be unsure about future payoffs within their partnership. Such uncertainty can make it difficult to sign complete formal contracts, especially if what might change in the relationship is difficult to communicate to an outside party. At the same time, a long-lasting stable relationship is crucial for the effective provision of informal incentives. If shocks to the productivity of a partnership may cause it to end or be less productive in the near future, players will have less incentive to work today, reducing relational gains and potentially hastening the partnership's demise in a vicious cycle.

Given the option to leave one's current relationship to costlessly and anonymously re-match, players are only willing to make cooperative sacrifices if those who leave a relationship face some *endogenous* cost of being re-matched. For this reason, in standard (non-stochastic) repeated games with re-matching, maximizing social welfare requires that partners fail to immediately achieve the full potential equilibrium benefits of their relationship, such as by burning money or enduring an initial "incubation period" with efforts and payoffs lower than could be supported in equilibrium; see e.g. Kranton (1996) and Carmichael and MacLeod (1997). In particular, social-welfare maximizing equilibrium play necessarily fails to be renegotiation-proof. This paper sheds new light on this classic result, by providing a sufficient condition ("meaningful first impressions") given which social-welfare maximizing equilibrium play is renegotiation-proof (Theorem 5). Along the way, I will also establish new results about *joint*-welfare maximizing play in complete information stochastic games with voluntary exit.

Each period in a given partnership, two players simultaneously decide how much effort to exert after observing a partnership-specific state. "Effort" can be interpreted broadly, e.g. to include relationship-specific investments. After observing efforts, the partners then decide whether to quit the relationship and whether to pay voluntary "wages". The partnership ends if either player quits or "dies", in which case each surviving player is costlessly and anonymously re-matched with a new partner.

The model imposes few substantive restrictions on stage-game payoffs or on the stochastic state of the partnership. Notably, stage-game payoffs are assumed to satisfy increasing differences in players' efforts and the state, while the stochastic process is assumed to satisfy a positive serial auto-correlation property that higher past states make higher future states more likely in the sense of first-order stochastic dominance. However, no substantive restrictions are placed on how efforts control the stochastic process. This allows for a rich set of potential applications from labor to macroeconomics and organizational economics, in which greater effort grows, depletes, or has a non-monotone effect on a payoff-relevant relational stock. For example, in a labor context, one could interpret the worker's (multi-dimensional) effort as including work intensity as well as investments in firm-specific human capital. The assumptions are sufficiently weak that the existing literature on comparative statics in stochastic games does not apply. (See the literature discussion below.)

Analysis of the model is divided in two parts. In the first part (Section 4.1), I derive a subgame-perfect equilibrium (SPE) that maximizes players' joint welfare among all SPE, *taking as given* the players' outside options (Theorem 1). Joint payoff in this equilibrium is non-decreasing in the state (Theorem 2), but higher states need *not* in general be associated with higher joint stage-game payoff or higher joint continuation payoff. Consequently, players in higher states may or may not exert more effort, may or may not exit with lower probability, etc. However, more comparative statics are available in the special case in which players' efforts have no effect on future states. In this case, partnerships in higher states will enjoy higher stage-game payoffs, higher continuation payoffs, and later stopping times in the sense of first-order stochastic dominance (Theorem 3).

In the second part (Section 4.2), I derive the maximal social welfare that can be supported in equilibrium, within a "partnership matching market" with costless and anonymous re-matching after partnership dissolution. If some player's partnership ends, whether because he quit, his partner quit, or his partner died, he is automatically re-matched with a new partner to begin the next period. This new partnership is assumed to be a "fresh start", in the sense that (i) the stochastic processes driving stage-game payoffs are iid across partnerships and (ii) players know nothing about their current partner's history before their partnership began, including his age, number of past partnerships, etc.[1] Expected payoffs in a new partnership generate outside options for each player should his current partnership end. The analysis endogenizes the maximal joint outside option that can be supported in any equilibrium of the partnership market, thereby closing the model (Theorem 4). Further, given an equal exogenous flow of births and deaths, I characterize the steady-state distribution of histories among active partnerships in the social-welfare maximizing equilibrium.

Patterns of performance and turnover in the social-welfare maximizing equilibrium shed light on well-known stylized facts about the dynamics of relationships, the so-called "survivorship bias" and "honeymoon effect". The survivorship bias is a broad empirical finding – documented in employment (Topel and Ward (1992)), marriage (Stevenson and Wolfers (2007)) and organizations (Levinthal (1991)) – that *older* partnerships tend to be more productive and less likely to dissolve in the near future. The honeymoon effect is a less general empirical finding – documented in organizations (Fichman and Levinthal (1991)) but *not* marriage (Stevenson and Wolfers (2007)) – that *brand-new* partnerships also tend to be more productive and less likely to dissolve in the near future, relative to those in "adolescence".[2]

The survivorship bias arises in the social-welfare maximizing equilibrium because partnership dissolution is triggered when the partnership-state falls below a threshold surface in the state-space. Thus, partnerships that have lasted a long time tend to be those that have received mostly positive shocks that made the partnership more profitable and less likely to end in the near future. On the other hand, in classic models of repeated games with costless and anonymous re-matching, maximizing equilibrium social welfare dictates that players fail to realize all potential equilibrium gains at the start of each relationship. Thus, new relationships may in fact tend to endure an "anti-honeymoon" before emerging into a more productive, established phase. (See e.g. Section 5.2 of Mailath and Samuelson (2006).) This

[1] If historical variables such as age could be observed, then market-wide welfare might be enhanced in "old-maid equilibria" in which players who are not newly-born are shunned.

[2] By contrast, Stinchcombe (1965) argues that partnerships can be especially unstable when they are young if, among other reasons, players are uncertain about match quality and quickly learn whether they are a good match. This insight is supported by this paper's analysis, once the stochastic state of the partnership is understood to reflect what players have learned about partnership quality.

paper qualifies this well-known result, by providing a sufficient condition – "meaningful first impressions" – given which equilibrium social welfare is maximized when players maximize equilibrium *joint* welfare within every partnership. In that case, the honeymoon effect arises in equilibrium due to a selection effect, just as in non-strategic models such as Fichman and Levinthal (1991). Namely, partnerships that last more than one period are those that generated sufficiently positive first impressions.[3]

In the joint-welfare maximizing equilibrium, there is typically a range of states in which partners exert zero effort but elect to remain together despite this failure to cooperate. Players endure such "hard times", rather than quitting, because of the *option value* associated with waiting to exit. However, this option value does not only arise as usual from exogenous variation in the productivity of the partnership itself. The option to exit later becomes more valuable, in equilibrium, because of the endogenous variability of players' behavior.

The rest of the paper is organized as follows. The introduction continues with discussion of some related literature. Section 2 then presents a self-contained and in-depth analysis of a simple illustrative example with Prisoners' Dilemma stage-game payoffs. This example highlights most of the paper's qualitative results and novel analytical methods in a setting of some independent interest. Sections 3-4 then generalize the model and analysis to a much richer setting that allows for more general stochastic processes and does not require many of the special features of the example. Section 5 concludes with some remarks and directions for future research. Some proofs are in an Appendix.

Related literature.

This paper synthesizes elements from the literatures on productivity shocks (e.g. Jovanovic (1979a)), relational contracts (e.g. Levin (2003)), and repeated games with rematching (e.g. Kranton (1996)), in a rich but tractable stochastic framework.

Jovanovic (1979a) considers a model in which a worker learns over time about the productivity of the match with his present firm and quits as soon as he becomes sufficiently

[3] Using a dating metaphor, one may expect a couple that goes on a second date to be very likely to go on a third (if first impressions are sufficiently more important than second impressions). However, in the "adolescence" of such a relationship, break-up becomes more likely as negative impressions have time to accumulate.

pessimistic about the match. Consequently, workers who have remained longer at the same firm are less likely to leave and more likely to be more productive.[4] The key difference here is that partners face an incentive problem as well as a learning problem. Whereas the worker in Jovanovic always enjoys the full gains from his current match, players here must work to enjoy those gains and choose how to distribute them through voluntary wages. Levin (2003) characterizes optimal "relational contracts" in a principal-agent context in which the agent's cost of effort is iid. Unlike Levin (2003), this paper allows for two-sided incentives and non-iid stage-game payoffs, and endogenizes players' outside options through a re-matching technology.[5]

The analysis here confirms and combines key qualitative findings from the literatures on productivity shocks and relational contracts. For example, I show that performance *inside* the partnership decreases with the attractiveness of players' outside options. This extends a well-known finding of the relational contracts literature (see e.g. MacLeod and Malcomson (1989) and Baker, Gibbons, and Murphy (1994)) to a richer stochastic setting. Similarly, the observation that partnerships can (under some conditions) exhibit the survivorship bias and honeymoon effect is qualitatively similar to Fichman and Levinthal (1991)'s findings about firm performance and survival when productivity follows a random walk.

On the other hand, some of these same findings are quite surprising when viewed from the perspective of repeated games with re-matching (see e.g. Kranton (1996), Datta (1996) and recently Eeckout (2006) and Fujiwara-Greve and Ohuno-Fujiwara (2009)). A key finding of this literature is that social welfare is maximized when partners *fail* to realize all potential equilibrium gains in their individual partnerships; instead, they burn money, forego profitable cooperation on the basis of payoff-irrelevant information, or enduring an unproductive "incubation" phase before transitioning to a maximally productive phase.[6] The analysis here

[4] Also closely related is Jovanovic (1982), in which each firm's growth rate and survival depends on what it learns about own productivity, and Jovanovic (1979b), in which similar effects arise as workers who choose to remain in their current job make firm-specific investments to improve the future performance of the match.

[5] However, Levin's analysis is *not* less general, as he allows for incomplete information and imperfect monitoring of effort.

[6] In an incomplete information setting, Ghosh and Ray (1996) and Watson (1999) provide a separate, signaling rationale for "starting small".

shows that such results hinge crucially on the assumption of non-random payoffs. When partnerships exhibit *initial randomness* in the form of "meaningful first impressions" (Assumption 6), social-welfare maximizing equilibria of the overall partnership market dictate renegotiation-proof play within each partnership.

This paper also adds to the "dynamic games" literature in which a payoff-relevant state follows a known stochastic process.[7] For example, a key insight in Haltiwanger and Harrington (1991) and Bagwell and Staiger (1997)'s models of collusion and the business cycle, that collusion thrives at those times when the *future* state is most likely to be conducive to collusion, is helpful for interpreting this paper's results as well. However, the focus here is on how players' ability to dissolve their partnership and re-match interacts with their incentive to exert costly effort. Also, by allowing for any persistent stochastic process, my analysis encompasses both the iid case (as in Rotemberg and Saloner (1986), Ramey and Watson (1997)) and the "positively autocorrelated" case considered by Bagwell and Staiger (1997), among others.

Like this paper, Roth (1996) shows how to construct joint-welfare maximizing equilibria in a dynamic partnership, using an algorithm in the spirit of Abreu, Pearce and Stacchetti (1990). Roth's model can be viewed as a special case of mine in which, among other things, the initial state is non-random, the state is one-dimensional and follows a simple random walk, and there is no feedback of effort on future states. Also, Roth does not account for the important distinction between joint-welfare vs. social-welfare maximizing play. Indeed, social welfare in his setting is *not* maximized by joint-welfare maximizing play when players' outside options are endogenous via the option to re-match. By contrast, this paper characterizes social-welfare maximizing play and provides sufficient conditions given which such play maximizes joint welfare within each partnership.

Recently, Chassang (2010a) and Bonatti and Horner (2010) have developed other theories of cooperation dynamics. Namely, Chassang (2010a) shows how players "build routines" in repeated games with incomplete information about payoffs, while Bonatti and Horner (2010) develop a theory of dynamic public good provision given unobserved efforts and

[7] A growing and less closely related literature considers dynamic games in the presence of imperfect information, e.g. Athey and Bagwell (2001) and Horner and Jamison (2007).

uncertainty about the quality of the public good. In each of these papers, the underlying environment does not change over time. This paper highlights dynamics that arise when payoffs are stochastic, while abstracting from (important) issues of incomplete information and imperfect monitoring.

More tangentially related is the existing literature on "stochastic games", especially those papers such as Amir (1996) and Curtat (1996) in which sufficient monotone structure is imposed to generate comparative statics. However, most of these papers focus on equilibria in Markov strategies, often proving uniqueness of such equilibria, whereas I consider subgame-perfect equilibria (SPE) and focus on the SPE that maximizes joint welfare among all SPE. Further, this literature imposes stronger assumptions than are needed here, in large part because they prove stronger results such as uniqueness.

Lastly, although the option to exit plays an important role in the analysis, the literature on so-called "option games" is not directly related. In an option game, players' payoffs depend upon who exercises a real option (e.g. exiting a market) and when they do so, and papers in this literature tend to focus on issues of strategic pre-emption or delay that arise when players prefer to be the first or last to exercise their option. See e.g. Grenadier (2002) and Chassang (2010b). By contrast, my focus is to endogenize the productivity of the match itself.

2 Dynamic Prisoners' Dilemma

This section provides a self-contained analysis of an illustrative and tractable special case – the "Dynamic Prisoners' Dilemma" – of the more general model of Section 3. Several of the results here are corollaries of more general findings presented in Section 4, but the simplifying features of this example allow for proofs that are simpler and more intuitive. My hope is that the analysis here will help build readers' intuition for the more general analysis to be presented later.

Model. Two symmetric players play a repeated "partnership game" that continues until some player quits or dies, after which any survivors are anonymously re-matched with new

	Work	Shirk
Work	$1, 1$	$-1 - c_t, 1 + c_t$
Shirk	$1 + c_t, -1 - c_t$	$0, 0$

Figure 1: Stage-game payoffs at time t, while the partnership persists.

partners. Each player seeks to maximize his expected total (undiscounted) lifetime payoff.[8]

Partnership stage-game. Each period $t \geq 0$ of the partnership proceeds as follows. First, the players observe state variable $c_t > 0$, which I shall call the "cost of effort". Second, the players simultaneously choose whether to work or shirk, observe these efforts, and receive Prisoners' Dilemma stage-game payoffs as in Figure 1.[9] Third, each player dies with exogenous probability $(1 - \gamma)$, iid across players and periods. Should either player die, the partnership ends and any survivor is costlessly and anonymously re-matched in a new partnership to begin the next period. (When a partnership begins, each player knows nothing about his new partner's history.) Otherwise, with probability γ^2, both players survive and simultaneously choose whether to stay or quit the partnership. If either player quits, the partnership ends and both players are costlessly and anonymously re-matched. If both stay, the partnership continues to the next period.

Stochastic process. The "cost of effort" $C_t > 0$ for all t and $\log(C_t)$ follows a random walk, i.e. $\frac{C_t}{C_{t-1}}$ are iid. Further, the players observe a public randomization $Z_0 \sim U[0, 1]$ at the start of their relationship, independent of $(C_t : t \geq 0)$. (The role of the public randomization will become clear.)

Simplifying features of this example. Analysis of the Dynamic Prisoners' Dilemma considered here is dramatically simplified by four features of this example that are all relaxed

[8] The analysis can be easily extended to settings in which players discount payoffs each period instead of (or in addition to) facing the risk of death.

[9] Such payoffs arise naturally in a context in which players bear all of the cost of their own effort but share equally the return to that effort. Suppose that each player generates a return equal to his cost when working alone, but generates an excess return of one when working together with the other player. The payoffs of Figure 1 then arise when the cost and return of individual effort is $2(1 + c_t)$.

in Sections 3-4. First, payoffs are symmetric and asymmetric play generates weakly lower joint stage-game payoff than symmetric play. (Joint payoff is zero when one player works and one shirks and at least zero when both players work or both players shirk.) Second, the productivity of the partnership does not depend on the state variable. (Joint payoff is two when both players work and zero otherwise, regardless of the cost of effort.) Third, the state variable C_t is exogenous, i.e. the distribution of the cost of effort in period t does not depend on players' efforts in previous periods. Fourth, $\log(C_t)$ is a random walk. Because of these simplifying features, some details of the model presented in Section 3 are not relevant here. In particular, because of players' symmetry and the optimality of symmetric play, it is without loss to restrict attention to symmetric equilibria with no voluntary transfers and equal outside options for "males" and "females". To avoid cluttering the exposition here, I will therefore make no reference to either "wages" or "gender".

2.1 Joint-welfare maximizing SPE.

Suppose that each player in a given partnership has an outside option worth $v \geq 0$ should he survive but the partnership end. (This outside option will be endogenized in Section 2.2.) I begin by showing that, for any given $v \geq 0$, the joint-welfare maximizing subgame-perfect equilibrium (SPE) of the partnership game is characterized by a pair of thresholds.

Definition 1 (Work threshold)**.** Both players "adopt work threshold c^W" if, at every time t, (i) both players work if $c_t \leq c^W$ and (ii) both players shirk if $c_t > c^W$.

Definition 2 (Exit threshold)**.** Both players "adopt exit threshold c^E" if, at every time t, (i) both players stay if $c_t \leq c^E$ and (ii) both players quit if $c_t > c^E$.

Should both players adopt work threshold c^W, standard real-options logic implies that expected joint payoff is maximized when players terminate their relationship according to a threshold rule, namely, when they both adopt an exit threshold $c^E(c^W; v)$ that depends on the work threshold c^W and the outside option v. Lemma 1 gathers together several useful facts about this "optimal exit threshold".

Lemma 1 (Optimal exit threshold)**.** *Suppose that both players adopt work threshold* c^W*. Joint payoff is maximized when they also adopt exit threshold* $c^E(c^W; v) = \alpha(v)c^W$*, where* $\alpha(v)$ *is non-increasing in* v*.*

Definition 3 (Threshold equilibrium)**.** A (c^W, c^E)-threshold equilibrium is a SPE in which both players adopt work threshold c^W and exit threshold c^E on the equilibrium path and, off the equilibrium path, both shirk and quit.

Proposition 1 (Joint-welfare maximizing SPE)**.** *Fix any outside option* $v \geq 0$*. There exists* $c^{*W}(v) \geq 0$ *such that* $(c^{*W}(v), \alpha(v)c^{*W}(v))$*-threshold equilibrium exists and achieves the maximal joint payoff among all SPE. (*$c^{*W}(v)$ *is the "optimal work threshold" and* $c^{*E}(v) = \alpha(v)c^{*W}(v)$ *is the "optimal exit threshold.")*

Proof. Optimal exit given a work threshold. Suppose that the players adopt work threshold c^W, so that each receives stage-game payoff one when $c_t \leq c^W$ and zero when $c_t > c^W$. By Lemma 1, joint payoff is maximized when both players adopt exit threshold $c^E(c^W; v)$. Indeed, since both players receive identical payoffs when both adopt work threshold c^W, the exit threshold $c^E(c^W; v) = \alpha(v)c^W$ maximizes players' individual payoffs so that each player is willing to quit iff the state exceeds this threshold. Thus, if any SPE exists in which the players adopt work threshold c^W, then $(c^W, c^E(c^W; v))$-threshold equilibrium exists that achieves the maximal joint payoff among all SPE in which players adopt work threshold c^W.

Optimality of threshold SPE. Fix any SPE. Let $W = \{c :$ both players work with positive probability at some time t, after some history, when $c_t = c\}$. Consider any time-t history at which both players sometimes work and $c_t \in W$. Shirking when one is supposed to work increases each player's stage-game payoff by at least c_t, after which that player enjoys continuation payoff (as evaluated immediately after time-t effort) of at least γv.[10] Thus, this SPE must generate continuation payoff after time-t efforts of at least $c_t + \gamma v$ for each player.

I claim that a $(\sup W, c^E(\sup W; v))$-threshold equilibrium exists. Since both players work whenever $c_t \leq \sup W$, such threshold strategies generate weakly greater joint stage-game payoffs than the original SPE while the partnership persists, in every state c_t. And, since the players' exit threshold $c^E(\sup W; v)$ maximizes their joint payoff given work threshold

[10] This continuation payoff is guaranteed if the player quits whenever he survives to the end of period t.

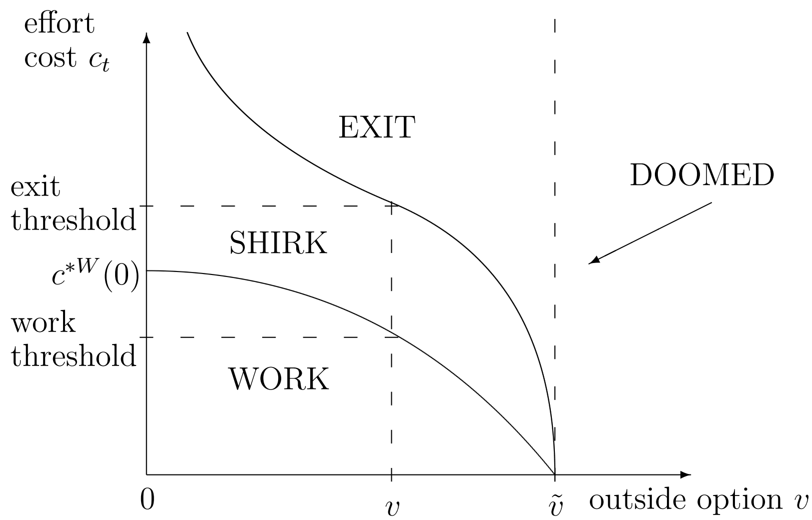

Figure 2: Summary of the optimal SPE of Proposition 1, for all $v \geq 0$.

$\sup W$, players' joint payoff from any on-path history in state c_t is weakly higher than in the original SPE, from any history in the same state c_t. In particular, for all $c_t \in W$, the players' joint continuation payoff after both work at time t is at least $2(c_t + \gamma v)$ when these threshold strategies are played. Since the players receive equal payoffs, finally, each player's continuation payoff is at least $c_t + \gamma v$ and he has sufficient incentive to work. This argument applies to all $c_t \in W$, and hence by continuity to $c_t = \sup W$. We conclude that a $(\sup W, c^E(\sup W; v))$-threshold equilibrium exists and that this SPE generates weakly greater joint payoff than the original SPE. $\square$

Players' behavior in the joint-welfare maximizing SPE of Proposition 1 is summarized by Figure 2. Since outside option v is fixed, the partnership state can be viewed as moving up and down a vertical slice of this figure. When the region labeled "EXIT" is reached, both players shirk and quit. Until then, both players work and stay when in the region labeled "WORK" while both shirk and stay when in the region labeled "SHIRK".

Figure 2 illustrates some noteworthy properties of the optimal work and exit thresholds. First, not surprisingly, there exists a critical outside option $\tilde{v}$ above which all partnerships are "DOOMED", i.e. the players shirk and immediately quit regardless of how small their initial cost of effort c_0. Exit is efficient when $v \geq \frac{1}{1-\gamma^2}$, but the threshold $\tilde{v} < \frac{1}{1-\gamma^2}$. Intuitively, the reason is that all partnerships operate under the "shadow of cooperation breakdown".

Should there be a negative shock that will induce exit, both partners will exert zero effort – and hence earn zero stage-game payoffs – in the last period of their relationship. Due to this risk, players only remain in a partnership if cooperation generates sufficient excess return over the outside option.

"Hard times". For all outside options $v < \tilde{v}$, the optimal work threshold $c^{*W}(v)$ and the "gap" $\frac{c^{*E}(v)}{c^{*W}(v)}$ between the optimal work and exit thresholds is strictly decreasing in v. (See Lemmas 1-2.) This gap represents what one might call "hard times", periods in a relationship when partners endure zero stage-game payoffs in hopes that cooperation will resume. As the outside option becomes more valuable, players are both less willing to work and also less willing to wait for their partnership to improve.

Partnership stopping time. Let T^* denote the stopping time of a partnership, when the joint-welfare maximizing SPE of Proposition 1 is played. The partnership may end due to (i) death of either partner or (ii) voluntary separation. To distinguish these, let T_i^{die} be the time at which player i dies, $T^{die} = \min_i T_i^{die}$ the first time at which either player dies, and T^{sep} the first time at which the partners would have separated absent death, i.e. $T^{sep} = \min\{t : C_t > c^{*E}(v)\}$. By definition, $T^* = \min\{T^{die}, T^{sep}\}$.

Since death is independent of separation, the hazard rate of partnership termination $\Pr(T^* = t|T^* \geq t) = 1 - (1 - \Pr(T^{die} = t|T^{die} \geq t))(1 - \Pr(T^{sep} = t|T^{sep} \geq t)$. Since each player survives each period with probability γ, $\Pr(T^{die} = t|T^{die} \geq t) = 1 - \gamma^2$ for all t. Since the players separate in the first period in which the cost of effort exceeds the exit threshold, (i) $\Pr(T^{sep} = 0) = \Pr(C_0 > c^{*E}(v))$ and (ii) $\Pr(T^{sep} = t|T^{sep} \geq t) = \Pr(C_t > c^{*E}(v)|\max\{C_0, ..., C_{t-1}\} \leq c^{*E}(v))$ for all $t \geq 1$. All together,

$$\Pr(T^* > 0) = 1 - \gamma^2 \Pr(C_0 \leq c^{*E}(v)) \tag{1}$$

$$\Pr(T^* > t|T^* \geq t) = 1 - \gamma^2 \Pr\left(C_t \leq c^{*E}(v)|\max\{C_0, ..., C_{t-1}\} \leq c^{*E}(v)\right). \tag{2}$$

"Survivorship bias". Since players separate once the cost of effort first exceeds an exit threshold, partnerships that have survived several periods will, more likely than not, have received mostly positive shocks that moved the cost of effort away from the exit threshold. This positive selection effect tends to make partnerships that have lasted a long time less likely to end

in the near future.[11] For example, suppose that $\log(C_t)$ follows a *symmetric* random walk with motion $\log(C_t) - \log(C_{t-1}) \sim U[-1, 1]$, and that $c_0 = c^{*E}(v)$ so that players are just barely willing to stay in the relationship. Table 1 documents the hazard rate of separation over time. For instance, conditional on partnership survival until time $t = 4$, the players will choose to separate that period approximately 12.8% of the time. The survivorship bias effect is present here, as the probability of separation decreases with partnership duration. (The fact that the hazard of separation at time t is approximately $\frac{1}{2t}$ follows from symmetry of the random walk; see Hughes (1995).)

Period	2	3	4	5	10	25
% partnership ends	25%	16.7%	12.8%	9.8%	5.0%	2.0%

Table 1: Probability of separation in period t, when $\log(C_t)$ follows a symmetric random walk, conditional on $c_0 = c^{*E}(v)$ and on partnership survival up to that point.

"Honeymoon effect". Similarly, partnerships dissolve immediately if $C_0 > c^{*E}(v)$. Thus, the set of partnerships that survive will be those for which $C_0 \leq c^{*E}(v)$. As long as the cost of effort varies sufficiently widely across partnerships while changing sufficiently slowly within each partnership (e.g. C_0 is atomless and $\frac{C_1}{C_0} \approx 1$), partnerships that do not dissolve at time $t = 0$ will likely not dissolve for several periods. Similarly, in such settings, partnerships that are productive at time $t = 0$ will likely remain productive for several periods.

Payoffs in the optimal equilibrium. Let $\overline{\Pi}^{eqm}(c_0; v)$ denote each player's expected payoff in the joint-welfare maximizing SPE of Proposition 1 given initial state $C_0 = c_0$. Each player's ex post payoff is equal to the number of productive periods enjoyed during the partnership *plus* the outside option v if player i survives the partnership's demise (either as a "widow" or as a "divorcee"):

$$E\left[\overline{\Pi}^{eqm}(C_0; v)\right] = E\left[\#\left\{t \in \{0, 1, ..., T^*\} : C_t \leq c^{*W}(v)\right\}\right] + v \Pr\left(T_i^{die} > T^*\right). \qquad (3)$$

[11] In general, the hazard of exit need not be monotone. For instance, suppose that $\log(C_t)$ is very likely to either fall by slightly less than two or rise by slightly more than one, and that $c_0 = c^{*E}(v)$. Conditional on both staying at time $t = 1$, the partnership is much more likely end at time $t = 3$ then at time $t = 2$.

The optimal work and exit thresholds. The work and exit thresholds $(c^{*W}(v), c^{*E}(v))$ in the joint-welfare maximizing SPE are characterized by a pair of indifference conditions. First, should the initial state equal the exit threshold ($c_0 = c^{*E}(v)$), each player is indifferent between staying and quitting. The benefit of staying is that each player will earn stage-game payoff one in each future period in which both players work. On the other hand, the benefit of quitting is that the player avoids the possibility of losing outside option v, should the partnership end with his death (RHS of (4)). Similarly, should the initial state equal the work threshold ($c_0 = c^{*W}(v)$), each player is indifferent between working and shirking. As before, the benefit of working is that each player will earn stage-game payoff one in each productive future period (LHS of (5)). On the other hand, the benefit of shirking – and thereby inducing the other player to quit and dissolve the partnership – is that one both saves the cost of effort and avoids losing the outside option to death (RHS of (5)).

Let $T(c^E)$ denote the stopping time of the partnership when both players adopt exit threshold c^E. (So, $T^* = T(c^{*E}(v)$.) The optimal work and exit thresholds $(c^{*W}(v), c^{*E}(v))$ solve the following pair of equations:[12]

$$E\left[\#\left\{t \in \{1,2,...,T(c^E)\} : C_t \leq c^W\right\} | c_0 = c^E\right] = v \Pr\left(T_i^{die} = T(c^E) | c_0 = c^E\right) \quad (4)$$

$$E\left[\#\left\{t \in \{1,2,...,T(c^E)\} : C_t \leq c^W\right\} | c_0 = c^W\right] = c^W + v \Pr\left(T_i^{die} = T(c^E) | c_0 = c^W\right) \quad (5)$$

Lemma 2 documents some useful facts about the optimal work threshold $c^{*W}(v)$. (See Lemma 1 for more on the optimal exit threshold $c^{*E}(v) = \alpha(v)c^{*W}(v)$.)

Lemma 2 (Optimal work threshold)**.** *The optimal work threshold $c^{*W}(v)$ is non-increasing in v and $\left|c^{*W}(v^h) - c^{*W}(v^l)\right| \in \left[0, v^h - v^l\right]$ for all $v^h > v^l$.*

Special case: worthless outside option. An interesting special case is that in which players' outside option is worthless ($v = 0$). Without loss, one may assume that players never voluntarily exit the partnership in the joint-welfare maximizing equilibrium ($c^{*E}(0) = \infty$), so the only question is when the players will be able to cooperate. Fortunately, the optimal work threshold $c^{*W}(0)$ can be characterized quite simply in terms of (i) the likelihood of

[12]Since $(C_t : t \geq 0)$ is a random walk, it is straightforward to show that (4-5) has a unique solution.

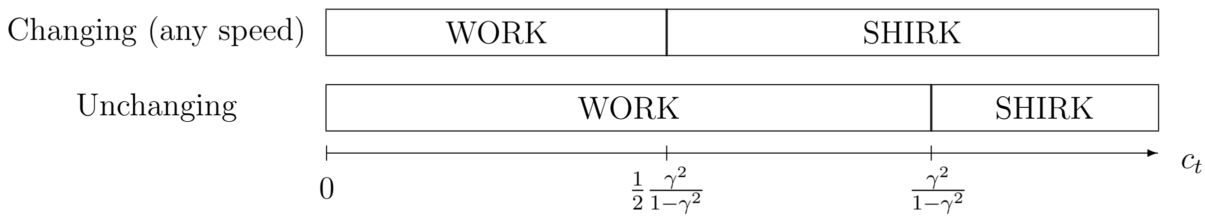

Figure 3: Illustration of Corollary to Claim 1.

death and (ii) the likelihood that the players' cost of effort will *cumulatively* increase over t periods, for all $t \geq 1$.

Claim 1. $c^{*W}(0) = \sum_{t \geq 1} \gamma^{2t} \Pr(C_t \leq C_0)$.[13]

Corollary. *(a)* $c^{*W}(0) = \frac{\gamma^2}{1-\gamma^2}$ *if* $C_t = C_0$ *for all* t. *(b)* $c^{*W}(0) = \frac{\gamma^2}{2(1-\gamma^2)}$ *if* $\log(C_t) = \log(C_{t-1}) + X_t$ *for all* $t \geq 1$, *where* $X_t \sim U[-\varepsilon, \varepsilon]$ *iid and* $\varepsilon > 0$.

Proof of Claim 1: $c^{*W}(0) = \max\{c^W : (c^W, \infty)\text{-threshold equilibrium exists}\}$. Since the partnership survives each round with probability γ^2 and both players work (shirk) when the cost of effort is less (greater) than the work threshold, each player's continuation payoff after time-0 efforts in state c_0 is

$$\sum_{t \geq 1} \gamma^{2t} \Pr(C_t \leq c^W | C_0 = c_0) = \sum_{t \geq 1} \gamma^{2t} \Pr(C_t / C_0 \leq c^W / c_0)$$
$$\geq (\leq) \sum_{t \geq 1} \gamma^{2t} \Pr(C_t \leq C_0) = c^{*W}(0) \text{ for all } c_0 \leq (\geq) c^W. \quad (6)$$

That is, each player's continuation payoff in any (c^W, ∞)-threshold equilibrium equals $c^{*W}(0)$ when his cost of effort $c_0 = c^W$ and exceeds $c^{*W}(0)$ when $c_0 < c^W$. In particular, a (c^W, ∞)-threshold equilibrium exists for all $c^W \leq c^{*W}(0)$ but not for any $c^W > c^{*W}(0)$. □

"Discontinuity" in maximal SPE payoffs.[14] The corollary to Claim 1 may be surprising at first, since the range of states in which cooperation can be supported in SPE is discontinuous

[13] This result does not depend on the fact that $\log(C_t)$ is a random walk. $c^{*W}(0) = \sum_{t \geq 1} \gamma^{2t} \Pr(C_t \leq C_0)$ (and the joint-welfare maximizing SPE is a threshold equilibrium) as long as $(C_t : t \geq 0)$ is an exogenous stochastic process with the "persistence" property that $\Pr(C_{t+1} > z | C_t = c)$ is non-decreasing in c for all t and all z, including the case of iid costs.

[14] The discontinuity of the optimal work threshold documented here hinges on the fact that there are finitely many actions. If one enriches the model to allow a continuum of efforts $e_t \in [0, 1]$, then there need not be any such discontinuity. For instance, suppose that stage-game payoffs take the form $\pi_{it}(e_{it}, e_{jt}) =$

in the "speed" $\varepsilon \geq 0$ at which the cost of effort evolves over time (Figure 3). In fact, this result is quite intuitive. When the cost of effort is exactly at the work threshold, players anticipate that they will only be able to cooperate in those future periods when the cost of effort is *not greater* than today. When the cost of effort does not change, it is obviously certain not to be greater than today. On the other hand, when the cost of effort follows a symmetric stochastic process, future costs are equally likely to be greater or less than today's cost. This shrinks by half the future value of the relationship, regardless of the speed at which the players' cost of effort changes over time. Consequently, cooperation can only be credibly sustained given cost of effort that is half as large.

2.2 Social-welfare maximizing equilibrium with re-matching

Players' outside option v is simply their ex ante expected equilibrium payoff in a new partnership. Thus, maximizing ex ante social welfare is equivalent to maximizing players' endogenous outside option.

Outside option v can be "generated by SPE play" if there exists a SPE of the partnership game *given outside option* v in which each player's ex ante expected payoff from a new match equals v. Recall that $E\left[\overline{\Pi}^{eqm}(C_0; v)\right]$ denotes each player's ex ante expected payoff in the joint-welfare maximizing SPE of Proposition 1, given outside option v. Thus, an outside option v cannot possibly be endogenously supported in equilibrium *unless* $v \leq E\left[\overline{\Pi}^{eqm}(C_0; v)\right]$, and

$$\overline{v} = \sup\left\{v : E\left[\overline{\Pi}^{eqm}(C_0; v)\right] \geq v\right\} \tag{7}$$

is an *upper bound* on the outside option that can be generated by SPE play.

Efficient benchmark. In the Dynamic Prisoners' Dilemma example considered here, players' joint stage-game payoff each period equals two should both work and equals zero should one or both shirk, regardless of the state variable c_t. (See Figure 1.) In particular, social welfare is maximized when all players work and stay in their current partnership until parted

$\max\{e_i, e_j\} \equiv e^{(1)}$ if $e_{it} = e_{jt}$, $\pi_{it}(e_{it}, e_{jt}) = e^{(1)} + c_t\left(e^{(1)} - e_{it}\right)$ if $e_{it} < e_{jt}$, and $\pi_{it}(e_{it}, e_{jt}) = -\pi_{jt}(e_{it}, e_{jt})$ if $e_{it} > e_{jt}$. The joint-welfare maximizing SPE specifies symmetric effort at every history that is both weakly decreasing in c_t and continuous in the speed parameter ε.

by death, at which point any "widows" re-match and work in their new relationship, and so on. Such play generates stage-game payoff one until death for each player, for an expected lifetime payoff

$$v^{eff} = \sum_{t \geq 0} \gamma^t = \frac{1}{1-\gamma} \tag{8}$$

representing the "efficient outside option". More generally, for any fixed outside option $v \leq v^{eff}$, social welfare is maximized when players always work and stay in their current relationship, in which case each player enjoys expected lifetime payoff

$$\Pi^{eff}(v) = 1 + \gamma(1-\gamma)v + \gamma^2 \Pi^{eff}(v) = \frac{1+\gamma(1-\gamma)v}{1-\gamma^2}. \tag{9}$$

(Each player gets continuation payoff zero upon death with probability $(1-\gamma)$, v upon being widowed with probability $\gamma(1-\gamma)$, and $\Pi^{eff}(v)$ should both players survive and the partnership continue. Note that $\Pi^{eff}\left(v^{eff}\right) = v^{eff}$ and $\Pi^{eff}(v) > v$ for all $v < v^{eff}$.)

The efficient outside option v^{eff} cannot be generated by SPE play. Indeed, both players shirk and the partnership is certain to end at time $t = 0$ in *all* SPE of the partnership game, given any outside option $v \geq v^{eff}$. More precisely, $\overline{\Pi}^{eqm}(c_0; v) = 0$ for all $c_0 > 0$ and all $v \geq v^{eff}$ so that $E[\overline{\Pi}^{eqm}(C_0; v^{eff})] = 0 < v^{eff}$. (The proof of this claim is straightforward and omitted.) Thus, $\overline{v} < v^{eff}$.

Achieving the upper bound $\overline{v}$. Proposition 2 shows that the upper bound outside option $\overline{v}$ can be generated by SPE play. (There are other sorts of SPE that also generate the upper bound outside option; see the discussion after Proposition 3.)

Proposition 2. *Outside option $\overline{v}$ can be generated by SPE play, in a SPE whose path of play proceeds as follows: (i) if the initial public randomization $z_0 \leq \frac{E\left[\overline{\Pi}^{eqm}(C_0;\overline{v})\right]-\overline{v}}{E\left[\overline{\Pi}^{eqm}(C_0;\overline{v})\right]-\gamma\overline{v}}$, then both players shirk and quit at time $t = 0$; (ii) otherwise, play proceeds as in the joint-welfare maximizing SPE of Proposition 1 given outside option $\overline{v}$.*

Proof. First, since shirking and quitting at time $t = 0$ is a SPE, the specified path of play in fact arises in a SPE of the partnership game given outside option $\overline{v}$. Let $\widehat{\Pi}^{eqm}(c_0, z_0, \overline{v})$ denote each player's interim expected payoff when starting a new partnership in which this equilibrium will be played, given initial cost of effort c_0 and public randomization realization

z_0. To simplify formulae, let $\hat{p} = \frac{E\left[\overline{\Pi}^{eqm}(C_0;\overline{v})\right]-\overline{v}}{E\left[\overline{\Pi}^{eqm}(C_0;\overline{v})\right]-\gamma\overline{v}}$ denote the probability with which the partners shirk and quit at time $t = 0$. Observe that[15]

$$E\left[\widehat{\Pi}^{eqm}(C_0, Z_0, \overline{v})\right] = \hat{p}\gamma\overline{v} + (1-\hat{p})E\left[\overline{\Pi}^{eqm}(C_0;\overline{v})\right] = \overline{v}. \tag{10}$$

This completes the proof. □

Renegotiation-proof play within partnerships. When $E\left[\overline{\Pi}^{eqm}(C_0;\overline{v})\right] > \overline{v}$, the upper bound outside option $\overline{v}$ is generated by SPE play in which players fail to realize all of the potential equilibrium gains from their relationship. Instead of immediately beginning a fully productive partnership, these players sometimes shirk and quit on the basis of a *payoff-irrelevant* public randomization. Such play is not renegotiation-proof, since other SPE exist in which both players immediately work and both receive strictly higher payoffs. This failure of renegotiation-proofness is an important feature of classic models of *non-stochastic* repeated games with re-matching (see e.g. Kranton (1996) and Carmichael and MacLeod (1997)). To induce players to work in such models, players who shirk and then quit to start a new partnership must be punished in some way. However, since players are assumed anonymous in each new partnership, it is impossible to punish past behavior directly. To deter exploitative behavior, some "friction" must therefore be introduced that makes new partnerships less valuable than existing partnerships. As long as new and existing partnerships have identical equilibrium productive possibilities, any such friction – necessary to maximize equilibrium social welfare in the matching market as a whole – *must* take the form of some failure to maximize joint welfare within at least some new partnerships.

One of the main contributions of this paper is to provide conditions under which this conflict between social-welfare and joint-welfare maximization disappears in *stochastic* repeated games with re-matching. When equilibrium productive possibilities vary from partnership to partnership – whether because partnerships differ at the start or because "things change"

[15] ALGEBRAIC DETAILS FOR REFEREES: Let $E = E\left[\overline{\Pi}^{eqm}(C_0;\overline{v})\right]$. Observe that

$$\hat{p}\gamma\overline{v} + (1-\hat{p})E = \frac{\gamma\overline{v}(E-\overline{v})}{E-\gamma\overline{v}} + \frac{\overline{v}(1-\gamma)E}{E-\gamma\overline{v}} = \frac{\overline{v}E-\gamma\overline{v}^2}{E-\gamma\overline{v}} = \overline{v}.$$

during the life of a partnership – new partnerships may differ from existing partnerships because of a selection effect. In particular, players in a sufficiently attractive existing partnership will have ample incentive to work, in order to avoid being tossed back into the matching market and a less desirable new partnership. In the context of the Dynamic Prisoners' Dilemma, Proposition 3 establishes that payoff-relevant variation known by players at the *start* of each partnership – "first impressions" that take the form of an atomless initial cost of effort C_0 – is sufficient to eliminate all conflict between social-welfare maximization and joint-welfare maximization. (Theorem 5 establishes a closely-related result in a more general setting.)

Proposition 3. *Suppose that C_0 is atomless. Then $E\left[\overline{\Pi}^{eqm}(C_0;\overline{v})\right] = \overline{v}$ and outside option $\overline{v}$ is generated by joint-welfare maximizing SPE play.*

Proof. $E\left[\overline{\Pi}^{eqm}(C_0;v)\right]$ is each player's ex ante expected payoff when both adopt the optimal work and exit thresholds $(c^{*W}(\overline{v}), c^{*E}(\overline{v}))$ given outside option $\overline{v}$. $E\left[\overline{\Pi}^{eqm}(C_0;\overline{v})\right] = \overline{v}$ means that outside option $\overline{v}$ is generated by joint-welfare maximizing SPE play. To show that $E\left[\overline{\Pi}^{eqm}(C_0;\overline{v})\right] = \overline{v}$, it suffices by the definition of $\overline{v}$ in (7) to show that $E\left[\overline{\Pi}^{eqm}(C_0;v)\right]$ is continuous in v.

Re-writing (3), $E\left[\overline{\Pi}^{eqm}(C_0;v)\right] = S(c^{*W}(v), c^{*E}(v); v)$ where

$$S(c^W, c^E; v) = \sum_{t\geq 0} \Pr\left(t \leq T(c^E)\right) \Pr\left(c_t \leq c^W | t \leq T(c^E)\right) + v \sum_{t\geq 0} \Pr\left(t = T(c^E)\right) \Pr\left(T_i^{die} > t | t = T(c^E)\right) \tag{11}$$

and $T(c^E)$ denotes the stopping time of the partnership given exit threshold c^E. Since C_0 is atomless and $\log(C_t)$ follows a random walk, the ex ante distribution of C_t is atomless for all t. Thus, all probability terms in (11) are continuous in both c^E and c^W. (Recall from (2) that $\Pr(t = T(c^E) | t \geq T(c^E)) = 1 - \gamma^2 \Pr\left(C_t \leq c^E | \max\{C_0, ..., C_{t-1}\} \leq c^E\right)$. Continuity of the probability terms in (11) then follows from the fact that $C_0, ..., C_t$ atomless implies $C_t | (\max\{C_0, ..., C_{t-1}\} \geq c^E)$ is atomless for all t and all c^E.) Thus, $S(c^W, c^E; v)$ is continuous in c^W, c^E, as well as obviously continuous in v. Next, recall that $c^{*W}(v)$ is continuous in v by Lemma 2. So, $S(c^{*W}(v), c^E; v)$ is continuous in v. By the Envelope Theorem, then, $E\left[\overline{\Pi}^{eqm}(C_0;v)\right] = \sup_{c^E \geq 0} S(c^{*W}(v), c^E; v)$ is continuous in v. This completes the proof. □

Example 1 shows that, without an atomless initial state as in Proposition 3, it may not be possible to generate outside option $\overline{v}$ by joint-welfare maximizing SPE play. By contrast, Example 2 provides a concrete illustration of how outside option $\overline{v}$ is generated by joint-welfare maximizing SPE play given an atomless initial state.

Example 1. Consider the simplest scenario of an unchanging, non-stochastic repeated game, i.e. $C_t = C_0$ for all t and $\Pr(C_0 = c) = 1$ for some $c > 0$. Define v^* implicitly by $\Pi^{eff}(v^*) = v^* + \frac{c}{\gamma^2}$. Given outside option v^*, each player is indifferent when the other player works between (i) working and staying given that both players will work and stay until parted by death or (ii) shirking and quitting. (Shirking induces the other player to dissolve the partnership with probability γ^2, when it would have otherwise survived. Thus, a "penalty" of $\frac{c}{\gamma^2}$ next period in a new partnership is just sufficient to induce players to work in their current relationship.) Thus, when $v = v^*$, a SPE exists in which both players work and stay until parted by death. In particular, $E\left[\overline{\Pi}(C_0; v^*)\right] = \Pi^{eff}(v^*) > v^*$. By contrast, given any outside option $v > v^*$, *all* SPE of the partnership game are such that both players shirk and the partnership is certain to end at time $t = 0$. Thus, $E\left[\overline{\Pi}(C_0; v)\right] = \gamma v < v$ for all $v > v^*$, and $\overline{v} = v^*$. Since joint-welfare maximizing play generates payoff $\Pi^{eff}(\overline{v}) > \overline{v}$, outside option $\overline{v}$ can only be generated by SPE that "wastes" some surplus that could have been achieved in equilibrium. This can be seen in Figure 4, where the surplus that must be wasted equals the difference $\Pi^{eff}(\overline{v}) - \overline{v}$.

Example 2. Consider now the next-simplest scenario of an unchanging repeated game with an atomless initial state, i.e. $C_t = C_0$ for all t and C_0 atomless. Each player strictly prefers to shirk and quit for realized payoff γv given any cost of effort $c_0 > \gamma^2\left(\Pi^{eff}(v) - v\right)$, since then each player's present gain from shirking outweighs the "penalty" of needing to start a new relationship. (See the discussion in Example 2.) On the other hand, given any $c_0 \leq \gamma^2\left(\Pi^{eff}(v) - v\right)$, a SPE exists in which both players work and stay until parted by death, for realized payoff $\overline{\Pi}(c_0; v) = \Pi^{eff}(v)$. In other words, the optimal work threshold $c^{*W}(v) = \gamma^2\left(\Pi^{eff}(v) - v\right) = \frac{\gamma^2(1-v(1-\gamma))}{1-\gamma^2}$ and (ii) the optimal exit threshold $c^{*E}(v) = c^{*W}(v)$.

Since $c^{*W}(v)$ is continuous in v and C_0 is atomless, each player's maximal *ex ante* expected

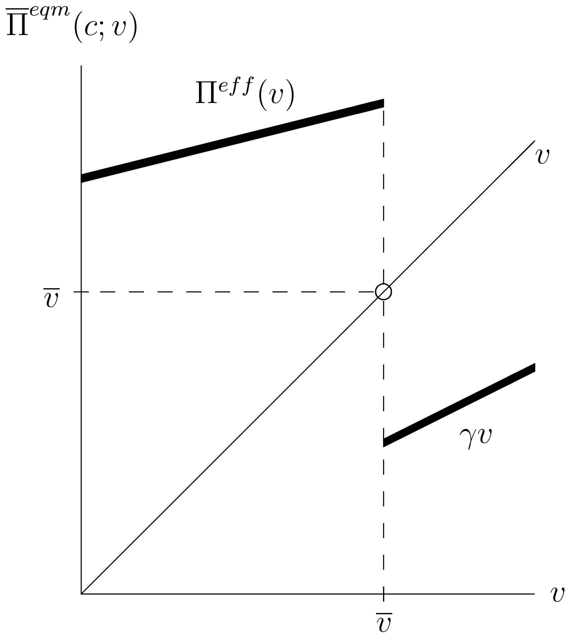

Figure 4: Maximal SPE payoff if $C_t = c$ for all t.

SPE payoff

$$E\left[\overline{\Pi}(C_0;v)\right] = \Pi^{eff}(v)\Pr\left(C_0 \leq c^{*W}(v)\right) + \gamma v \Pr\left(C_0 > c^{*W}(v)\right)$$

is continuous in v. By the definition of $\overline{v}$ in (7), then, $E\left[\overline{\Pi}(C_0;\overline{v})\right] = \overline{v}$ and the maximal equilibrium outside option $\overline{v}$ is generated by joint-welfare maximizing SPE play that proceeds as follows. At the start of their relationship, partners observe their match-specific cost of effort c_0. If $c_0 \leq c^{*W}(\overline{v})$, then these partners work and stay until parted by death; otherwise, the players exert no effort, dissolve their partnership at the first opportunity, and seek out a new partner. This "dating" process continues for each player until he finds a sufficiently attractive mate, or dies trying.

Further discussion. The conflict between social-welfare maximization and joint-welfare maximization in Example 1 arises from an underlying discontinuity of the maximal equilibrium payoff $\overline{\Pi}^{eqm}(c;v)$ with respect to the outside option v. In particular, $\lim_{\varepsilon\to 0}\overline{\Pi}^{eqm}(c;v^*+\varepsilon) = \gamma v^* < v^* < \Pi^{eff}(v^*) = \overline{\Pi}^{eqm}(c;v^*)$; see Figure 4. Generating outside option v^* therefore requires that players fail to realize $\Pi^{eff}(v^*) - v^* > 0$ in potential equilibrium gains. More intuitively, suppose for the sake of contradiction that all players were to work in the first period of every partnership in Example 1, as is possible in equilibrium given outside option $\overline{v}$. Each player would then strictly benefit by becoming a "serial shirker", who shirks and quits in a succession of partnerships for stage-game payoff $1 + c$ every period. To deter such exploitative behavior, at least some partnerships must fail to maximize equilibrium joint surplus.

In settings such as Example 1, there can be several distinct ways by which to generate outside option $\overline{v}$ in equilibrium. Yet all such approaches share one crucial feature: each player faces an endogenous "switching cost" should he quit his current partnership and seek a new match. Such switching costs are essential to provide equilibrium incentives to work in the current match. Consider an augmented version of Example 1 in which, at the start of each partnership, players (i) observe a public randomization (as here) and (ii) have an opportunity to "burn money" publicly. Three very different sorts of equilibria generate outside option $\overline{v}$ in this augmented example.

- *"Burn money."* At the start of period $t = 0$, each player burns money equal to $\frac{c}{\gamma^2}$. As long as both burn this amount of money, play proceeds as in the joint-welfare maximizing SPE, with both players working and staying until parted by death. Otherwise, both players shirk and quit immediately. Burning $\frac{c}{\gamma^2}$ in one's *next* partnership deters players from shirking in their current relationship, by making established partnerships worth $\frac{c}{\gamma^2}$ more than new partnerships. Since each partnership survives with probability γ^2 each period, this premium is just sufficient to deter players from shirking and leaving for a new match. Consequently, players have an incentive to work and stay in each partnership until parted by death.

- *"Incubation period."* Both players shirk for S^* periods, after which they transition to joint-welfare maximizing SPE play. The "incubation period" S^* is chosen so that each player would be willing to pay $\frac{c}{\gamma^2}$ in order to transition immediately to joint-welfare maximizing SPE play at time $t = 0$.[16]

[16] S^* may depend on the public randomization, but it suffices to restrict attention to support of the form $\text{supp}(S^*) = \{s, s+1\}$ for some $s \geq 0$. In this case, players always prefer to remain in their current relationship, rather than leave to start a new one. (Once players with a $(s+1)$-period incubation have

- *<u>"Dating."</u>* If the public randomization exceeds a threshold τ^*, then play immediately proceeds according to the joint-welfare maximizing SPE; otherwise, both players shirk and quit. I refer to this sort of equilibrium as "dating" since players engage in a sequence of unproductive, one-period relationships until they find a "mate" with whom they immediately enter into a fully productive match. Unlike the burning-money and incubation equilibria, players in the dating equilibrium dissolve relationships on the equilibrium path of play. However, as in these other equilibria, τ^* is chosen so that each player would be willing to pay $\frac{c}{\gamma^2}$ in order to play the joint-welfare maximizing SPE with the first partner they meet.

While there can be many sorts of equilibria that generate the maximal equilibrium outside option $\overline{v}$ when the initial state C_0 fails to be atomless, Proposition 3 shows that only one sort of equilibrium generates $\overline{v}$ when C_0 is atomless – the "dating" equilibrium. (See also Example 2.) This is quite intuitive, since dating not only imposes an endogenous switching cost through a costly search process – just as burning money or an incubation period impose endogenous costs associated with starting a new partnership – but also serves a sorting function whereby players only "mate" with a sufficiently attractive partner.

Further, since partnerships are differentiated by match-quality – "good" partnerships are those with cost of effort less than the work threshold – players have sufficient incentive to work in good partnerships without the need to burn money or otherwise fail to realize all potential equilibrium gains in such partnerships. Fear of being tossed back onto the "dating market" is sufficient to induce players in good partnerships to work. On the other hand, in bad partnerships, there does not exist any SPE in which either player works, and shirking and quitting to try a new date maximizes the players' equilibrium joint payoff.

3 More General Model

The model has two parts: a "partnership game" played by two asymmetric players, and a "partnership matching market", which generates outside options in the partnership game by

endured one period, they prefer waiting s more periods rather than starting a new partnership in which they will have to wait at least s more periods.)

allowing players to anonymously re-match with a new partner should their current partnership end. Each player dies with exogenous probability $(1-\gamma)$ each period, iid across periods, and seeks to maximize expected total lifetime payoffs. Likewise, each partnership ends whenever either player dies or quits, in which case each surviving player starts a partnership game with a new partner.

Note on notational shorthand. To improve clarity and shorten equations, I have adopted several notational conventions throughout the paper. First, random variables are capitalized while realizations are in lower case. Second, variables specific to a player and time have two subscripts, e.g. e_{it} for player i's effort in period t of the partnership. Vectors of such variables for all players *at one time* t are unbolded with one subscript, e.g. $e_t = (e_{it}, e_{jt})$, while those for all players *at all times no later than* t are bolded with one subscript, e.g. $\mathbf{e}_t = (e_0, ..., e_t)$. Finally, *sums* are denoted by a summation subscript, e.g. $\pi_{\Sigma t}(e_t; x_t) = \sum_i \pi_{it}(e_t; x_t)$.

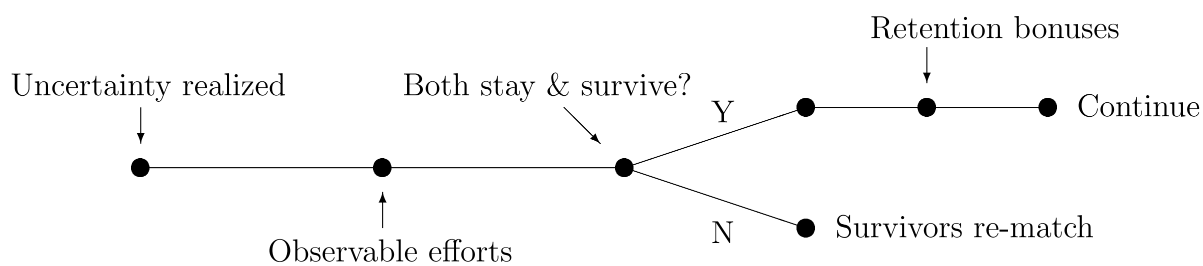

Figure 5: Timing of the partnership stage-game in period $t = 0, 1, 2, ...$

Partnership game. Each period $t = 0, 1, 2, ...$ of a partnership proceeds as follows; see Figure 5.[17] First, a payoff-relevant state $x_t \in \mathcal{X}_t = \text{supp}(X_t)$ is realized and publicly observed. $(\mathcal{X}_t, \succeq)$ is a partially ordered set. Second, each player i simultaneously decides what effort $e_{it} \in \mathcal{E}_{it}$ to exert, where efforts may control the stochastic process $(X_t : t \geq 0)$. Efforts are then publicly observed and each player i receives stage-game payoff $\pi_{it}(e_t; x_t)$. $(\mathcal{E}_{it}, \succeq)$ is a partially ordered, finite set having minimal element "0".

Assumption 1 (Stage-game payoffs)**.** For each player i, $\pi_{it}(e_t; x_t)$ is weakly decreasing in e_{it}

[17] "Time" t captures the duration of a partnership, not any notion of real time in the partnership market. See Assumption 5.

and weakly increasing in e_{jt}, and $\pi_{it}(0,0;x_t) = 0$ for all x_t. Further, joint payoff $\pi_{\Sigma t}(e_t;x_t)$ is uniformly bounded.[18]

Assumption 2 (Increasing differences)**.** π_{it} has weakly increasing differences in $(e_t;x_t)$. That is, $e_t^H \succ e_t^L$ and $x_t^H \succ x_t^L$ implies $\pi_{it}(e_t^H;x_t^H) - \pi_{it}(e_t^L;x_t^H) \geq \pi_{it}(e_t^H;x_t^L) - \pi_{it}(e_t^L;x_t^L)$.

Definition 4 (Cost of effort)**.** Let $c_{it}(e_t;x_t) = \sup_{\tilde{e}_{it}} (\pi_{it}(\tilde{e}_{it}, e_{jt};x_t) - \pi_{it}(e_t;x_t))$ denote each player's "cost of effort e_{it}" when player j exerts effort e_{jt} at time t in state x_t.

By Assumption 1, each player has a weakly dominant strategy to exert zero effort in each effort stage-game, so the cost of effort $c_t(e_t;x_t) = \pi_{it}(0, e_{jt};x_t) - \pi_{it}(e_t;x_t)$. By Assumption 2, $x_t' \succ x_t$ implies

$$\pi_{it}(e_t;x_t') \geq \pi_{it}(e_t;x_t) \text{ for all } e_t \quad (12)$$

$$c_t(e_t;x_t') \leq c_t(e_t;x_t) \text{ for all } e_t \quad (13)$$

for all e_t. That is, stage-game payoffs are weakly increasing in the state *and* players' incentive to exert less effort is weakly decreasing in the state. (Increasing differences implies (12) when we set $e_t^H = e_t$ and $e_t^L = (0,0)$ and implies (13) when we set $e_t^H = e_t$ and $e_t^L = (0, e_{jt})$.)

Third, each player dies with exogenous probability $(1-\gamma)$, iid across players and periods. Should both players survive, each then simultaneously decides whether to stay or quit the partnership. The partnership ends if either quits *or* if either dies. If so, each surviving player i receives an outside option having value $v_i \geq 0$. Otherwise, the partnership remains active in period $t+1$.

Utility is assumed to be transferable, and players can make voluntary wage transfers to one another at any time. However, I will show that it is without loss to restrict attention to equilibria in which players pay wages each period only in the form of "retention bonuses", after and only if both players stay (Lemma 3).

Stochastic process in each partnership. The stochastic process $(X_t : t \geq 0)$ has the property that future states are more likely to be higher when the current state is higher,

[18] The analysis extends easily to settings in which joint payoff is not bounded (as when productivity follows a random walk), as long as the maximum feasible expected lifetime joint payoff is finite from every history.

a serial auto-correlation property that I will refer to as "persistence". Two definitions are needed to make this precise.

Definition 5 (Increasing subset)**.** Let $(\mathcal{Z}, \geq)$ be any partially-ordered set. $\mathcal{Y} \subset \mathcal{Z}$ is an "increasing subset of $\mathcal{Z}$" if $a_1 \in \mathcal{Y}$, $a_2 \in \mathcal{Z}$, and $a_2 \geq a_1$ implies $a_2 \in \mathcal{Y}$.

Definition 6 (Generalized first-order stochastic dominance[19])**.** Let A_1, A_2 be random variables with support in partially ordered set $(\mathcal{Z}, \geq)$. A_1 "first-order stochastic dominates" (FOSD) A_2 if $\Pr(A_1 \in \mathcal{Y}) \geq \Pr(A_2 \in \mathcal{Y})$ for all increasing subsets $\mathcal{Y} \subset \mathcal{Z}$.

Assumption 3 (Persistence)**.** $x_t' \succeq x_t$ implies $X_{t+1}|(x_t', \mathbf{x}_{t-1}, \mathbf{e}_t)$ FOSD $X_{t+1}|(x_t, \mathbf{x}_{t-1}, \mathbf{e}_t)$ for all $\mathbf{x}_{t-1}, \mathbf{e}_t$.

Finally, it will be convenient to assume that partners have access to a public randomization at the start of their relationship. (Assumption 4 simplifies the characterization of the maximal social welfare that can be supported in equilibrium. See the discussion below.)

Assumption 4 (Public randomization)**.** $X_0 = (Y_0, Z_0)$ where $Z_0 \sim U[0,1]$ is independent of $(Y_0, X_t : t \geq 1)$ and payoff-irrelevant.

Partnership market. Players' outside options are generated endogenously from their ability to start a new partnership, within the following context.

Matching and re-matching. There is a unit mass of atomless players, half "male" and half "female" who are paired to play the partnership game, with an equal flow of $(1-\gamma)$ births and deaths each period.[20] Any player who is newly-born or whose partnership ended in the previous period (whether due to the death of a partner or due to endogenous exit) is automatically and costlessly matched with a new partner. Further, each such match is a "fresh start" in two senses. First, players know nothing about their current partner's history before their partnership, including his age, number of past partnerships, and so on. Second,

[19]When $\mathcal{Z} = \mathbf{R}$, this condition reduces to the familiar requirement that $\Pr(A_1 \geq z) \geq \Pr(A_2 \geq z)$ for all $z \in \mathbf{R}$. There is more than one natural way to generalize FOSD to multi-dimensional settings, some more restrictive than the notion used here. See e.g. Stoyan and Daley (1983).

[20]Gender captures the possibility that certain players may be matched and re-matched into specific roles, e.g. buyer and supplier, worker and firm, entrepreneur and investor.

players' history before the current partnership began has no impact on the state of that partnership. (In particular, Assumption 5 implies that partnerships are iid.)

Assumption 5 (Fresh start)**.** The distribution of X_t depends (only) on partnership duration t, the history of states in the current partnership, and the history of efforts in this partnership.

Solution concept. The solution concept is subgame-perfect equilibrium (SPE) in each partnership game, with the extra requirement that players' outside options arise endogenously as the expected present value of starting a new partnership in which that SPE will be played. In particular, for a given SPE of the partnership game with given outside options $v = (v_1, v_2)$, let $E\left[\Pi_{i0}^{eqm}(X_0; v)\right]$ denote player i's expected equilibrium payoff when starting a new partnership at time $t = 0$.

Definition 7 (Partnership-market equilibrium)**.** A *"partnership-market equilibrium"* consists of a SPE of the partnership game and a profile of outside options $v = (v_1, v_2)$ such that

$$v_i = E\left[\Pi_{i0}^{eqm}(X_0; v)\right] \text{ for } i = 1, 2. \tag{14}$$

Discussion of Assumptions 3-5: By Assumption 3, the partnership is weakly more likely to transition to a higher state tomorrow from a higher state today, holding fixed the history of players' efforts. The fact that no assumptions are made on how efforts impact future states allows for great flexibility, e.g. the model can accommodate settings in which effort grows, depletes, or has a non-monotone effect on a payoff-relevant stock. On the other hand, Assumption 3 does rule out a variety of potential applications in which payoffs are stochastic but not persistent. For instance, suppose that $\mathcal{X}_t = \{\text{low}, \text{high}\}$ for all t as in Bagwell and Staiger (1997). Assumption 3 fails in the case of negative serial auto-correlation.

Here are some simple examples of state processes $(X_t : t \geq 0)$ satisfying Assumption 3. In each case, $\mathcal{X}_t \subset \mathbf{R}^K$. Examples (A-C) are exogenous Markov processes, (D) is a non-Markov exogenous process, (E) is a non-trivially controlled process.

(A) X_t are iid.

(B) X_t reverts to mean μ, e.g. $X_t = \rho\mu + (1-\rho)X_{t-1} + \sigma\varepsilon_t$, where ε_t are iid mean zero.

(C) $g(X_t)$ is a random walk on $\mathbf{R}^K$, where $g : \mathbf{R}^K \to \mathbf{R}^K$ is any non-decreasing function relative to the usual product order on $\mathbf{R}^K$.

(D) $X_t = (Y_0, ..., Y_t)$ is a sequence of publicly observed estimates of K unobserved parameters, e.g. unknown productivity of the match à la Jovanovic (1979a).

(E) X_t is a capital stock with a random growth rate, e.g. $X_{t+1} = Y_t(X_t + \sum_i e_{it})$, where $(Y_t : t \geq 0)$ is an exogenous stochastic process as in any of the previous examples.

By Assumption 4, no two partnerships are identical (with positive probability). Even if two partnerships are payoff-identical, the players in those partnerships will observe different realizations of the public randomization. Access to a public randomization can allow strictly greater expected welfare to be supported in partnership-market equilibria. Intuitively, if the randomization is used to coordinate on more or less desirable equilibria of the partnership game, players in a productive relationship will treat their current partner well for fear that they may get a "bad draw" in future partnerships. This role of public randomization is well-known from the literature on non-stochastic repeated games, and guarantees that the maximal joint outside option $\overline{v}_\Sigma$ (to be defined in (24)) can be supported in partnership-market equilibrium. By giving players access to a public randomization, I focus on the more novel aspects of this paper's analysis, such as whether $\overline{v}_\Sigma$ can be supported by (renegotiation-proof) joint-welfare maximizing equilibrium play.

Assumption 5 imposes at least two substantive economic restrictions. First, shocks to a partnership are idiosyncratic to the players in that partnership. This rules out the possibility of market-wide shocks (correlated across partnerships active at the same time), which would of course be interesting to study in the context of enriching existing models of the business cycle. Indeed, extending the present analysis to allow for correlated shocks appears to be an important and promising direction for future research. Second, shocks to a partnership have no bearing on future partnerships in which those players might participate. Thus, the "state" here does not capture any payoff-relevant characteristics of the players themselves, such as intelligence, beauty, or skills.

4 Welfare-maximizing equilibrium

The analysis here has two parts. Section 4.1 characterizes the joint-welfare maximizing subgame-perfect equilibria (SPE) of the partnership game for any given outside options $v = (v_1, v_2)$ (Theorem 1), and develops welfare and turnover comparative statics (Theorems 2-3). Section 4.2 then characterizes the maximal social welfare that can be supported in the partnership market (Theorem 4), and explores some properties of the partnership-market equilibria that support this maximal social welfare. Namely, I provide sufficient conditions for equilibrium social welfare to be maximized by joint-welfare maximizing SPE play within each partnership.

4.1 Joint-welfare maximizing subgame-perfect equilibrium

Let $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ denote the maximal joint payoff that can be achieved in *any* SPE of the partnership game given outside options $v = (v_i, v_j)$, as evaluated before efforts at time t from payoff-relevant history $(x_t, \mathbf{e}_{t-1})$.[21] I will demonstrate an equilibrium that achieves this maximal SPE joint payoff at every history reached on the equilibrium path.

By deviating from time-t effort-profile e_t with zero effort and then quitting, player i can increase his time-t stage-game payoff by $c_{it}(e_t; x_t)$ and then enjoy outside option v_i with probability γ (should he not die that period). Thus, to support effort-profile e_t, each player i's equilibrium continuation payoff (including wage transfers) after time-t efforts must be at least $\gamma v_i + c_{it}(e_t; x_t)$. In particular, costly efforts can only be sustained if joint equilibrium continuation payoff inside the partnership exceeds players' (survival-weighted) joint outside option *plus* their joint cost of effort. Assuming that continuation play after time-t efforts maximize players' joint equilibrium continuation payoff, equilibrium joint welfare is

[21] Without loss, one may restrict attention to SPE in which payoffs depend on the history of past states $\mathbf{x}_t$ only through the current state x_t. Indeed, the current state may depend on the full history of past states. For example, if $x_t = (x_{t-1}, y_t)$ for all $t > 0$, then the distribution of X_t can depend on all of the "new information" $(x_0, y_1, ..., y_{t-1})$ learned during the course of the partnership. Further, it is without loss to restrict attention to SPE in which payoffs do not depend on the history of wages.

maximized given time-t efforts $e_t(x_t, \mathbf{e}_{t-1})$, defined as

$$e_t(x_t, \mathbf{e}_{t-1}) = \arg\max_{e_t} \left(\pi_{\Sigma t}(e_t; x_t) + \gamma \max\left\{v_\Sigma, E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right]\right\}\right) \quad (15)$$

$$\text{subject to } c_{\Sigma t}(e_t; x_t) \leq \gamma \max\left\{0, E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right] - v_\Sigma\right\}. \quad (16)$$

In fact, this maximal joint payoff can be realized in SPE, with efforts $e_t(x_t, \mathbf{e}_{t-1})$ played at each history $(x_t, \mathbf{e}_{t-1})$ on the equilibrium path.

Theorem 1 (Joint-welfare maximizing SPE)**.** *A SPE exists that maximizes joint payoff among all SPE at every history. On the equilibrium path of play, (i) players exert efforts* $e_t(x_t, \mathbf{e}_{t-1})$ *(defined in (15)) at every history* $(x_t, \mathbf{e}_{t-1})$*, (ii) both stay iff doing so is efficient given their joint equilibrium continuation payoff, and (iii) wages are paid (if at all) at the end of each period only if both players survive and stay.*

Comparative statics. Theorem 1 is proven by an algorithmic argument (outlined in the text below) in the spirit of Abreu, Pearce, and Stacchetti (1990). A side-benefit of this algorithmic style of proof is that I am also able to establish comparative statics properties of the joint-welfare maximizing equilibrium, by showing that these properties are preserved at every step of the algorithm, as well as in the limit.

Theorem 2 (Welfare increasing in the state)**.** *The maximal joint welfare* $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ *that can be realized in SPE from history* $(x_t, \mathbf{e}_{t-1})$ *is weakly increasing in* x_t*, for all* $\mathbf{e}_{t-1}$*.*

The model imposes essentially no restriction on how efforts can control the stochastic process. Consequently, there is little that one can say in general about how efforts in the welfare-maximizing SPE vary with the state, nor on how the history of efforts impacts equilibrium variables such as players' payoffs, efforts, and exit. Indeed, although Theorem 2 establishes that players' joint payoff in the joint-welfare maximizing SPE is increasing in the state x_t, neither joint stage-game payoff nor joint continuation payoff need be increasing in x_t. Consequently, partners may exert lower efforts and even be more likely to exit in higher states. However, additional comparative statics are available in a notable special case, when players' efforts have *no impact* on future states.

Definition 8 (Exogenous stochastic process)**.** $(X_t : t \geq 0)$ is an *exogenous stochastic process* if the distribution of X_t depends only on (t, x_{t-1}).

Given exogeneity, players' effort-decisions at time t have no impact on the set of SPE continuation payoffs. Thus, in any joint-welfare maximizing SPE, players will choose whatever efforts maximize joint stage-game payoff, among those satisfying the relevant incentive-compatibility constraint.

Theorem 3 (Comparative statics with an exogenous state)**.** *Suppose that $(X_t : t \geq 0)$ is an exogenous stochastic process. In the joint-welfare maximizing SPE, at every history reached on the equilibrium path: (i) players' joint stage-game payoff and joint continuation payoff are each weakly increasing in x_t; and (ii) partnership stopping time conditional on x_t is weakly increasing in x_t, in the sense of first-order stochastic dominance.*

Proof sketch for Theorems 1-2. The rest of this section provides a sketch of the proof of Theorems 1-2. (The proof of Theorem 3 is in the Appendix.)

Part I: Credibility and optimality of efforts $e_t(x_t, \mathbf{e}_{t-1})$ and associated "retention bonuses". The first part of the proof hinges on an important preliminary result.

Lemma 3 (Joint-welfare maximizing SPE play)**.** *Suppose that SPE exist such that, at time $t+1$ from each history $(x_{t+1}, \mathbf{e}_t)$, players' joint payoff is $\Pi^{eqm}_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$. (i) A SPE exists such that, at time t from history $(x_t, \mathbf{e}_{t-1})$, players' joint payoff $\Pi^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ solves*

$$\Pi^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) = \max_{e_t} \left(\pi_{\Sigma t}(e_t; x_t) + \gamma \max\left\{v_\Sigma, E\left[\Pi^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t\right]\right\}\right) \tag{17}$$

$$\textit{subject to } c_{\Sigma t}(e_t; x_t) \leq \gamma \max\left\{0, E\left[\Pi^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t\right] - v_\Sigma\right\} \tag{18}$$

(ii) In this SPE, wages are paid (if at all) at the end of each period only if both players survive and stay.

Lemma 3 vastly simplifies the analysis since it implies that an effort-profile can be implemented in SPE iff its joint cost of effort is less than the partnership's excess return over the players' joint outside option. In other words, one may view partners as choosing their effort-profile optimally subject to the endogenous incentive constraint (18),[22] i.e. they will play efforts $e_t(x_t, \mathbf{e}_{t-1})$ at every history $(x_t, \mathbf{e}_{t-1})$.

Part II: Algorithmic characterization of the optimal SPE. Next, I develop an algorithmic argument in the spirit of APS to characterize the joint-welfare maximizing SPE of the partnership game. APS characterize the set of all SPE strategies as the limit of a decreasing sequence of sets of strategy profiles. The approach developed here differs in two ways. First, I focus on the simpler issue of characterizing just the *maximal joint payoff* that can be supported in SPE. Conceptually, at each stage of the APS algorithm,[23] identify the maximal joint payoff that can be achieved by any remaining strategy profiles. Clearly, the sequence of such upper bounds on joint payoff is decreasing and converges to the maximal SPE joint payoff. Second, and more important, the additional structure here allows me to establish new results about the joint-welfare maximizing SPE. Conceptually, by keeping track of the strategies that achieve the upper bound on joint payoffs at every step of the APS algorithm, and by showing that these strategies always satisfy certain properties, I can prove by induction that the joint-welfare maximizing SPE strategies also possess those properties. This allows me to prove, for example, that maximal SPE joint payoff is weakly increasing in the state (Theorem 2) as well as, later, additional welfare and turnover comparative statics when the state follows an exogenous stochastic process (Theorem 3).

Lemma 3 maps the maximal joint payoff that can be supported at time $t+1$ to the maximal joint payoff that can be supported at time t. Thus, one can construct a sequence of upper bounds for every history, $\left\{\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) : k = 1, 2, ...\right\}$, such that $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is non-increasing in k and converges to the maximal SPE joint payoff at history $(x_t, \mathbf{e}_{t-1})$.

[22] The incentive condition (18) arises naturally in repeated games with voluntary transfers. More novel here is the observation regarding the optimal *timing* of wages. In repeated games with exit and iid payoffs, wages paid at any time can provide equivalent incentives to work and stay in the relationship (see Levin (2003) for a discussion). By contrast, in this paper's stochastic environment, the timing of wages matters. Whereas Lemma 3 establishes that it is without loss to assume that wages are paid at the very end of each period, McAdams (2010) provides an example showing that paying wages just after efforts are observed (in the form of "performance bonuses") *does* entail loss of generality.

[23] The approach developed in APS can be extended in a natural way to dynamic repeated games. See e.g. Chapter 5.7.1 of Mailath and Samuelson (2006).

Also importantly for this paper's purposes, these upper bounds exhibit a monotonicity in the state, i.e. $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is weakly increasing in x_t for all k as well as in the limit.

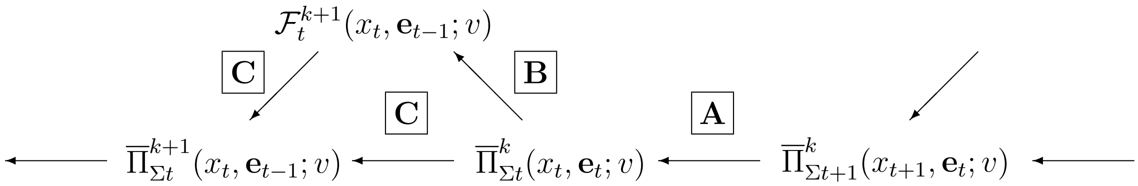

Figure 6: Key steps of the algorithmic argument.

Key steps of the algorithmic argument (illustrated in Figure 6): Suppose that there exist upper bounds $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ on SPE joint payoff from all histories at all times and that these upper bounds are weakly increasing in the current state x_t. The essence of the proof is to use these upper bounds to derive *weakly lower* upper bounds $\overline{\Pi}^{k+1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ at all histories that remain weakly increasing in x_t. Here in the text, I provide the inductive step to construct this sequence of upper bounds on joint payoff and show that monotonicity in x_t is preserved along this sequence. In the Appendix, I prove that this sequence of bounds is weakly decreasing in k and that it converges to joint payoff that can be realized in SPE.

Step A. Given bounds $\overline{\Pi}^k_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$ on joint continuation payoff at time $t+1$, joint continuation payoff after time-t efforts is bounded by

$$\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v) = \gamma \max\left\{v_\Sigma, E\left[\overline{\Pi}^k_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right]\right\}. \tag{19}$$

Observe that

$$E\left[\overline{\Pi}^k_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right] = \int_0^\infty \Pr\left(\overline{\Pi}^k_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) \geq z|x_t, \mathbf{e}_t\right) \mathrm{d}z. \tag{20}$$

By presumption, $\overline{\Pi}^k_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$ is weakly increasing in x_{t+1}. Thus, the set $\{x_{t+1} : \overline{\Pi}^k_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) \geq z\}$ is an increasing subset of $\mathcal{X}_{t+1}$ for all z. By Assumption 3, then, each of the probability terms inside the integral in (20) is weakly increasing in x_t. Thus, $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)$ is weakly increasing in x_t.

Step B. Let $\mathcal{F}_t^{k+1}(x_t, \mathbf{e}_{t-1}; v)$ be the set of time-t efforts that can be supported in SPE given joint continuation payoffs $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)$ after effort, i.e. those satisfying the IC-constraint

(18) given these expected joint continuation payoffs after time-t effort. Since (18) slackens with higher continuation payoffs, the fact that $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)$ is weakly increasing in x_t implies that $\mathcal{F}^{k+1}_t(x_t, \mathbf{e}_{t-1}; v)$ is weakly increasing in x_t, relative to the set inclusion order.

Step C. By Lemma 3, we may define new upper bounds on time-t SPE joint payoff,

$$\overline{\Pi}^{k+1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) = \max_{e_t \in \mathcal{F}^{k+1}_t(x_t, \mathbf{e}_{t-1}; v)} \left(\pi_{\Sigma t}(e_t; x_t) + \overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v) \right). \tag{21}$$

$\overline{\Pi}^{k+1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is weakly increasing in x_t since both $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)$ and $\mathcal{F}^{k+1}_t(x_t, \mathbf{e}_{t-1}; v)$ are weakly increasing in x_t, while $\pi_{\Sigma t}(e_t; x_t)$ is weakly increasing in x_t by Assumption 2.

The remainder of the proof is in the Appendix. □

4.2 Social-welfare maximizing partnership-market equilibrium

A social-welfare maximizing partnership-market equilibrium is one that maximizes players' joint outside option, among all partnership-market equilibria. Recall that $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ denotes the maximal SPE joint payoff at history $(x_t, \mathbf{e}_{t-1})$ given outside options $v = (v_i, v_j)$. Lemma 4 establishes that this maximal joint payoff depends on players' outside options only through their *sum*.

Lemma 4. *$v'_\Sigma = v_\Sigma$ implies $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v') = \overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$.*

Proof. The proof of Lemma 4 is immediate from the algorithmic construction in the proof of Theorems 1-2. Lemma 4 can also be viewed as a corollary of Lemma 3, once one observes that players' outside options do not appear in the objective (17) or in the constraint (18) except through the sum v_Σ. Intuitively, asymmetries in players' outside options have no impact on what can be achieved in equilibrium, since any such asymmetries can be counter-balanced by appropriate retention bonuses. □

Definition 9 (Maximal SPE joint payoff)**.** Let $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v_\Sigma)$ denote the maximal joint payoff in any SPE from history $(x_t, \mathbf{e}_{t-1})$, as a function of players' *joint* outside option v_Σ.

Players' joint outside option cannot exceed $\sup\{v_\Sigma : v_\Sigma \leq E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)\right]\}$. Theorem 4 establishes that this maximal joint outside option can in fact be supported in equilibrium, and shows one way in which to do so.

Theorem 4 (Maximal social welfare)**.** *In social-welfare maximizing partnership-market equilibria, players' endogenous joint outside option is*

$$\overline{v}_{\Sigma} = \sup\left\{v_{\Sigma} : v_{\Sigma} \leq E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_{\Sigma})\right]\right\}. \tag{22}$$

Further, play in one such equilibrium proceeds as follows: if the public randomization $z_0 \leq \frac{E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0;\overline{v}_{\Sigma})\right]-\overline{v}_{\Sigma}}{E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0;\overline{v}_{\Sigma})\right]-\gamma\overline{v}_{\Sigma}}$*, then both players exert zero effort and quit immediately; otherwise, continuation play maximizes SPE joint welfare as in Theorem 1.*

Discussion of Theorem 4: Suppose for the moment that the maximal expected joint payoff that can be supported in SPE, $E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_{\Sigma})\right]$, is continuous in the players' joint outside option v_{Σ}. In this case,

$$\overline{v}_{\Sigma} = E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; \overline{v}_{\Sigma})\right] \tag{23}$$

and Theorem 4 implies that equilibrium social welfare is maximized only when players maximize equilibrium joint welfare within each partnership.

This finding sheds new light on a well-known result in the literature on repeated games with re-matching. In non-stochastic repeated games with re-matching, maximizing social welfare always requires that partners fail to maximize joint welfare. (See Section 5.2 of Mailath and Samuelson (2006) for a particularly clear exposition.) A key insight that emerges here is that the tension between market-wide and individual partnership performance disappears once partnerships are *not* created equal, namely, when "first impressions" are payoff-relevant. Indeed, even quite modest initial stochastic variation across partnerships eliminates the need to perform sub-optimally at the start of relationships, as when the initial state is augmented with a "partnership type" $S_0 \sim U[0,1]$ as follows.

Assumption 6 (Meaningful first impressions)**.** $X_t = (Y_t, S_0)$ for all t, where $S_0 \sim U[0,1]$ and, for all t, y_t, $(y_t, s_0') \succ (y_t, s_0)$ for all $s_0' > s_0$ and $\pi_{it}(e_t'; s_0, y_t) - \pi_{it}(e_t; s_0, y_t)$ is *strictly* increasing in s_0 for all i and $e_t' \succ e_t$.

S_0 is a public signal (that I refer to as "the first impression" or as "the partnership type") observed by players at the start of their relationship, capturing an aspect of match quality that, by definition, increases players' stage-game payoffs and decreases their cost of effort.

Note that Theorem 5 holds no matter how small the impact of the partnership type on payoffs.

Theorem 5. *Given meaningful first impressions, a full measure of partnerships achieve the maximal SPE joint payoff in any social-welfare maximizing partnership-market equilibrium.*

Discussion of Theorem 5. The presence of meaningful first impressions eliminates the need to "waste surplus" at the start of relationships. Intuitively, the reason is that observing a payoff-relevant type at time $t = 0$ breaks players' *indifference* over potential partners. When players do not care about the identity of their partner, each player will naturally be concerned that his current partner will cheat him and then re-match with an equally attractive replacement. Burning money at the start of every partnership allows players to assuage this concern. Once players strictly prefer some partners over others, however, the re-matching market no longer provides "easy pickings" for a cheater. In particular, players will reject any partner who is not a sufficiently good fit, and fear of future rejection provides a deterrent against misbehavior in any sufficiently well-matched partnership. Of course, the threshold for a "sufficiently good fit" is endogenous. It is determined so that players at this threshold are indifferent between (i) staying and playing the joint-welfare maximizing SPE of Theorem 1 or (ii) quitting to re-match. (For a worked-out example, see Section 2.2.)

When first impressions matter, Theorem 5 implies that all social-welfare maximizing partnership-market equilibria specify joint-welfare maximizing play.[24] Theorem 6 establishes that joint-welfare maximizing play is also *sufficient* to maximize equilibrium social welfare in this case.

Theorem 6. *Given meaningful first impressions,* $\overline{v}_\Sigma$ *is the* unique *solution to*

$$v_\Sigma = E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)\right]. \tag{24}$$

Discussion of Theorem 6: Outside options $v = (v_i, v_j)$ are generated by joint-welfare maximizing SPE play if $v_i = E\left[\overline{\Pi}^{eqm}_{i0}(X_0; v)\right]$ for $i = 1, 2$. Theorem 6 implies that only one *joint* outside option can be generated by joint-welfare maximizing SPE play. (This result is

[24] If (15) has multiple solutions at some history, then there will exist different social-welfare maximizing SPE in which each of these optimal IC efforts is played. Otherwise, efforts are unique.

not obvious, since equilibrium play depends on the outside option.) Technically, the proof proceeds by showing that the maximal excess joint return of a new partnership over the players' joint outside option, $E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)\right] - v_\Sigma$, is strictly decreasing in v_Σ. Intuitively, as players' outside options become less valuable, they have more reason to work and invest in their relationship. Thus, even if falling outside options are bad news in the sense of lowering equilibrium payoffs, this loss is mitigated by the fact that the players' partnership becomes stronger and more productive.

Social-welfare maximizing partnership-market equilibria could potentially differ in how the surplus is divided between the players. For example, if a wife anticipates that her *next* husband will pay her a handsome wage, then her current husband might need to pay her a wage to induce her to stay. In this way, even if there exists a social-welfare maximizing partnership-market equilibrium in which no wages are paid, other such equilibria might exist in which either player receives the lion's share of the surplus.

Steady-state distribution over partnership histories. One may view a partnership as a Markov chain over *histories* $h_t = (x_t; \mathbf{e}_{t-1})$, where any partnership that ends at time t is understood to transition to a brand new partnership (with new partners).

Suppose that a partnership is currently in history h_t. Let $e_t(h_t)$ be the effort-profile played in the optimal SPE at this history, as characterized in the proof of Theorem 2. Similarly, let $p^{exit}_t(h_t)$ be the probability that at least one player would quit at time t after this history, and let $X_{t+1}(h_t) \sim X_{t+1}|(h_t, e_t(h_t))$ denote next-period's state should the partnership persist to that time. Transition probabilities among histories may be fully described as follows:

- With probability $1 - \gamma^2$, the partnership will end due to death, after which a new partnership will be created having random initial history $H_0 = X_0$.
- With probability $\gamma^2 p^{exit}_t(x_t, \mathbf{e}_{t-1})$, the partnership will end due to some partner's endogenous departure, after which a new partnership will again be created.
- With probability $(1-\gamma^2)(1-p^{exit}_t(x_t, \mathbf{e}_{t-1}))$, the partnership will continue to time $t+1$, with an augmented random history $H_{t+1} = (h_t; e_t(h_t); X_{t+1}(h_t))$.

Note that, through the process of death and re-birth, all histories that are reached on the equilibrium path communicate and are positively recurrent. Thus, this Markov chain is ergodic and there exists a unique steady-state distribution over histories.

Claim 2 (Steady-state distribution)**.** *For every partnership-market equilibrium, there exists a unique steady-state distribution over partnership histories.*

Proof. This result follows from standard Markov-chain methods; details are omitted to save space. See Sections 4.3 and 4.6.2 of Ross (1996), especially Theorem 4.3.3. □

In the remainder of this section, I discuss some qualitative features of a "typical" player's life experience, assuming welfare-maximizing partnership-market equilibrium play.

Dating. At time $t = 0$, players will immediately exit any relationship in which the realized initial state is in a decreasing subset of $\mathcal{X}_0$. Consequently, any player who is seeking a new partner will typically experience several partnerships that each last exactly one period – and in which both players exert zero effort because they anticipate no future interaction – before finding a partner who they do not immediately leave.

Honeymoon. In any partnership that continues to a second period, players obviously expect higher continuation payoffs than during their unsuccessful dating phase. In fact, such "newly-joined" partners will also enjoy higher stage-game payoffs than when they were unsuccessfully dating, for two reasons. First, the initial state in a "successful date" will be higher than in an unsuccessful one, allowing players to generate higher stage-game payoffs from any time-0 efforts (Assumption 2). Second, since the players view their future relationship as generating higher continuation payoffs than their outside options, they can also support non-trivial effort at time $t = 0$.

Of course, there is no guarantee that a surviving partnership in its earliest periods will be very profitable or very stable. For instance, it could be that the initial state lies very close to the threshold below which the partnership would not have formed, and that there is a high likelihood of break-up in the near future. However, since this caveat applies equally to non-strategic models that identify a "honeymoon effect", such as Fichman and Levinthal

(1991),[25] one can say that social-welfare equilibrium play here also exhibits a "honeymoon effect" – *if* players form meaningful first impressions, so that social-welfare maximizing play dictates joint-welfare maximizing play from the start of each relationship. Otherwise, social-welfare maximizing play dictates that players in new partnerships receive artificially depressed payoffs, a sort of "anti-honeymoon effect".[26]

Hard times. The state of a partnership may rise and fall many times, in ways that affect the extent of cooperation that can be supported in the welfare-maximizing equilibrium. This volatility of players' willingness to cooperate creates payoff volatility that in turn creates an endogenous option value to remaining in the relationship. Consequently, players tend to remain in relationships even when stage-game payoffs are low, in hopes that their partner's behavior will improve.

Good times and golden years. Players stay in the partnership during hard times in the hope that they will enjoy positive shocks that will enable them to enjoy higher profits and greater stability in the future. Indeed, depending on the details of the stochastic process $(X_t : t \geq 0)$, there may be an increasing subset of the state-space from which the partnership is certain never to end, save by exogenous death. Such "golden years" can arise for two sorts of reasons. First, there may be an absorbing portion of the state-space, that is everywhere high enough to support continuation of the partnership. Second, equilibrium efforts in high enough states may be sufficiently high and feedback from profitable efforts may be positive enough to overwhelm any exogenous shocks that might cause the relationship to deteriorate.

4.3 Extension: positive matching cost

The analysis of Sections 4.1-4.2 can be readily adapted to an extension of the model in which players must pay $m > 0$ whenever (re-)matched. The main difference is that players'

[25]Fichman and Levinthal (1991) consider an organization's decision to form and disband, when profits follow an exogenous random walk. The analysis here differs in several ways, the most important being that (i) profits are endogenous and (ii) the outside option is endogenous.

[26]The term "anti-honeymoon" is not standard, but this idea is well-known. See e.g. Section 5.2 of Mailath and Samuelson (2006).

outside options depend on whether there is an active re-matching market. If players of either gender do not expect to recoup the matching cost in a new partnership, no new partnerships will form and each player's outside option is zero. In particular, for large enough matching costs, all partnership-market equilibria are trivial ones in which (i) partnerships never form and (ii) should they form, play proceeds as in a SPE of the partnership game given zero outside options. For small enough matching costs, however, the social-welfare maximizing partnership-market equilibrium is non-trivial, with non-negative outside options generated by the prospect of starting a new partnership:

$$v_i = E\left[\Pi_{i0}^{eqm}(X_0; v)\right] - m \geq 0 \text{ for } i = 1, 2, \tag{25}$$

where $\Pi_{i0}^{eqm}(x_0; v)$ denotes player i's payoff in some SPE of the partnership game given initial state x_0 and outside options $v = (v_i, v_j)$. (The set of such SPE payoffs is characterized in Section 4.1.)

Claim 3. *Given meaningful first impressions, $\overline{v}_\Sigma(m)$ is strictly decreasing in m in a neighborhood of $m = 0$.*

Discussion: An increase in matching costs has competing effects on payoffs in the social-welfare maximizing equilibrium. While players must pay more to form each partnership, such costs can act as an exit deterrent and hence encourage players to work and invest in their current partnership. However, this benefit of better partnership performance only arises if the *overall* effect of higher matching costs is to lower players' outside options. Thus, this overall effect must be weakly negative on ex ante payoffs. Indeed, the overall effect of higher matching cost is *strictly* negative since players will respond to higher partnership formation costs by sampling strictly fewer partners, leading to a strictly worse average fit among active partnerships when first impressions are payoff-relevant.

By contrast, in non-stochastic repeated games with re-matching, maximizing social welfare requires that players "waste" some expected surplus at the start of their partnership, in order to incentivize effort in established relationships. Matching costs then serve as a substitute for such waste, allowing players to support the same social welfare for all small enough matching costs.

5 Concluding Remarks

In their study of relational contracts in developing economies, Johnson, McMillan, and Woodruff (2002) emphasize the importance of *established* relationships in supporting the "trust" necessary to work together in an environment lacking a reliable court system. The theory of repeated games with re-matching has advanced two alternative explanations for why players may only trust those with whom they already have a working relationship. According to one view (see e.g. Kranton (1996)), social custom may require that players incur costs / forego potential equilibrium benefits ("burn money") when establishing a relationship. Since players already in a relationship prefer to avoid burning money a second time, they will be careful to treat their current partner well. According to a second view (see e.g. Sobel (1985)), players' actions may signal information about their motives to their current partner, so that surviving partnerships are only those in which both players have proven themselves sufficiently trustworthy.

One of the novel findings of this paper is that burning money is never socially optimal when (i) players have no private information and (ii) players form "meaningful first impressions" that are at least somewhat informative of future payoffs. Put differently, increasing the cost of *forming* a new relationship unambiguously lowers social welfare under these conditions. Thus, this paper sheds light on the set of circumstances in which we should expect costly courtship. In addition to settings with private information, in which a suitor may feel compelled to prove his love, courtship may arise in environments in which players can only learn about the quality of their match *after* forming a new relationship.

Separately, a broad empirical literature from Topel and Ward (1992) on employment, Levinthal (1991) on firm survival, and Stevenson and Wolfers (2007) on marriage have established certain stylized facts about relationship dynamics. For instance, partnerships often exhibit a "honeymoon effect" and a "survivorship bias" in that very young and very old partnerships are often more profitable and less likely to end in the near future than those of intermediate age. A rich theoretical literature has provided a foundation for such dynamics in a context with one-sided incentives. For instance, in a labor search context (e.g. Pissarides (1994)), workers will only start a new job and/or leave their current job when presented with a sufficiently attractive new opportunity, so that new jobs will tend at first to be highly productive honeymoons. Similarly, when each firm's productivity is subject to persistent random shocks (e.g. Jovanovic (1982)), longer-lived firms will tend to be those that have received mostly positive shocks and hence be more likely to survive in the near future.

This paper extends this existing literature by adding two-sided incentives and a rich stochastic structure. The resulting theory generates potentially testable predictions about the dynamics and duration of partnerships (and of search interludes between matches) in novel applications ranging from supply-chain and customer relationships to joint ventures, as well as potentially enriching the study of classic applications in labor and macroeconomics.

I conclude with a brief discussion of a few directions that I hope to pursue in future work.

Macroeconomic shocks. One interesting direction for future research would be to consider the interaction between macroeconomic shocks and partnership performance and turnover dynamics. For instance, suppose that all active partnerships are subject to common multiplicative shocks to productivity,[27] but that matching costs do not change over time. In this context, positive shocks naturally induce greater search, as players care relatively more about finding a better match. Such intensified search will lead to (i) less stable and hence less productive partnerships but also (ii) higher-quality matches, with implications for how macroeconomic shocks affect equilibrium social welfare.

Transitional dynamics. Similarly, it would be interesting to adapt this paper's analysis to characterize transitional dynamics, when the partnership market is not in steady state after a macroeconomic shock. This could shed new light, for instance, on how the activity of labor markets varies over a recessionary cycle.

Changing individuals. In this paper, each player's next partnership is stochastically identical to his current one. In other words, all shocks are to partnerships, not to the individuals in those partnerships. Of course, individuals may also change in ways that will persist in a new match. Enriching the model to allow for such personal characteristics is important

[27] That is, stage-game payoffs take the form $\lambda_t \pi_{it}(e_t; x_t)$ where x_t is a partnership-specific state and λ_t is a common factor in all active partnerships.

and could have profound implications for the steady-state distribution of the partnership market. For one thing, the set of players seeking to re-match will be adversely selected. This could increase active partners' desire to avoid the re-matching market, creating a still deeper adverse selection in this market.

Endogenous learning. The model here can capture a wide variety of "learning" settings in the spirit of Jovanovic (1979a), including ones in which players make investments to increase the precision of a public signal about an unobserved payoff-relevant parameter. (Such investment could be one aspect of players' multi-dimensional effort.) When investing in a more precise signal of the underlying state, players create short-term volatility in their beliefs about the state. Such volatility can increase the value of the players' option to exit, but could also be harmful if it disrupts an otherwise productive partnership. This suggests a speculation, that players in a stable relationship may actively seek to avoid uncovering new information, while players in a rocky relationship may seek to uncover as much new information as possible.

Appendix

Proof of Lemma 1.

Proof. Because $(C_t : t \geq 0)$ is a Markov process, the optimal exit rule is memoryless: there exists an "optimal exit-set" $E \subset R_{++}$ such that the players terminate the partnership in the first period t in which $c_t \in E$. A necessary condition of optimality is that, given work threshold c^W and exit-set E, expected joint payoff is maximized by terminating the partnership when the current state is in E and by otherwise not terminating the partnership. Indeed, given that $(\log(C_t) : t \geq 0)$ is a random walk and joint stage-game payoff each period is non-increasing in the state c_t, this necessary condition pins down the optimal exit-set as of the form $E = [c^E(c^W; v), \infty)$, for some exit threshold $c^E(c^W; v)$. (The details are straightforward and omitted to save space. See e.g. Dixit and Pindyck (1994).) To complete the proof, I need to show that $c^E(c^W; v) = \alpha(v)c^W$, where $\alpha(v)$ is non-increasing in v.

Suppose for the moment that $c^E(c^W; v) > 0$. A necessary condition of optimality is that, conditional on $c_0 = c^E(c^W; v)$, players' joint payoff is the same whether they both stay or quit, given that continuation play will be according to thresholds $(c^W, c^E(c^W; v))$. (If $c^E(c^W; v) = 0$, joint payoff must be weakly higher when players quit, and the argument is easily modified.) In particular,

$$\begin{aligned} &E[\#\{t \in \{1, 2, ..., T(c^E(c^W; v))\} : C_t \leq c^W\} | c_0 = c^E(c^W; v)] \\ &\qquad = v \Pr\left(T_i^{die} = T(c^E(c^W; v)) | c_0 = c^E(c^W; v)\right). \end{aligned} \tag{26}$$

where $T(c^E)$ denotes the stopping time of the partnership when both players adopt exit threshold c^E, and T_i^{die} is the time of player i's death. (26) is an indifference condition. LHS of (26) captures the benefit of staying in the partnership, that each player will enjoy some future periods in which both will work (when $c_t \leq c^W$) with stage-game payoff of one. RHS of (26) captures the benefit of quitting, that each player will avoid the possibility of losing their outside option v to death. (If $T_i^{die} > T(c^E(c^W; v))$, then player i survives as either a "widow" or "divorcee".)

By definition, $T(c^E)$ is the first time at which $c_t > c^E$ and/or some player dies. Conditional on $c_0 = c^E$, then, $T(c^E)$ is the first time at which $\frac{C_t}{C_0} > 1$ and/or some player dies. Since $\log(C_t)$ is a random walk, the distribution of $\frac{C_t}{C_0}$ is independent of C_0. Thus, the distribution of $T^{sep} = \min\{t : \frac{C_t}{C_0} > 1\} | (c_0 = c^E)$ does not depend on c^E. (T^{sep} is mnemonic for "time of separation".) Since $T(c^E) = \min\{T^{sep}, T_i^{die}, T_j^{die}|\}$ and death is independent of separation, we conclude that (i) $T(c^E) | (c_0 = c^E)$ does not depend on c^E and (ii) $\Pr\left(T_i^{die} = T(c^E) | c_0 = c^E\right)$ does not depend on c^E. In particular, the RHS of (26) does not depend on $c^E(c^W; v)$, and the LHS of (26) depends on $c^E(c^W; v)$ only through the ratio $\frac{c^E(c^W;v)}{c^W} \equiv \alpha(v)$ (again, because $(\log(C_t) : t \geq 0)$ is a random walk). We conclude that, for all work thresholds $c^W > 0$, the optimal exit threshold $c^E(c^W; v) = \alpha(v) c^W$ for some $\alpha(v) \geq 0$.

The key observation to prove that $c^E(c^W; v)$ is non-increasing in v is that the LHS of (26) is non-increasing in the exit threshold, i.e. $E[\#\{t \in \{1, 2, ..., T(c^E)\} : C_t \leq c^W\} | c_0 = c^E]$ is non-increasing in c^E. To see why, note that $(\log(C_t) : t \geq 0)$ being a random walk implies (i) the distribution of $T(c^E) | (c_0 = c^E)$ does not depend on c^E and (ii) $\Pr(C_t \leq c^W | c_0)$ is non-increasing in c_0. On the other hand, as discussed above, $\Pr\left(T_i^{die} = T(c^E) | c_0 = c^E\right)$ does not depend on c^E. By (26), then, $c^E(c^W; v)$ is non-increasing in v. □

Proof of Lemma 2.

Proof. As shorthand, let "v-SPE" refer to a SPE of the partnership game when the players' outside option equals v.

Part I: $c^{*W}(v^h) \leq c^{*W}(v^l)$. Suppose there exists a v^h-SPE in which players adopt work threshold $c^W(v^h)$. I will show that there exists a v^l-SPE in which players adopt the same work threshold $c^W(v^h)$ and the v^l-optimal exit threshold $\alpha(v^l)c^W(v^h)$ (see Lemma 1). This will imply that $c^{*W}(v^l) \geq c^{*W}(v^h)$.

As an intermediate step, suppose that players were to adopt the v^h-SPE strategies given outside option v^l. That is, the players both adopt work threshold $c^W(v^h)$ and quit at exactly the same histories as in the v^h-SPE. Given outside option v^h, at every on-path history at which $c_t \leq c^W(v^h)$, each player gets continuation payoff after time-t efforts of at least $c_t + \gamma v^h$. (Otherwise, each player would prefer to shirk and then quit should he survive to the end of period t.) Given outside option v^l, each player's stream of payoffs differs only in that he gets $v^h - v^l$ less when enjoying his outside option. Since he survives after time-t efforts only with probability γ, each player's continuation payoff is at most $\gamma(v^h - v^l)$ lower than given outside option v^h. At every on-path history at which $c_t \leq c^W(v^h)$, then, each player gets continuation payoff after time-t efforts of at least $c_t + \gamma v^l$. Thus, each player is willing to adopt work threshold $c^W(v^h)$ given outside option v^l *if* players were to follow the quitting strategies of the v^h-SPE.

Now, suppose that both players adopt work threshold $c^W(v^h)$ as in the v^h-SPE but now adopt v^l-optimal exit threshold $\alpha(v^l)c^W(v^h)$. Since this exit threshold maximizes each player's payoff from every history, each player's continuation payoff from every history is no less than when v^h-SPE quitting strategies were followed. Thus, both players remain willing to adopt work threshold $c^W(v^h)$. Further, by the proof of Lemma 1, both players are willing to adopt exit threshold $\alpha(v^l)c^W(v^h)$ when they both adopt work threshold $c^W(v^h)$. Namely, $(c^W(v^h), \alpha(v^l)c^W(v^h))$-threshold equilibrium exists given outside option v^l.

Part II: $c^{*W}(v^l) - c^{*W}(v^h) \leq v^h - v^l$. Let $\hat{c}^W = c^{*W}(v^l) - (v^h - v^l)$ and recall that $\alpha(v)c^W$ is the optimal exit threshold given a work threshold c^W and outside option v. To show that $c^{*W}(v^h) \geq \hat{c}^W$, it suffices to show that there exists $(\hat{c}^W, \hat{c}^E(v^h))$-threshold equilibrium given outside option v^h, where I will use notation $\hat{c}^E(\tilde{v}) = \alpha(\tilde{v})\hat{c}^W$. To establish such equilibrium existence, in turn, it suffices to check that players are willing to work should $c_0 = \hat{c}^W$, conditional on continuation play according to the work and exit thresholds $(\hat{c}^W, \hat{c}^E(v^h)$. (For details of this step, see the proof of Proposition 1.) Since $\hat{c}^E(v^h)$ is the optimal exit threshold given work threshold $\hat{c}^W$ and outside option v^h, each player's continuation payoff inside the partnership is weakly greater than under the sub-optimal exit threshold $\hat{c}^E(v^l)$. Thus, it suffices to check that players are willing to work should $c_0 = \hat{c}^W$, conditional on continuation play according to thresholds $(\hat{c}^W, \hat{c}^E(v^l)$.[28] That is, I need only show that

$$E\left[\#\left\{t \in \{1,2,...,T(\hat{c}^E(v^l))\} : C_t \leq \hat{c}^W\right\} | c_0 = \hat{c}^W\right] \geq \hat{c}^W + v^h \Pr\left(T_i^{die} = T(\hat{c}^E(v^l)) | c_0 = \hat{c}^W\right) \tag{27}$$

Using the fact that $\log(C_t)$ is a random walk, $\frac{\hat{c}^E(v^l)}{\hat{c}^W} = \frac{c^{*E}(v^l)}{c^{*W}(v^l)} = \alpha(v^l)$ implies that the LHS of (27) equals $E[\#\{t \in \{1,2,...,T(c^{*E}(v^l))\} : C_t \leq c^{*W}(v^l)\}|c_0 = c^{*W}(v^l)]$. Next, the RHS of (27) is at most $c^{*W}(v^l) + v^l \Pr(T_i^{die} = T(c^{*E}(v^l))|c_0 = c^{*W}(v^l))$ since (i) $\hat{c}^W = c^{*W}(v^l) - (v^h - v^l)$ and (ii) $\Pr(T_i^{die} = T(c^{*E}(v^l))|c_0 = c^{*W}(v^l)) = \Pr\left(T_i^{die} = T(\hat{c}^E(v^l))|c_0 = \hat{c}^W\right) \leq 1$. The desired inequality (27) then follows from the fact that (5) holds with equality when $c^W = c^{*W}(v^l)$, $c^E = c^{*E}(v^l)$, and $v = v^l$. □

Proof of Lemma 3.

Proof. Let $\Pi_{it}^{eqm}(x_t, \mathbf{e}_t; v) = \gamma E\left[\Pi_{\Sigma t+1}^{eqm}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right]$ be shorthand for player i's expected time-t continuation payoff, after efforts e_t from history $(x_t, \mathbf{e}_{t-1})$, should both players subsequently choose to stay to period $t+1$ upon surviving period t. Figure 7 illustrates the key idea of Lemma 3. As long as $\Pi_{\Sigma t}^{eqm}(x_t, \mathbf{e}_t; v)$ exceeds the players' joint payoff after exit, γv_Σ, *plus* their joint incentive to shirk from efforts e_t, $c_{\Sigma t}(e_t; x_t)$, there exists a retention bonus promise given which both players have sufficient incentive to exert efforts e_t and then stay. Further, this promise is credible since each player promises less than his willingness to pay to avoid cooperation breakdown.

[28] The argument here does *not* presume existence of $(\hat{c}^W, \hat{c}^E(v^l))$-threshold equilibrium given outside option v^h. (Indeed, such strategies do not constitute an equilibrium, since players have an incentive to quit sooner given a higher outside option.)

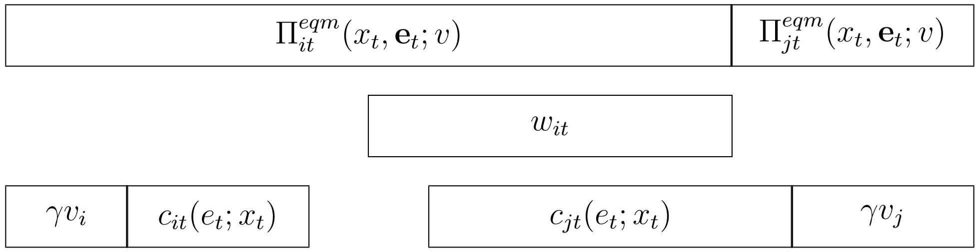

Figure 7: Efforts e_t are incentive-compatible when player i pays wage w_{it} (and $w_{jt} = 0$).

Let $\Delta_{it}(e_t) = \Pi_{it}^{eqm}(x_t, \mathbf{e}_t; v) - \gamma v_i - c_{it}(e_t; x_t)$ denote player i's "excess continuation payoff", the extra profit that he enjoys inside the partnership after efforts e_t, relative to deviating with zero effort and then quitting the relationship. $\Delta_{it}(e_t)$ is the most that player i can credibly promise to pay player j as a reward for not deviating from the prescribed efforts e_t and then not quitting.[29]

Without loss, suppose that $\Delta_{it}(e_t) \geq \Delta_{jt}(e_t)$. If $\Delta_{it}(e_t) + \Delta_{jt}(e_t) < 0$, then at least one player who exerts positive effort must strictly prefer to deviate by exerting zero effort and then quitting, given any credible wage. Otherwise, any retention bonus $w_{it} \in [\max\{0, -\Delta_{jt}(e_t)\}, \Delta_{it}(e_t)]$ from player i to player j can credibly support efforts e_t. Thus, effort-profile e_t can be supported in some SPE iff $\Delta_{it}(e_t) + \Delta_{jt}(e_t) \geq 0$, i.e. iff e_t satisfies (18). This completes the proof, since then the maximal SPE joint welfare given the specified continuation payoffs is the solution to (17). □

Proof of Theorems 1-2.

Proof. Let $\overline{\Pi}_t^{eqm}(x_t, \mathbf{e}_{t-1}; v) \in \mathbf{R}^2$ be the players' payoff profile in a SPE that maximizes *joint* welfare among all SPE from history $(x_t, \mathbf{e}_{t-1})$, and let $\overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v) = \Sigma_i \overline{\Pi}_{it}^{eqm}(x_t, \mathbf{e}_{t-1}; v)$.

Outline of proof. I will construct a monotonically decreasing sequence of bounds on SPE joint welfare from each history, $(\overline{\Pi}_{\Sigma t}^{k}(x_t, \mathbf{e}_{t-1}; v) : k \geq 0)$, that converges pointwise to

[29] Should efforts e_t be played, player i becomes willing to pay up to $\Delta_{it}(e_t) + c_{it}(e_t; x_t)$ to avoid exit. Then, should both players survive and stay to period $t+1$, player i becomes willing to pay more still to avoid a transition to an optimal punishment continuation SPE in which both players exert zero effort, pay zero wages, and exit for certain at time $t+1$. Thus, player i has sufficient incentive to exert his prescribed effort, then stay, then pay the specified bonus.

$\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$, and show that this maximal joint payoff is implemented by SPE strategies as specified in Theorem 1. Further, $\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is non-decreasing in x_t for each k, as well as in the limit $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$, establishing Theorem 2.

Part I: Decreasing sequence of bounding payoff-profile sets. By Assumption 1, there exists a uniform upper bound M on players' joint payoff at any history. Define $\overline{\Pi}^{0}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) = M$ at all histories. Clearly, $\overline{\Pi}^{0}_{\Sigma t} \geq \overline{\Pi}^{eqm}_{\Sigma t}$. Next, for all $k \geq 1$, define $\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ recursively as follows (using shorthand $\mathbf{e}_t = (\mathbf{e}_{t-1}, e_t)$):

$$\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) = \max_{e_t} \left(\pi_{\Sigma t}(e_t; x_t) + \gamma \max \left\{ v_\Sigma, E\left[\overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] \right\} \right) \tag{28}$$

$$\text{subject to } c_{\Sigma t}(e_t; x_t) \leq \gamma \max \left\{ 0, E\left[\overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] - v_\Sigma \right\}. \tag{29}$$

Assuming that $\overline{\Pi}^{k-1}_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$ are upper bounds on joint payoff at time $t+1$, then (29) is a necessary condition for efforts e_t to be supported in any SPE from history $(x_{t+1}, \mathbf{e}_t)$. *Proof:* Players expect joint "inside continuation payoff" of at most $\gamma E\left[\overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t\right]$ should they choose effort-profile e_t and then both stay should both survive. If players' joint outside option v_Σ exceeds this bound, then at least one player strictly prefers to quit and neither player can be incentivized to exert any costly effort. Otherwise, players' joint cost of effort $c_{\Sigma t}(e_t; x_t)$ must be less than or equal to the amount by which their joint inside continuation payoff exceeds their joint outside option. (If not, at least one player would strictly prefer to deviate by exerting zero effort and then quitting.) Indeed, by Lemma 3, (29) is also a sufficient condition to support time-t efforts e_t in SPE given continuation payoffs $\overline{\Pi}^{k-1}_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$.

Since $\overline{\Pi}^{0}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is an upper bound on joint payoff, $\overline{\Pi}^{0}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}^{1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$. By induction, $\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is non-increasing in k. (The value of the maximization (28) is non-decreasing in continuation payoffs. Thus, $\overline{\Pi}^{k}_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v) \leq \overline{\Pi}^{k-1}_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$ for all $(x_{t+1}, \mathbf{e}_t)$ implies $\overline{\Pi}^{k+1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) \leq \overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$.) Further, by induction, $\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ for all k. (Higher-than-equilibrium payoffs can be supported given higher-than-equilibrium continuation payoffs. Thus, the fact that $\overline{\Pi}^{k-1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ implies $\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$.)

Part II: These upper bounds on joint welfare are non-decreasing in x_t.

Base step: $k = 0$. $\overline{\Pi}_t^0(x_t, \mathbf{e}_{t-1}; v) = M$ is constant and hence trivially non-decreasing in x_t.

Induction step: $k \geq 1$. Suppose that $\overline{\Pi}_t^{k-1}(x_t, \mathbf{e}_{t-1}; v)$ is non-decreasing in x_t for all t. Observe that, for any $x_t^H \succeq x_t^L$,

$$\begin{aligned} E[\overline{\Pi}_{\Sigma t+1}^{k-1}(X_{t+1}; \mathbf{e}_t; v)|x_t^H, \mathbf{e}_t]) &= \int_0^\infty \Pr(\overline{\Pi}_{\Sigma t+1}^{k-1}(X_{t+1}; \mathbf{e}_t; v) > z|x_t^H, \mathbf{e}_t; v)\mathrm{d}z \\ &\geq \int_0^\infty \Pr(\overline{\Pi}_{\Sigma t+1}^{k-1}(X_{t+1}; \mathbf{e}_t; v) > z|x_t^L, \mathbf{e}_t; v)\mathrm{d}z \qquad (30) \\ &= E[\overline{\Pi}_{\Sigma t+1}^{k-1}(X_{t+1}; \mathbf{e}_t; v)|x_t^L, \mathbf{e}_t]) \end{aligned}$$

By the induction hypothesis, $\{x_{t+1} \in \mathcal{X}_{t+1} : \overline{\Pi}_{t+1}^{k-1}(X_{t+1}; \mathbf{e}_t; v) > z\}$ is an increasing subset of $\mathcal{X}_{t+1}$ for all z. Inequality (30) now follows from Assumption 3. Thus, for any given effort-history $\mathbf{e}_t$, $\max\left\{v_\Sigma, \delta E\left[\overline{\Pi}_{\Sigma t+1}^{k-1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right]\right\}$ is non-decreasing in x_t, so that higher x_t slackens the IC-constraint (29) while increasing the second term of (28). Finally, the first term of (28) is non-decreasing in x_t by Assumption 2. All together, we conclude that the value of the maximization (28) is non-decreasing in x_t. This completes the desired induction.

Let $\overline{\Pi}_{\Sigma t}^\infty(x_t, \mathbf{e}_{t-1}; v)$ denote the pointwise limit of $\overline{\Pi}_{\Sigma t}^k(x_t, \mathbf{e}_{t-1}; v)$ as $k \to \infty$. Since $\overline{\Pi}_{\Sigma t}^k(x_t, \mathbf{e}_{t-1}; v)$ is non-decreasing in x_t for all k, $\overline{\Pi}_{\Sigma t}^\infty(x_t, \mathbf{e}_{t-1}; v)$ inherits this monotonicity as well.

Part III: Limit of upper bounds can be achieved in SPE. It suffices now to show that $\overline{\Pi}_t^\infty(x_t, \mathbf{e}_{t-1}; v) = \overline{\Pi}_t^{eqm}(x_t, \mathbf{e}_{t-1}; v)$. As shown earlier, $\overline{\Pi}_t^\infty(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}_t^{eqm}(x_t, \mathbf{e}_{t-1}; v)$. Let $e_t(x_t, \mathbf{e}_{t-1})$ denote a limit of any sequence of solutions to (28) subject to (29), as $k \to \infty$. By construction, efforts $e_t(x_t, \mathbf{e}_{t-1})$ are incentive-compatible if players expect continuation play in later periods that generates time-$(t+1)$ payoffs of $\overline{\Pi}_{t+1}^\infty(x_{t+1}, \mathbf{e}_t; v)$ for each player. Again by construction, these efforts generate continuation payoffs $\overline{\Pi}_{t+1}^\infty(x_{t+1}, \mathbf{e}_t; v)$; thus, these strategies constitute a joint-welfare maximizing SPE. Thus, $\overline{\Pi}_t^\infty(x_t, \mathbf{e}_{t-1}; v) \leq \overline{\Pi}_t^{eqm}(x_t, \mathbf{e}_{t-1}; v)$. This completes the proof. □

Proof of Theorem 3

By Theorem 2, joint payoff $\overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v)$ in the joint-welfare maximizing SPE is weakly increasing in x_t for all $(\mathbf{e}_{t-1}, v)$. Given an exogenous stochastic process, further, such payoffs do not depend on the history of efforts. Since outside options $v = (v_i, v_j)$ are held fixed, I will henceforth use the simpler notation $\overline{\Pi}^{eqm}_{\Sigma t}(x_t)$ here.

Proof of (i). Recall that $\overline{\Pi}^{eqm}_{\Sigma t}(x_t) = \max_{e_t}\left(\pi_{\Sigma t}(e_t; x_t) + \gamma \max\left\{v_\Sigma, E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1})|x_t\right]\right)\right\}$ subject to the IC-constraint $c_{\Sigma t}(e_t; x_t) \leq \gamma \max\left\{0, E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1})|x_t\right] - v_\Sigma\right\}$. Joint continuation payoff should period $t+1$ be reached,

$$E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1})|x_t\right] = \int_0^\infty \Pr\left(\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1}) > z|x_t\right) \mathrm{d}z, \tag{31}$$

is weakly increasing in x_t: $\{x_{t+1} : \overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1}) > z\}$ is an increasing subset of $\mathcal{X}_{t+1}$ so that, by Assumption 3, each of the probability terms in (31) is weakly increasing in x_t. Finally, since efforts do not control future payoffs, time-t efforts in the optimal SPE will be chosen to maximize joint stage-game payoff subject to the IC-constraint. Since joint continuation payoff is weakly increasing in x_t, so is the set of effort-profiles e_t satisfying the IC-constraint. Consequently, realized joint stage-game payoff is weakly increasing in x_t.

Proof of (ii). Let $QUIT_t = \{x_t \in \mathfrak{X}_t : \overline{\Pi}^{eqm}_{\Sigma t}(x_t) < v_\Sigma\}$ and $STAY_t = X_t \backslash\ QUIT_t$ denote the set of time-t states in which both players quit and stay, respectively, in the joint-welfare maximizing SPE of Theorem 1. Since joint continuation payoff is weakly increasing in x_t by Theorem 3(i), $STAY_t$ is an increasing set for all t.

Let $p^k_t(x_t)$ denote the probability that the partnership will survive until at least time $t+k$, conditional on $X_t = x_t$. I need to show that, for each $k \geq 1$, $p^k_t(x_t)$ is weakly increasing in x_t. The proof is by induction.

Base step. At any time t, the partnership is certain to end if $x_t \in QUIT_t$ and otherwise ends with probability $1-\gamma^2$ if $x_t \in STAY_t$. Thus, $p^1_t(x_t)$ being weakly decreasing in x_t follows directly from $STAY_t$ being an increasing subset of $\mathfrak{X}_t$.[30]

Induction step. As the induction hypothesis, suppose that $p^m_t(x_t)$ is weakly increasing in x_t for all t and all $m = 1, ..., k-1$. I need to show that $p^k_t(x_t)$ is weakly increasing in x_t for all t. Note that

$$p^k_t(x_t) = p^1_t(x_t) E\left[p^{k-1}_{t+1}(X_{t+1})|X_t = x_t\right] \tag{32}$$

[30]DETAILS FOR REFEREES: (i) $p^1_t(x_t) \in \{1-\gamma^2, 1\}$ for all x_t, (ii) $p^1_t(x_t) = 1$ implies $p^1_t(x'_t) = 1$ for all $x'_t \prec x_t$, and (iii) $p^1_t(x_t) = 1-\gamma^2$ implies $p^1_t(x'_t) = 1-\gamma^2$ for all $x'_t \succ x_t$.

(The partnership survives for k periods iff it survives for $k-1$ periods after first surviving for one period.) The base step showed that $p_t^1(x_t)$ is weakly increasing in x_t. It suffices now to show the same of the expectation term

$$E\left[p_{t+1}^{k-1}(X_{t+1})\middle|X_t = x_t\right] = \int_0^1 \Pr\left(p_{t+1}^{k-1}(X_{t+1}) > p\middle|X_t = x_t\right)\mathrm{d}p \tag{33}$$

By the induction hypothesis, each set $\{x_{t+1} \in \mathcal{X}_{t+1} : p_{t+1}^{k-1}(x_{t+1}) > p\}$ is an increasing subset of $\mathcal{X}_{t+1}$. By Assumption 3, we conclude that each of the probability-terms in (33) is weakly increasing in x_t. This completes the proof. $\square$

Proof of Theorem 4

As argued in the text, no joint outside option greater than $\overline{v}_\Sigma$ can possibly be supported in partnership-market equilibrium. To complete the proof, it suffices to verify that the strategies specified in Theorem 4 constitute a SPE and generate outside options $\overline{v}_\Sigma$. (The theorem specifies play on the equilibrium path; augment this with shirking and quitting to start a fresh relationship should either player deviate from this path of play.)

Let $\hat{p} = \frac{E\left[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0;\overline{v}_\Sigma)\right]-\overline{v}_\Sigma}{E\left[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0;\overline{v}_\Sigma)\right]-\gamma\overline{v}_\Sigma}$ be the probability with which players shirk and quit immediately based on the public randomization. Note that, by construction,

$$\hat{p}\gamma\overline{v}_\Sigma + (1-\hat{p})E\left[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0;\overline{v}_\Sigma)\right] = \overline{v}_\Sigma. \tag{34}$$

Thus, if players adopt the specified strategies, market-wide play generates ex ante expected joint payoff $\overline{v}_\Sigma$ at birth, supporting the maximal joint outside option $\overline{v}_\Sigma$. It suffices now to show that these strategies constitute a SPE. First, players are willing to shirk and quit when the public randomization is less than $\hat{p}$, since they expect uncooperative continuation play in the current relationship. Second, given joint outside option $\overline{v}_\Sigma$, Theorem 1 specifies SPE continuation play should the public randomization be more favorable. This completes the proof. $\square$

Proof of Theorem 5

Proof. To complete the proof, it suffices to show that $E[\overline{\Pi}_{\Sigma 0}^{eqm}(S_0, Y_0; v_\Sigma)]$ is continuous in v_Σ. (Recall that $X_t = (S_0, Y_t)$ where $S_0 \sim U[0,1]$ is the partners' "first impression"; see

Assumption 6.) If so, the maximization (24) requires that $\overline{v}_\Sigma = E[\overline{\Pi}^{eqm}_{\Sigma 0}(S_0, Y_0; \overline{v}_\Sigma)]$, which in turn is only possible if a probability-one measure of partnerships achieve the maximal joint equilibrium payoff $\overline{\Pi}^{eqm}_{\Sigma 0}(S_0, Y_0; \overline{v}_\Sigma)$ given their endogenous outside options.

An increase in joint outside option from v_Σ to $v_\Sigma + \varepsilon$ has two effects on the maximal SPE joint payoff. First, the direct effect is that players enjoy higher joint payoff when quitting and quit whenever they were previously almost indifferent to doing so. This direct effect increases joint payoff by at most ε. Second, since $E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(h_t, e_t(h_t; v_\Sigma), Y_{t+1}; v_\Sigma)\right] - \gamma v_\Sigma$ is non-increasing in v_Σ (see Part I of the proof of Theorem 6, which does not depend on Theorem 5), an indirect effect is that players can support (weakly) fewer effort-profiles at every history $h_t = (s_0, y_t, \mathbf{e}_{t-1})$. This decreases payoffs at those histories, inducing more exit and less effort at previous histories, and so on in a backward cascade that decreases joint payoff. This indirect effect of higher v_Σ may have a discontinuous effect on ex post payoffs but I will show that, when there are meaningful first impressions, it has a continuous effect on ex ante expected payoffs.

Recall that players' efforts $e_t(h_t; v_\Sigma)$ maximize joint payoff subject to the IC-constraint that joint continuation payoff is greater than or equal to joint outside option plus joint cost of effort:[31]

$$c_{\Sigma t}(e_t(h_t; v_\Sigma); h_t) \leq E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(h_t, e_t(h_t; v_\Sigma), Y_{t+1}; v_\Sigma)\right] - \gamma v_\Sigma. \qquad (35)$$

I begin by showing that (35) binds with zero probability on the equilibrium path. Fix any joint outside option v_Σ, effort-profile e_t, history of effort profiles $\mathbf{e}_{t-1}$, and state $x_t = (s_0, y_t)$. By Assumption 6, $c_{it}(e_t; s_0, y_t)$ is strictly decreasing in s_0 for each player i while, by the proof of Theorem 2, $E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(s_0, y_t, \mathbf{e}_{t-1}, e_t, Y_{t+1}; v_\Sigma)\right]$ is weakly increasing in s_0. Thus, if the IC-constraint (35) binds for some efforts e_t at history $(s_0, y_t, \mathbf{e}_{t-1})$, then for all $s_0^l < s_0 < s_0^h$ it fails at history $(s_0^l, y_t, \mathbf{e}_{t-1})$ and is strictly satisfied at history $(s_0^h, y_t, \mathbf{e}_{t-1})$. Since by assumption there are finitely many effort-levels, (35) binds on $e_t(s_0, y_t, \mathbf{e}_{t-1})$ for finitely many partnership types $s_0 \in \mathbf{R}$. We conclude that, with probability one in the joint-welfare maximizing SPE,

[31] To simplify the presentation, I focus on the case in which there is a unique such maximizer at almost all histories reached on the equilibrium path. More generally, the proof extends almost unchanged, when one recognizes that a discontinuity of $E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(S_0, Y_0; v_\Sigma)\right]$ in v_Σ requires that the IC-constraint be binding on *all* such maximizers at a set of histories reached with positive probability.

the IC-constraint will not be binding on *any* effort-profile prescribed on the equilibrium path.

Next, I prove right-continuity, that $\lim_{\varepsilon\to 0}\overline{\Pi}^{eqm}_{\Sigma t}(h_t; v_\Sigma+\varepsilon) = \overline{\Pi}^{eqm}_{\Sigma t}(h_t; v_\Sigma)$ for all v_Σ and all histories h_t reached with probability one on the equilibrium path. For this step, I employ a variation on the algorithm used in the proof of Theorem 2 (illustrated in Figure 6). Fix $\hat{v}_\Sigma$. For all histories h_t and $\varepsilon \geq 0$, define

$$\overline{\Pi}^{1}_{\Sigma t}(h_t; \hat{v}_\Sigma+\varepsilon) = \overline{\Pi}^{eqm}_{\Sigma t}(h_t; \hat{v}_\Sigma) + \varepsilon.$$

Since the positive "direct effect" of higher joint outside option discussed earlier is at most ε and the "indirect effect" is always negative, $\overline{\Pi}^{1}_{\Sigma t}(h_t; \hat{v}_\Sigma+\varepsilon) > \overline{\Pi}^{eqm}_{\Sigma t}(h_t; \hat{v}_\Sigma+\varepsilon)$. Also, clearly, $\overline{\Pi}^{1}_{\Sigma t}(h_t; v_\Sigma)$ is right-continuous at $v_\Sigma = \hat{v}_\Sigma$ for all histories h_t.

As in Steps A-C of the algorithm illustrated in Figure 6, define

$$\begin{aligned}
\overline{\Pi}^{1}_{\Sigma t}(h_t, e_t; v_\Sigma) &= \gamma \max\left\{v_\Sigma, E\left[\overline{\Pi}^{1}_{\Sigma t+1}(h_t, e_t, X_{t+1}; v_\Sigma)|h_t, e_t\right]\right\}\\
\mathcal{F}^2_t(h_t; v_\Sigma) &= \left\{e_t : \gamma v_\Sigma + c_{\Sigma t}(e_t; x_t) \leq \overline{\Pi}^{1}_{\Sigma t}(h_t, e_t; v_\Sigma)\right\}\\
\overline{\Pi}^{2}_{\Sigma t}(h_t; v_\Sigma) &= \max_{e_t\in\mathcal{F}^2_t(h_t;v_\Sigma)} \left(\pi_{\Sigma t}(e_t; x_t) + \overline{\Pi}^{1}_{\Sigma t}(h_t, e_t; v_\Sigma)\right)
\end{aligned}$$

As argued above, the IC-constraint (35) is not (exactly) binding for *any* effort-profile at a probability-one set of histories reached on the equilibrium path. At each such history, $\mathcal{F}^2_t(h_t; v_\Sigma)$ is unchanging in a neighborhood of $\hat{v}_\Sigma$. Thus, the right-continuity of $\overline{\Pi}^{1}_{\Sigma t}(h_t, e_t; v_\Sigma)$ in v_Σ implies right-continuity of $\overline{\Pi}^{2}_{\Sigma t}(h_t; v_\Sigma)$ in v_Σ, at a probability-one set of equilibrium histories. Repeating this argument for all $k \geq 1$, we conclude that $\overline{\Pi}^{k}_{\Sigma t}(h_t; v_\Sigma)$ is right-continuous in v_Σ at $\hat{v}_\Sigma$ at a probability-one set of equilibrium histories. Such continuity carries over to the limit as well, so that maximal equilibrium joint payoff $\overline{\Pi}^{eqm}_{\Sigma t}(h_t; v_\Sigma)$ is right-continuous in v_Σ at a probability-one set of histories. In particular, $E[\overline{\Pi}^{eqm}_{\Sigma t}(S_0; v_\Sigma)$ is right-continuous in v_Σ at $\hat{v}_\Sigma$. The proof of left-continuity is similar, and omitted to save space. □

Proof of Theorem 6

Proof. Part I: $v_\Sigma - E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ *is strictly increasing in* v_Σ. In a slight variation on the notation used in the text, let $\overline{\Pi}^{eqm}_{\Sigma t}(h_t; v^h_\Sigma)$ denote the maximal SPE joint payoff from history

$h_t = (x_t, \mathbf{e}_{t-1})$ given joint outside option v^h_Σ. Consider now a lower joint outside option $v^l_\Sigma \in [0, v^h_\Sigma)$ and let $\widetilde{\Pi}_{\Sigma t}(h_t; v^l_\Sigma)$ denote the joint payoff that would result should players with joint outside option v^l_Σ mimic welfare-maximizing play as if it were v^h_Σ. Note that the stage-game payoff process and the partnership stopping time T are identically distributed when players follow the same strategies. Thus, the only difference in payoffs arises from the fact that players only get v^l_Σ when they survive but the partnership ends instead of v^h_Σ. In particular, for all histories h_t,

$$\overline{\Pi}^{eqm}_{\Sigma t}(h_t; v^h_\Sigma) - \widetilde{\Pi}_{\Sigma t}(h_t; v^l_\Sigma) = \gamma(v^h_\Sigma - v^l_\Sigma) \sum_{t' \geq t} \gamma^{t'-t} \Pr(T = t'|h_t) \leq \gamma \left(v^h_\Sigma - v^l_\Sigma\right) \tag{36}$$

Let $e_t(v^h_\Sigma)$ denote the efforts played in the optimal SPE given joint outside option v^h_Σ. Observe that these efforts remain incentive-compatible given lower joint outside option v^l_Σ:

$$E\left[\widetilde{\Pi}_{\Sigma t+1}(H_{t+1}; v^l_\Sigma)|h_t, e_t(v^h_\Sigma)\right] \geq E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(H_{t+1}; v^h_\Sigma)|h_t, e_t(v^h_\Sigma)\right] - \gamma\left(v^h_\Sigma - v^l_\Sigma\right) \tag{37}$$

$$\geq \gamma v^h_\Sigma + c_{\Sigma t}(e_t(v^h_\Sigma); x_t) - \gamma\left(v^h_\Sigma - v^l_\Sigma\right) \tag{38}$$

$$= c_{\Sigma t}(e_t(v^h_\Sigma); x_t) + \gamma v^l_\Sigma.$$

((37) follows from (36). (38) follows from the incentive-compatibility constraint (18) as applied to the optimal equilibrium given v^h_Σ.) By similar logic, staying is incentive-compatible given these mimicking strategies whenever players stay in the optimal equilibrium given joint outside option v^h_Σ. (Details omitted to save space.) Thus, these mimicking strategies constitute a SPE given v^l_Σ. In particular, $E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v^l_\Sigma)] \geq E[\widetilde{\Pi}_0(X_0; v^l_\Sigma)]$. Thus, $E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v^h_\Sigma)] - E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v^l_\Sigma)] \leq \gamma\left(v^h_\Sigma - v^l_\Sigma\right)$ and we conclude that $\gamma v_\Sigma - E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ is non-decreasing in v_Σ. Since $\gamma < 1$, this implies that $v_\Sigma - E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ is strictly increasing in v_Σ.

Part II: $v_\Sigma = E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ has a unique solution $\overline{v}_\Sigma$. (a) Given zero outside option, each player's expected payoff is non-negative in any SPE of the partnership game. In particular, $0 \leq E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; 0)]$. (b) Since joint payoffs are bounded (Assumption 1), $v_\Sigma \geq E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ for all large enough v_Σ. (c) By Part I and the proof of Theorem 5, $v_\Sigma - E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ is strictly increasing and continuous in v_Σ. Thus, $v_\Sigma = E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ has a unique solution. □

Proof of Claim 3

Proof. For all sufficiently small matching costs m, $\overline{v}_\Sigma(m)$ is implicitly defined by $\overline{v}_\Sigma(m) = E[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; \overline{v}_\Sigma(m))] - 2m$. (Each player pays $m \geq 0$ to start a new partnership having joint outside option $\overline{v}_\Sigma(m)$. When m is sufficiently large, an active matching market cannot be supported in equilibrium and $\overline{v}_\Sigma(m) = 0$.) In Part I of the proof of Theorem 6, I show that $v_\Sigma - E[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma)]$ is strictly increasing in v_Σ. Thus, $\overline{v}_\Sigma(m)$ is strictly decreasing in m for all small enough m. □

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