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Financing Constraints and Relational Contracts[*]

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Abstract

We consider a model in which a principal must both repay a loan and motivate an agent to work hard. Output is non-contractible, so the principal faces a commitment problem with both her creditor and her agent. In a profit-maximizing equilibrium, the agent's productivity is initially low and increases over time. Productivity continues increasing even after the debt has been repaid, eventually converging to a steady state that is independent of the size of the initial loan. We apply the model to argue that a firm that relies on external debt will typically under-invest in the scale of its existing businesses, but might either over- or under-invest when expanding into new lines of business.

1 Introduction

Firms rely on funding from capital markets to invest and expand, which requires that they credibly promise to repay their investors (see Shleifer and Vishny (1997) for evidence of this commitment problem). But securing funding does not guarantee profitability; companies must also motivate their employees to work hard in order to be successful. The promises a firm makes to its creditors can influence the credibility of the compensation it promises workers. As a result, a firm's survival, productivity, and growth depend on whether it can fulfill its financial obligations while simultaneously motivating workers to exert more than perfunctory effort.

This paper examines how a firm's obligations in the capital market constrain the incentive contracts it can offer its workers in a setting with limited commitment. In our model, a liquidity-constrained firm purchases an asset by borrowing money from a competitive capital market. After this asset is purchased, the firm must repeatedly motivate its workers to exert continuous and observable effort, the proceeds of which can be used to both pay the worker and repay an external creditor. If realized profit is not contractible—for example, because the firm's manager can divert cash flows for her own private benefit—then the firm's promises to motivate its workers and to satisfy its debtors must be credible within the context of these ongoing relationships.

External financing and internal incentives are linked in two ways in this setting. First, the principal is liquidity-constrained and so must use her realized profits to satisfy the demands of both her creditor and her worker. Second, the principal can simultaneously renege on her obligations in both relationships, in which case she declares bankruptcy but is otherwise protected by limited liability. The firm's promises to one party can therefore affect the credibility of her promises to the other. Consequently, a firm's productivity is tightly linked to its financial position: more indebted firms are less productive, a firm's productivity increases as it pays off its external debts, and productivity can remain temporarily depressed even after debts are repaid. The resulting tension determines whether the firm invests, how it repays its debts, and how it motivates its workers over time.

As a real-world example of the logic that drives our results, consider Lincoln Electric's decision to borrow money in 1992 following devastatingly poor performance in its international ventures. This decision massively increased the company's long-term debt, which had two effects on the firm's

relationship with its workers in the United States. First, Lincoln Electric had to pay smaller performance bonuses to have enough money to repay the bank: former CEO Donald Hastings recounts that "the priorities were reducing the debt and increasing capital expenditures .... Accordingly, we would have to reduce the individual bonuses paid that year." (Hastings, 1999) Taking on debt also influenced worker morale, as Lincoln Electric faced the possibility that its entire relational incentive system would unravel if withheld bonuses from workers. Indeed, in a radical departure from the company's usually cooperative worker-manager relations, hundreds of Lincoln workers met with union representatives in the years after the firm borrowed money. One worker at the time underscored the growing discontent caused by more debt and lower compensation by noting, "there's a lot of disgruntlement....A lot of people feel ripped off." (Feder, 1994). Our paper models these dual concerns and explores their implications.

In our framework, the principal motivates the agent most effectively by backloading compensation. To ensure that she has no other outstanding obligations when she makes these deferred payments, the principal optimally repays the creditor as quickly as possible. Driven by this intuition, our profit-maximizing relational contract consists of three phases. In the first phase, the loan has not yet been completely repaid and so the principal pays her entire profit in each period to the creditor. The agent is paid nothing in these periods and is instead motivated only by the prospect of future rents. In the second phase, the principal promises rent to the agent in order to compensate him for his effort in the first phase. However, she is liquidity-constrained and so cannot simply pay the agent higher wages. Instead, we show that the agent continues to produce low output in these periods, which decreases total surplus but increases the agent's payoff. Thus, output is relatively low in the second phase even though the principal's debt has already been completely paid. After several periods of low output, the relationship finally converges to a third, stationary phase in which profit is maximized in each period. The principal earns positive profits and compensates the agent for his cost of effort in each period of this phase.

Our results imply that a firm's debt has substantial and persistent effects on its productivity. Holding output fixed, a larger loan takes longer to repay. But because the agent's compensation is deferred until after the loan is repaid, his incentive to work hard—and hence the firm's productivity—is decreasing in the size of the initial loan. As a result, a larger loan leads to a longer period of lower output. In extreme cases, a firm might be completely unable to borrow because

the resulting negative effect on productivity would make it impossible for the firm to repay the initial loan. While persistent, these productivity effects are not permanent, since the relationship eventually converges to a steady state that is independent of the initial loan.

We extend this baseline model to consider ways in which the investment can affect the firm's production function. In addition to the direct effect of investment on profits, a firm that borrows money to grow potentially faces two conflicting equilibrium effects on productivity. First, consistent with the insights of Klein and Leffler (1981) and Bernheim and Whinston (1990), a larger up-front investment can have a positive effect on productivity because a firm that engages in additional (profitable) activities stands to lose more if it declares bankruptcy. Thus, the principal is willing to pay more to both creditors and employees in equilibrium, which in turn increases output. However, a larger investment requires more up-front borrowing, which has a negative effect on productivity. Consistent with our baseline results and the experience of Lincoln Electric, higher debt can undermine the relationship between the principal and her agent and lead to lower output. We show that the size of each of these effects depends on the nature of the investment and the characteristics of the principal's relationship with her agent. The latter, negative effect dominates if the investment scales both the profits and costs of the existing enterprise, leading to under-investment in equilibrium. If investment takes the form of an expansion of business that affects profits but not the costs faced by the agent, then either effect might dominate and so the equilibrium might entail under- or over-investment. Thus, the productivity dynamics we analyze can have substantial and nuanced effects on a firm's decision to scale up production or to expand into new lines of business.

In our paper, the principal loses future profits if she reneges on her promised payments today. Thus, our analysis builds on the classic relational contracting papers by Bull (1987), MacLeod and Malcomson (1989), Levin (2003) and others; see Malcomson (2013) for an overview. Many of the classic papers in this literature focus on (i) a bilateral relationship between a single principal and agent, and (ii) settings in which stationary contracts are optimal. In contrast, our setting entails (i) a principal who must satisfy the demands of her relationships with both an outside creditor and an agent, which lets us (ii) explore a host of new dynamics and empirical predictions about debt and productivity. Simultaneous relationships with multiple agents have been studied by Levin (2002) and more recently by Andrews and Barron (2015) and Barron and Powell (2015). Li and Matouschek (2013) have explored exogenous, stochastic financing constraints in a bilateral

3

relationship between a principal and agent. To the best of our knowledge, this is the first paper that studies multiple relationships in which one of those relationships entails an agency problem and the other requires the repayment of a loan.

Our analysis is also closely related to the literature on financial frictions. The seminal analyses by Jensen and Meckling (1976), Myers (1977), Townsend (1979), Gale and Hellwig (1985), and others have spurred a massive literature on the interaction between investment and incentives; see Stein (2003) for an overview. Broadly, many papers (including Aghion and Bolton (1992), Hart (1995), and Holmstrom and Tirole (1997)) focus on agency problems between an entrepreneur and an outside investor who seeks to recoup her initial loan from the entrepreneur's realized profits. A key lesson from these papers is that commitment problems—in particular, the fact that the entrepreneur has limited funds and can declare bankruptcy—imply that the investor might not be able to recoup the full return on her capital, which leads to underinvestment in equilibrium.

We depart from this literature by adding a third party: a worker who is not a residual claimant on profits, but whose effort is nevertheless required to make the entrepreneur profitable. Thus, we take the standard assumption that an entrepreneur has limited liability and can divert a firm's profits to her private benefit and analyze how this same constraint influences incentive contracts and productivity within the firm. The principal faces a commitment problem because her payments are not contractible: she can renege on promised bonuses to her agents by diverting funds and declaring bankruptcy.

Albuquerque and Hopenhayn (2004) studies firm productivity dynamics in the presence of moral hazard and commitment problems, which is related to our own focus on costly effort provision. In their model, a liquidity-constrained entrepreneur must be induced not to steal working capital by the promise of future rent. The entrepreneur's temptation to steal capital—and hence the rent she must be promised—increases in the current size of the firm. Thus, the creditor offers more working capital at those histories in which the entrepreneur requires a reward for past performance, and offers less working capital following low output. Clementi and Hopenhayn (2006) also consider firm dynamics, though they focus on asymmetric information rather than moral hazard. We differ from these papers by explicitly modeling the incentive problem between the entrepreneur and her employees. As a result, our equilibrium entails productivity dynamics, even though the investor offers funds only at the very beginning of the game. Furthermore, a firm's profits can remain

4

(temporarily) depressed in our model even after its debt has been fully repaid.

The next section introduces our model. Section 3 contains our main analysis. We first solve two benchmark models to highlight the forces that drive our equilibrium, then prove the main results. Section 4 extends our model to consider firm scale and expansion into new lines of business. We discuss and conclude in Section 5.

2 Model

Consider a repeated game between a principal and an agent, both of whom are liquidity-constrained. At the start of the relationship, the principal must purchase an asset at price $K > 0$. She can borrow funds R from a competitive market to finance this purchase, which she promises to repay with a sequence $\{\bar{r}_t\}_{t=0}^{\infty}$ with $\bar{r}_t \geq 0$, $\forall t$.

If the principal does not buy the asset, then the game ends and all players earn 0. Otherwise, the principal and agent repeatedly play the following stage game with perfect monitoring and a common discount factor δ:

1. Principal and agent simultaneously choose whether or not to continue. The game ends and both players earn payoff 0 if either chooses not to continue.

2. The agent chooses output y_t at cost $c(y_t)$, with $c', c'' > 0$, $c(0) = 0$, and $\lim_{y \to \infty} c'(y) > 1$.

3. The principal chooses to pay amounts $r_t \geq 0$ and $b_t \geq 0$ to the capital market and agent, respectively. The sum of these payments cannot exceed the principal's realized profits: $r_t + b_t \leq y_t$.

4. If $r_t \neq \bar{r}_t$, then the asset is seized and both principal and agent earn 0 in the continuation game.

In each period t, the principal and agent's stage-game payoffs equal

$$
\begin{aligned}
\pi_t &= y_t - r_t - b_t \\
u_t &= b_t - c(y_t)
\end{aligned}
,
$$

5

respectively. We solve for a Subgame Perfect Equilibrium that maximizes the principal's ex ante expected payoff, subject to the constraint that the principal repays the capital market: $\sum_{t=0}^{\infty} \delta^t r_t \geq R$.

Define first-best output y^{FB} as the unique solution to

$$y^{FB} = \arg \max_y y - c(y) \tag{1}$$

and let $S^{FB} = \frac{1}{1-\delta}(y^{FB} - c(y^{FB}))$ be the first-best discounted surplus. We can immediately conclude that $R = K$: the principal borrows just enough money to purchase the asset. Note that in this formulation, we do not allow the principal to borrow any money apart from the initial amount, nor do we allow her to save profits across periods. We discuss this assumption further in Section 5.

Two key contracting frictions play an important role in our analysis. First, both the principal and agent are liquidity constrained, so both bonuses b_t and repayments r_t must be paid out of the realized profit y_t. This constraint implies the principal is liquidity constrained, which provides a rationale for why she must rely on external financing. Second, the principal cannot commit to a formal bonus scheme as a function of realized output because y_t is non-contractible. Strictly speaking, we model this non-contractibility by giving the principal the option to retain the entire realized profit in a period. More generally, we can interpret this assumption as the principal having the option to invest profits in wasteful projects that generate private benefits for her (at a rate of one util of private benefits for every \$1).[1] This assumption is similar to Albuquerque and Hopenhayn (2004), who assume that the principal can divert part of a firm's working capital, and Holmstrom and Tirole (1997), who assume that the principal can earn private benefits by mis-directing investments.

3   Financing Constraints and Relational Contracts

In this section, we show that a profit-maximizing equilibrium potentially entails three sources of inefficiency. First, the principal might not be able to purchase an asset, even if her project would

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[1]More formally: suppose that in each period, the principal privately observes a signal $s \in \{1, ..., N\}$ distributed uniformly at random and chooses an action $a \in \{1, ..., N\}$ after the continuation decision but before the agent chooses y. If $a = s$, then y equals realized profit; otherwise, y is a non-pecuniary payoff earned by the principal. Realized profit is contractible, but private benefits, a, and s are not. All of our results would continue to hold in the alternative model.

6

have strictly positive net present value. Second, conditional on purchasing the asset, productivity and profits start low and only slowly converge to a steady state. Finally, even in the steady state, output might be below first-best because the principal cannot commit to an incentive scheme that induces first-best output.

We begin by solving two benchmark cases to highlight the sources of these inefficiencies. First, we assume that output is contractible and show that first-best is attainable: the principal buys the asset whenever it has positive net present value, and the agent exerts first-best effort in each period. Second, we suppose output is not contractible, but assume that the principal directly chooses output so that the agent plays no role in the analysis. In this benchmark, some projects with positive net present value do not receive funding. However, *conditional* on investment occurring, the principal chooses first-best output in each period independent of the level of debt. Finally, we show how the full model with limited commitment can lead to both under-investment and productivity, bonuses, and repayments that evolve over time. All proofs are relegated to Appendix A.

We begin by describing the optimal program in the game with limited commitment, assuming that the principal purchases the asset. It is always an equilibrium for the principal and agent to simultaneously choose not to continue their relationship. This equilibrium is the optimal penal code because it results in a payoff of 0 for each player, which is the lowest feasible and individually rational payoff. Given these punishment payoffs following a deviation, a profit-maximizing equilibrium solves

$$\max_{\{r_t\},\{b_t\},\{y_t\}} \sum_{t=0}^{\infty} \delta^t (y_t - b_t - r_t) \tag{2}$$

subject to the following constraints:

1. **Incentive Compatibility.** the agent is willing to accept production and choose y_t in each period: $\forall T \geq 0$,
$$\sum_{t=T}^{\infty} \delta^{t-T}(b_t - c(y_t)) \geq 0. \tag{3}$$

2. **Dynamic Enforcement.** The principal is willing to pay $b_t + r_t$ in each period: $\forall T \geq 0$,

$$b_T + r_T \leq \sum_{t=T+1}^{\infty} \delta^{t-T}(y_t - b_t - r_t). \tag{4}$$

7

3. **Loan Repayment.** The principal borrows enough money to purchase the asset, and repays the borrowed amount:

$$\sum_{t=0}^{\infty} \delta^t r_t \geq R \geq K. \tag{5}$$

4. **Limited Liability**. $\forall T \geq 0$,

$$
\begin{aligned}
b_T &\geq 0 \\
r_T &\geq 0 \qquad . \\
b_T + r_T &\leq y_T
\end{aligned}
\tag{6}
$$

3.1 Benchmark 1: Output is Contractible

Our first benchmark demonstrates that first-best is attainable if output is contractible. Suppose that r_t and b_t can be written as a function of realized output in each period. Because output is contractible, the principal cannot declare bankruptcy unless she actually lacks the profit to make her scheduled repayment. Therefore, the dynamic enforcement constraints need not be satisfied, since the principal will pay b_T and r_T in period T whenever $b_T + r_T \leq y_T$. The optimal contract must continue to satisfy the other constraints.

Under these assumptions, we show that the principal induces the agent to produce first-best output and borrows money whenever doing so has positive net present value.

Proposition 1 *Suppose R and y_t are contractible and the principal can write a long-term contract at the start of the game. If $K < S^{FB}$, then the principal purchases the asset in any optimal contract, and $y_t = y^{FB}$ in every $t \geq 0$.*

Proof. *See Appendix A.* ■

The optimal contract exactly compensates the agent for his cost of output, so $b_t = c(y_t)$. The agent is indifferent between all output levels under this contract, so in particular is willing to choose first-best effort. The remaining profit in each period $y^{FB} - c(y^{FB})$ is used to repay the loan, with any residual accruing to the principal. If the firm has positive net present value, then the payments are enough to both repay the initial loan and guarantee the principal a positive payoff.

This benchmark shows that limited liability constraints alone do not lead to either production or investment inefficiencies in our model. Actions are perfectly observed in our model, so a formal contract can exactly compensate the agent for the cost of his actions. The principal earns all

of the surplus from the relationship and is willing to pay any amount less than that surplus to the creditor in exchange for up-front funding. Therefore, the limited liability constraints do not impose any investment inefficiencies. Furthermore, because output is contractible, the optimal contract optimally induces first-best output in each period. These two features contrast with our main results, which focus on the investment and production inefficiencies that arise if output is not contractible.

3.2 Benchmark 2: Principal Produces Output

Before turning to our main analysis, we consider a second benchmark in which output is not contractible but is chosen by the principal in each period, so the agent takes no actions. We show that an asset with strictly positive net present value might nevertheless not be purchased in equilibrium. However, conditional on borrowing money, there are no production inefficiencies; the firm produces first-best output in each period.

Suppose that output is not contractible, but the principal can produce output without relying on an agent. Formally, the principal chooses output y_t in each period and bears the cost of output $c(y_t)$ directly, while the agent takes no actions and receives no payments ($b_t = 0$ in each t). Because output is not contractible, the principal is willing to repay her loan only if she earns a strictly positive continuation payoff. The principal's rent creates a wedge between the total value created by the firm and the income that can be pledged to repay the loan. However, the principal fully internalizes the returns to output and so will always choose first-best output conditional on being funded.

Formally, the optimal equilibrium solves

$$\max_{\{y_t\},\{r_t\}} \sum_{t=0}^{\infty} \delta^t (y_t - c(y_t) - r_t)$$

subject to (5), (6) with $b_t = 0$ in each $t \geq 0$, and a modified Dynamic Enforcement constraint: $\forall T \geq 0$,

$$r_T \leq \sum_{t=T+1}^{\infty} \delta^{t-T} (y_t - c(y_t) - r_t).$$

We show that uninvestment is a feature of this benchmark, in the sense that an investment

9

might generate strictly positive surplus but not occur in equilibrium. However, the principal has an incentive to produce first-best output in each period, since doing so both increases her profits and relaxes all constraints.

Proposition 2 *In any optimal equilibrium of this benchmark, (i) $y_t = y^{FB}$ in each $t \geq 0$, and (ii) the project is funded only if $K \leq \delta S^{FB}$.*

Proof. *See Appendix A.* ∎

Consider the constrained maximization problem for this benchmark game. An increase in y_t both increases profit and relaxes all constraints if $y_t < y^{FB}$, so $y_t = y^{FB}$ in any optimal equilibrium. However, the principal's ability to repay a loan is constrained by her dynamic enforcement constraint. In particular, it must be that

$$\sum_{t=0}^{\infty} \delta^t r_t = K \leq \delta S^{FB}.$$

But the value of a funded firm equals S^{FB}. If $S^{FB} > K > \delta S^{FB}$, then purchasing the asset would maximize total surplus, but cannot occur in equilibrium. This underinvestment result is similar to Holmstrom and Tirole (1997), who also argue that the principal's inability to commit limits their ability to access the credit market. However, conditional on being funded, a firm produces first-best output regardless of its financial obligations.

3.3 The Effects of Finance on Productivity

We now turn to the baseline model and prove our main result. If the principal cannot commit to either pay the agent or repay the creditor, then her promised payments must be credible in the context of the ongoing relationship. The resulting dynamic enforcement constraints determine both the firm's ability to borrow money and how its productivity evolves over time.

If players are sufficiently patient, then the optimal formal contract also satisfies the dynamic enforcement constraint and so first-best is attainable. We focus on the dynamics that arise if first-best is unattainable, a sufficient condition for which is that no stationary contract attains first-best. The following assumption, which is maintained for the rest of the paper unless explicitly noted otherwise, formalizes this condition.

Assumption 1

$$\max\left\{\frac{\delta y^{FB} - c\left(y^{FB}\right)}{1 - \delta}, 0\right\} < K.$$

Note that Assumption 1 is violated if the principal does not need to borrow any money ($K = 0$). Indeed, if $K = 0$ it is straightforward to show (by adapting Levin's (2003) classic result) that the principal's payoff is maximized by a stationary contract—the players take the same actions in each period on the equilibrium path. So long as Assumption 1 is satisfied, however, we can show that first-best surplus is not attainable in equilibrium. In that case, stationary contracts are strictly sub-optimal and the profit-maximizing equilibrium necessarily entails dynamics. In particular, any stationary contract is strictly dominated by a non-stationary contract that frontloads payments to the creditor and backloads payments to the agent.

Our next goal is to characterize the dynamics in a profit-maximizing equilibrium. Define

$$S^* = \max_{\sigma^* | \sigma^* \text{ is an SPE}} \left\{E_{\sigma^*}\left[\sum_{t=0}^{\infty} \delta^t (y_t - c(y_t))\right] | K = 0\right\}$$

as the maximum total surplus attainable if the principal does not need to borrow any money. Let y^* equal the corresponding stationary output level.

Our first result characterizes an optimal payment path for a fixed sequence of equilibrium outputs. In equilibrium, the principal's payments balance two conflicting goals: the agent must be sufficiently motivated to produce output, and the creditor must earn enough money to justify his investment. Lemma 1 describes how the principal satisfies these two requirements over time.

Lemma 1 *Suppose Assumption 1 holds. Then without loss: (i) there exists $t^* > 0$ such that for all $t < t^*$, $b_t = 0$ and $r_t = y_t$, and for all $t > t^*$, $r_t = 0$; (ii) if $t > t^*$ and total continuation surplus equals S^*, then the continuation equilibrium is stationary. Further, in any optimal equilibrium: (iii) in each period, either $y_t = y^{FB}$ or the dynamic enforcement constraint binds; (iv) in each period $t \geq t^*$, either total continuation surplus equals S^* or $b_t + r_t = y_t$; (v) constraint (3) binds at $t = 0$.*

Proof. *See Appendix A.* ∎

Lemma 1 describes the optimal payments given a fixed sequence of outputs $\{y_t\}_{t=0}^{\infty}$. Initially, the principal devotes her entire profit to repaying her loan, while promising the agent future rents

11

to motivate him. Once the loan is paid off in period t^*, the principal must follow through on her promise to compensate the agent. She does so by paying him the entire output y_t and asking him to produce output such that his IC constraint is slack. Asking the agent to underprovide output is inefficient, but it is the only way to give the agent rent. Once the agent has earned enough rent to compensate him for his effort in periods $t < t^*$, the equilibrium converges to a steady state in which either first-best is attained or the agent's IC constraint binds.

To see why (i) is true, note that the agent's incentive constraint depends on present and future bonuses. As a result, the principal would like to backload payments to the agent by a logic familiar from dynamic moral hazard problems with limited liability. In contrast, only the discounted sum of payments r_t matters in the loan repayment constraint; the timing of those payments is irrelevant. Because only the sum $b_t + r_t$ matters in the other constraints, the optimal equilibrium defers the agent's compensation until the loan has been fully repaid in period t^*. Moreover, r_t can be frontloaded without affecting the dynamic enforcement constraint, so $r_t = y_t$ in the periods before t^*.

Suppose $t > t^*$ so that the loan has already been paid. If continuation surplus equals S^*, then it can be shown that continuation surplus must also equal S^* in all subsequent periods. So we can implement an equilibrium attaining S^* as a stationary equilibrium, proving (ii). For (iii), if the dynamic enforcement constraint does not bind and $y_t < y^{FB}$, then the principal could demand the agent produce slightly higher output and compensate him for the cost of that output. This perturbation would increase the principal's payoff while continuing to satisfy all constraints. Result (iv) follow from the fact that if the agent earns a strictly positive payoff, then it must be that his incentive compatibility constraint is slack. So he would be willing to produce more output, which would decrease his payoff but increase both total surplus and the principal's payoff. As a result, total surplus is strictly increasing in the principal's payoff if $y_t < y^{FB}$. If $b_t + r_t < y_t$ and continuation play does not attain first-best, then the principal could increase the agent's payoff today while decreasing his continuation payoff, leading to a more efficient continuation. Finally, y_0 affects only the principal's payoff and the incentive compatibility constraint in $t = 0$, which implies (v).

Our next step is to analyze the stationary equilibrium to which the optimal equilibrium eventually converges. Lemma 1 implies that continuation play at this steady state generates total surplus

12

S^*. The agent receives a non-negative continuation payoff, so $b_t \geq c(y_t)$ and hence the Limited Liability constraint does not bind. As a result, steady state output equals y^*, where either $y^* = y^{FB}$ or y^* solves

$$c(y^*) = \frac{\delta}{1-\delta}(y^* - c(y^*)).$$

It is straightforward to show that there exists a stationary equilibrium with output y^* in which the agent earns no surplus, so the principal's continuation payoff equals S^*.

This argument fully characterizes equilibrium play once the steady state is reached. Next, we analyze how the equilibrium transitions to the steady state, focusing on how the initial loan affects the evolution of productivity. Our main Proposition builds on Lemma 1 to solve for a profit-maximizing sequence of equilibrium outputs. Intuitively, this result demonstrates that a firm's financial position - how much debt it must repay - substantially impacts its productivity and profitability. Productivity in turn affects the principal's ability to repay her debts, so a firm's external financial obligations are inextricably linked to its internal incentive systems.

Proposition 3 *There exists an optimal equilibrium in which $\{y_t\}_{t=0}^{\infty}$ is determined by two parameters: the period T in which total surplus first equals S^*, and the output y_T produced in that period. In each period $t < T$, $y_t = \delta^{T-t}y_T$. In each period $t > T$, output is a fixed amount $y^* \geq y_T$ that is independent of K. T and y_T are chosen to solve*

$$\max_{T,y_T} \delta^T y_T \tag{7}$$

subject to

$$\sum_{t=0}^{T-1} \delta^t(\delta^{T-t}y_T - c(\delta^{T-t}y_T)) + \delta^{T+1}S^* - \delta^T c(y_T) = K, \tag{8}$$

$$y_T \leq y^*. \tag{9}$$

Proof. *See Appendix A.* ∎

Before the optimal equilibrium converges to its steady state, Lemma 1 implies that both (4) and the upper bound of (6) must bind in each period. Together, these binding constraint pin down output in period t as a function of the principal's continuation surplus from period $t+1$ onwards. Furthermore, the principal earns 0 in each period $t < T$ because $b_T + r_T = y_T$ in these periods. As a

result, the principal's profit can be written entirely as a function of output in period T, yielding the objective. This logic also uniquely pins down output in each period $t < T$ in terms of the principal's continuation surplus, which in turn is a function of T and y_T. Together, these arguments yield the relaxed problem (7)-(9).

The constraint (8) can be interpreted as an adding-up constraint for the output created by the relationship. Writing out S^* and rearranging yields the (slightly unwieldy) expression

$$\left(\sum_{t=0}^{T}\delta^t(\delta^{T-t}y_T) + \sum_{t=T+1}^{\infty}\delta^t y^*\right) = \left(\sum_{t=0}^{T}\delta^t c(\delta^{T-t}y_T) + \sum_{t=T+1}^{\infty}\delta^t c(y^*)\right) + K + \delta^T y_T \qquad (10)$$

Recall that $y_t = \delta^{T-t}y_T$ for all $t \leq T$. Hence, the left-hand side of (10) equals the total output produced in the course of this relationship, which must be split between the principal, the agent, and the bank. The agent earns no ex ante rent because (3) binds in $t = 0$, but must still be compensated for his cost of effort. This cost equals the first term on the right-hand side of (10). The second term on the right-hand side, K, captures the proceeds used to repay the bank. The final term equals the principal's profit, $\delta^T y_T$. Thus, this expression simply says that the amount of output produced must equal the sum of payments to each party. Constraint (9) is implied by the dynamic enforcement constraint.

Intuitively, Proposition 3 implies that the firm's productivity begins at a low level and increases at a constant rate $\frac{1}{\delta}$ in each period until it reaches the steady–state level y^*. Note that y^* is independent of K, so the effects of borrowing money on productivity eventually dissipate. However, large loans both increase the amount of time it takes to attain the steady state (T) and depress output in every period until the steady-state is reached. Furthermore, the loan is typically repaid several periods before the relationship converges to a steady-state. In other words, the principal's loan can have a lingering effect on productivity even after it has been fully repaid. This lingering productivity effect occurs because the principal needs to offer the agent rent in order to compensate him for his past efforts. Because the principal is liquidity-constrained, he cannot simply give the agent rent by paying him a high wage up-front; instead, he must ask the agent to shirk, which increases the agent's payoff but decreases total surplus.

The next two corollaries follow from (7)-(9) and constitute the main results of this section. First, we formalize the link between debt and productivity by performing comparative statics on

14

K.

Corollary 1 *There exists a profit-maximizing equilibrium in which (i) $y_T \in [\delta y^*, y^*)$, (ii) T is increasing in K in an optimal equilibrium, (iii) in each period t, y_t is decreasing in K.*

Proof. *See Appendix A.* ∎

The upper bound $y_T \leq y^*$ follows because y^* is the largest possible output that can be sustained in a relational contract. If $y_T < \delta y^*$, then there exists an alternative equilibrium in which $\tilde{T} = T + 1$ and $y_{\tilde{T}} = \frac{y_T}{\delta}$. By construction, this alternative leads the principal to earn the same profit, and it can be shown that this alternative relaxes the first constraint of Proposition 3. The second constraint is satisfied because $y_{\tilde{T}} < y^*$.

The solution to the program in Proposition 3 replicates as closely as possible an output path in which output at time $t \leq T + 1$ equals $\delta^{T+1-t} y^*$ (and hence $y_T = \delta y^*$). However, it might be impossible to satisfy the first constraint of Proposition 3 with equality for this output path, since T must be an integer. Thus, the optimal output in time T, y_T, is chosen to be as close as possible to δy^*, subject to the integer constraint on T. If $y_T > \delta y^*$, then a marginal increase in the initial debt K leads to lower output y_T and hence lower output in every previous period. If $y_T = \delta y^*$, then an increase in the debt leads to a longer horizon T, which also depresses output in all previous periods.

An increase in K increases the output that is devoted to repaying the bank. If output remains fixed, then either the principal's or the agent's payoff must fall. But it is not profit-maximizing for the principal's payoff to absorb this decrease, so the agent's payoff will fall in equilibrium, implying that he must exert less effort. Because T and y_T determine output in each period $t \leq T$, the agent's output decreases proportionally in every period $t \leq T$.

In addition to the effects on productivity, the principal's lack of commitment also leads to under-investment as in Proposition 2. The principal is willing to follow through on her debts only if she is promised strictly positive future profits, so the creditor cannot hope to be repaid the entire net present value of a project. Any repayment scheme that did so would violate the principal's Dynamic Enforcement constraint and so could not be sustained in equilibrium. Consequently, an asset with positive net present value might not be purchased in equilibrium.

Corollary 2 *Define $S(K) = \sum_{t=0}^{\infty} \delta^t (y_t - c(y_t))$ as the expected net present value of the optimal*

equilibrium given asset price K. Then (i) $S(K)$ is strictly decreasing in K, and (ii) There exists a $\tilde{K}$ such that $S(\tilde{K}) > \tilde{K}$, but no investment occurs for any $K \geq \tilde{K}$.

Proof. *See Appendix A.* ■

Corollary 2 demonstrates that "under-investment"—in the sense that not all projects with positive net present value are implemented—occurs in equilibrium. Part (i) says that the total surplus in the principal-agent relationship is decreasing in the debt level (ignoring the cost of the debt itself), and is an immediate implication of part (iii) of Corollary 1. Part (ii) follows for reasons similar to Proposition 2. The principal must earn enough surplus in equilibrium to be deterred from diverting profits to her private benefit, and therefore some of the profit created by the initial investment K cannot be recouped by the creditor.

4   Firm Scale and Expansion

We can extend the logic of our basic analysis to consider how a firm that faces this commitment problem might choose to scale its current operations or expand into new lines of business. To do so, we allow the principal to choose a one-time investment size at the start of the game, which determines both her profitability and the amount she must borrow.

Formally, suppose that the principal can choose an investment cost $K \geq 0$ at the very start of the game. This investment determines the firm's production function, but this cost must be covered by borrowed money. The game then proceeds as in the baseline model, with the sole exception that output and costs are determined by the firm's initial investment. If the firm is liquidated, then total output and payoffs equal 0 regardless of the investment. It is therefore an optimal punishment for the creditor to liquidate the firm following any deviation. We return to this assumption after we state the main result.

Intuitively, investment can potentially have two conflicting effects on productivity. First, as in the baseline model, larger investments require more debt and so can lead to lower output in the early periods of the relationship. Second, certain kinds of investments might relax the principal's dynamic enforcement constraints because she stands to lose more surplus if the firm is liquidated, which can lead to higher bonuses and output.

To explore these conflicting effects, we separately consider two different production functions.

In the first, the investment determines the **scale** of the firm: both cost and benefits are multiplied by a factor $f(K)$ that is increasing in the size of the initial investment K. The second considers an investment to **expand** the firm's business: the principal earns an additional source of profit $f(K)$ that is independent of the current relationship.

Definition 1 *Let $f(K)$ be strictly increasing, strictly concave, and twice continuously differentiable, with $f(0) = 0$. An investment K is a:*

1. **Scale investment** *if output equals $f(K)y_t$ and the agent's cost equals $f(K)c(y_t)$ in each period t.*

2. **Expansion investment** *if output equals $f(K) + y_t$ and the agent's cost equals $c(y_t)$ in each period t.*

A scale technology represents an investment that improves the productivity of the current workforce, while an expansion investment represents a new opportunity—for example, an international expansion or option to enter a new industry—that is unrelated to profitability in the existing business. In principle, we could consider many other types of investments, but these two examples provide natural benchmarks that demonstrate our core intuition.

For a scale technology, the surplus-maximizing investment depends on the agent's expected output choices in equilibrium. Fixing $S = \sum_{t=0}^{\infty} \delta^t(y_t - c(y_t))$, we can write total surplus as $f(K)S - K$. Let $K^{scale}(S)$ maximize total surplus given a fixed total discounted expected surplus S, so $K^{scale}(S)$ solves $f'(K^{scale}(S)) = \frac{1}{S}$. In contrast, the return on an expansion investment is independent of productivity in the existing relationship. Define K^{xp} as the first-best expansion investment, defined by $f'(K^{xp}) = 1$.

Our main result in this section analyzes the principal's investment in the profit-maximizing equilibrium of the games with each of these technologies. If the principal chooses a scale investment, then she always has a weak incentive to underinvest, and she strictly underinvests if first-best cannot be attained. The principal sometimes has an incentive to underinvest in expansion, as well, though for other parameters she has an incentive to overinvest instead.

Proposition 4   *1. Consider a scale investment. For profit-maximizing equilibrium σ^*, define K^* as the investment level and $S^* = E_{\sigma^*}\left[\sum_{t=0}^{\infty} \delta^t(y_t - c(y_t))\right]$. If Assumption 1 holds at*

$K = K^{scale}(S^{FB})$, then $K^* < K^{scale}(S^*)$ in any profit-maximizing equilibrium. Otherwise, $K^* = K^{scale}(S^*)$ and $S^* = S^{FB}$.

2. Consider an expansion investment. There exists an open set of parameters such that in any profit-maximizing equilibrium, $K < K^{xp}$. However, there exists another open set of parameters such that $K > K^{xp}$ in any profit-maximizing equilibrium.

Proof. See Appendix A. ∎

As in Proposition 3, the profit-maximizing equilibrium converges to a steady state in some period T. Define $\tilde{S}(K) = \sum_{t=T+1}^{\infty} \delta^{t-T-1}(y^* - c(y^*))$ as output minus cost in that steady state, ignoring the proceeds $f(K)$ from the investment.

Consider part 1 of Proposition 4. In the steady state, a larger scale investment K proportionally increases both output and the costs borne by the agent. Because the loan has already been paid when the steady state is reached, $f(K)$ cancels from all constraints (2)-(6) and hence $\tilde{S}(K) \equiv \tilde{S}$ is constant in K. Consequently, the analogue to constraint (8) from Proposition 3 is

$$\sum_{t=0}^{T} \delta^t (\delta^{T-t} y_T - c(\delta^{T-t} y_T)) + \delta^{T+1}\tilde{S} \geq \frac{K}{f(K)}$$

in the scale investment extension. The left-hand side of this expression is constant in K, while the right-hand side is strictly increasing in K because $f(K)$ is concave. Hence, a larger scale investment leads to lower output in the early periods of the relationship. The profit-maximizing equilibrium balances this productivity effect and the direct cost K of a scale investment against the gains from that investment, leading to under-investment relative to K^{scale}.

In contrast, an expansion investment has more complicated effects on equilibrium behavior. As with a scale investment, an expansion requires debt and so tends to lower productivity. However, an expansion investment also increases the amount of surplus that the principal loses following a deviation, since she loses the additional proceeds of the expansion if she declares bankruptcy. Unlike a scale investment, this additional surplus is not offset by higher costs because the agent's costs are independent of K. The profit-maximizing equilibrium might entail under- or over-investment depending on which of these two forces dominates.

The effect of an expansion investment on productivity is determined by the analogue to con-

straint (8),

$$\sum_{t=0}^{T} \delta^t (\delta^{T-t} y_T - c(\delta^{T-t} y_T)) + \delta^{T+1} \tilde{S}(K) \geq K + \delta^T f(K) - \frac{1}{1-\delta} f(K). \tag{11}$$

This constraint is vacuous if the right-hand side is negative at K^{xp}, in which case first-best is attainable. Otherwise, the principal has an incentive to under-invest if this constraint tightens as K increases to K^{xp}, and otherwise has an incentive to over-invest. The size of the expansion K affects this constraint in two places. First, $\tilde{S}(K)$ is weakly increasing in K because a larger expansion relaxes the dynamic enforcement constraint (4) once the steady state is reached. Second, for a fixed T, the right-hand side of (11) is locally increasing in K near K^{FB}. If the first effect dominates, then the principal has an incentive to over-invest in equilibrium to relax this constraint; otherwise, the principal has an incentive to under-invest. The proof for part 2 of Proposition 4 characterizes parameter ranges for which each of these effects dominate.

We can interpret these two different production technologies more broadly. For instance, a scale investment might determine the size of the firm's workforce, where per-worker output (and costs) exhibit decreasing returns to scale. In this interpretation, K equals the number of workers, each of whom requires a machine that must be purchased with borrowed funds. The principal engages in a symmetric, multilateral relational contract with these workers (a la Levin (2002)), where any deviation is jointly punished by the entire workforce. Similarly, an expansion investment can be interpreted as a reduced-form model of a second worker or division that manufactures a technologically independent output. Then $f(K)$ represents the net profit generated by this second output.

In the expansion extension, the possibility of over-investment is driven by the assumption that the principal loses all future revenue if she reneges. Thus, the principal might over-invest in the expansion to create slack that can be used in her relationship with the agent. We believe the assumption that the principal loses all future surplus following a deviation to be reasonable in many settings, particularly since it serves as an optimal penal code of the game (Abreu (1988)). However, this assumption might be unrealistic in some settings; for instance, the creditor might only be able to liquidate the expansion following a deviation by the principal. In that case, the principal would still have an incentive to under-invest in expansion, but she would never over-invest.

5   Discussion and Conclusion

Our analysis assumes that the principal accesses the credit market only once and does not save across periods. In practice, we could imagine a firm using several different forms of savings. We discuss two here: (i) a savings account that can be accessed by the principal at any time, or (ii) an escrow account that she can access only in certain periods, and only if she remains solvent. For the purposes of our model, the important difference between these two possibilities is that the principal can freely divert money from a savings account, but not from an escrow account. As a result, these two means of storing money have very different implications for equilibrium dynamics.

Suppose the principal can save her profits at interest rate $r = \frac{1-\delta}{\delta}$. Formally, we can model savings by replacing the period-by-period budget constraint $b_T + r_T \leq y_T$ with the requirement that accumulated payments cannot exceed accumulated profit: for all $T \geq 0$, $\sum_{t=0}^{T} \delta^t (y_t - r_t - b_t) \geq 0$. The principal can access these savings in each period, and in particular can steal these savings if she chooses to renege on her promises.

The principal does not need to save in the steady-state equilibrium because the upper bound of the limited liability constraint is slack. Before the equilibrium converges to its steady-state, the principal optimally uses her entire profit to either repay the money she borrowed, or pay the agent a bonus in exchange for a continuation equilibrium that favors the principal. In either case, the principal optimally pays her entire output in each period and so would have no use for savings. Hence, allowing savings of this form would not change our equilibrium dynamics, and all of our original results would continue to hold.

In contrast, the profit-maximizing equilibrium might differ substantially from the baseline model if the principal can store money in an escrow account that can only be accessed in certain periods and is lost in bankruptcy. As an illustration, assume first-best is unattainable and consider the profit-maximizing equilibrium after the debt has been repaid and play has converged to the steady state. The principal pays $b_t = c(y^*)$ and output equals y^* in each period, where y^* is chosen so that the dynamic enforcement constraint binds. Suppose the principal puts some money in an escrow account that she can remove several periods later if (and only if) she has not declared bankruptcy. This stored money relaxes the principal's dynamic enforcement constraint because she earns it on the equilibrium path but would lose it following a deviation. Intuitively, this escrow account

serves as a "hostage" (in the sense of Williamson (1983)) to induce the principal not to renege on her promises. Since the profit-maximizing equilibrium eventually converges to this steady state, an escrow account would substantially change equilibrium dynamics. Note that this type of escrow account could potentially mitigate or eliminate commitment problems in other kinds of relational contracts, as well.

Managers must balance the promises they make to motivate their workers against their obligations to external creditors. This papers has explored one mechanism by which debt can influence productivity in a setting with limited commitment. More generally, external finance can have a substantial impact on how a firm chooses to attract, retain, and motivate its workers. A firm's relationships with its employees are constrained by the demands of its relationships with banks and other creditors. Understanding these links can shed light on how firms invest, motivate their workers, and grow.

6   References

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7 Appendix A: Proofs

7.1 Proof of Proposition 1

Suppose $\frac{1}{1-\delta}(y^{FB}-c(y^{FB})) \geq K$ and consider the following long-term formal contract: the principal borrows $R = K$ at the start of the game and pays $b_t = c(y^{FB})$, $r_t = (1-\delta)K$ in each period $t \geq 0$. The agent chooses $y_t = y^{FB}$. The principal can commit to a long-term contract and so has no profitable deviation. Following any deviation in output, the agent is paid $b_t = 0$ and chooses $y_t = 0$ in each subsequent period.

We show that this strategy is an equilibrium with the desired properties. The agent's IC constraint is satisfied in each period because $b_t - c(y^{FB}) = 0 \geq 0$. The loan repayment constraint is satisfied because $\sum_{t=0}^{\infty} \delta^t r_t = \frac{1-\delta}{1-\delta}K = K = R$. For limited liability, $b_t, r_t \geq 0$ are implied by $K > 0$ and $c(y^{FB}) > 0$, while the assumption that the firm has positive net present value implies $\sum_{t=0}^{\infty} \delta^t(y^{FB} - c(y^{FB})) \geq K$, which in turn implies $c(y^{FB}) + (1-\delta)K \leq y^{FB}$ or $b_t + r_t \leq y_t$.

The principal earns her maximum feasible surplus $\Pi^{FB} - K$ from this long-term contract. So this contract is an equilibrium, and any other optimal contract must have the same properties. ∎

7.2 Proof of Proposition 2

Suppose $y_t < y^{FB}$ for some period $t \geq 0$. Consider the perturbation in which $y_t = y^{FB}$ but all other variables remain the same. This perturbation increases the principal's payoff, strictly relaxes the DE and Limited Liability constraints, and hold the Loan Repayment constraint constant. So $y_t \geq y^{FB}$ in any optimal equilibrium.

A necessary condition for DE to hold in period $t = 0$ is that

$$K = \sum_{t=0}^{\infty} \delta^t r_t \leq \frac{\delta}{1-\delta}(y^{FB} - c(y^{FB})).$$

If this condition does not hold, then Loan Repayment cannot be satisfied and so the firm does not start. If this expression is satisfied, consider an equilibrium with $r_t = (1-\delta)K$ and $y_t = y^{FB}$. This clearly maximizes the objective function and satisfies Loan Repayment with equality. It satisfies DE because

$$(1-\delta)K \leq \frac{\delta}{1-\delta}(y^{FB} - c(y^{FB})) - \delta K$$

is implied by the necessary condition for an equilibrium. Limited Liability is also satisfied because $(1-\delta)K \leq y^{FB} - c(y^{FB}) \leq y^{FB}$. Thus, if the firm can secure a loan, then $y_t = y^{FB}$ in each period of any optimal equilibrium. ∎

7.3 Proof of Lemma 1

(i) Let t^* be the final period in which $r_t > 0$, where $t^* = \infty$ if no such period exists. For $t < t^*$, suppose $b_t > 0$. Consider the following perturbation: $\tilde{b}_t = b_t - \epsilon$, $\tilde{r}_t = r_t + \epsilon$, $\tilde{r}_{t^*} = r_{t^*} - \delta^{t-t^*}\epsilon$, and $\tilde{b}_{t^*} = b_{t^*} + \delta^{t^*-t}\epsilon$, with no other changes. This perturbation holds the principal's payoff constant while weakly relaxing all IC constraints and continuing to satisfy DE, loan repayment, and limited liability. Therefore, this perturbation is also an optimal SPE. This perturbation can be performed until either $r_t^* = 0$ (in which case either t is the final period with $r_t > 0$, or we can continue the perturbation using some other period $t' \in (t, t^*)$), or $b_t = 0$.

Suppose $r_t < y_t$ in some period $t < t^*$. Consider the perturbation $\tilde{r}_t = r_t + \epsilon$ and $\tilde{r}_{t^*} = r_{t^*} - \delta^{t^*-t}\epsilon$. This perturbation satisfies limited liability by assumption, weakly relaxes DE, and satisfies all other constraints by construction. This perturbation can be performed until $r_{t^*} = 0$ (in

which case we can continue to perturb the equilibrium as in the previous argument) or $r_t = y_t$.

(ii) Assumption 2 immediately implies (by DE and IC constraints) that $y_t < y^{FB}$ in each period t. Suppose there exists a period $t \geq 0$ such that DE does not bind. Consider a perturbation in period t to $\tilde{y}_t$ such that $c(\tilde{y}_t) = c(y_t) + \epsilon$ and $\tilde{b}_t = b_t + \epsilon$, with no other changes. This perturbation strictly increases the principal's payoff because $y_t < y^{FB}$. It continues to satisfy loan repayment, and DE is slack by assumption. The IC constraint is satisfied because b_t and $c(y_t)$ increase by the same amount. Because y_t increases by strictly more than b_t, the feasibility constraint continues to be satisfied. So this alternative is an equilibrium that strictly dominates the original equilibrium. Contradiction; DE must bind.

(iii) We first claim that total surplus is strictly increasing in the principal's continuation payoff. We can ignore the loan repayment constraint in this argument because $t > t^*$ and hence $r_{t'} = 0$ in all $t' \geq t$. Suppose the agent earns continuation payoff $U > 0$ in some period $T \geq t$. This payoff may be written

$$U \equiv \sum_{t=T}^{\infty} \delta^{t-T}(b_t - c(y_t)) > 0.$$

In particular, the agent's IC constraint is slack in period T. But then there exists $\tilde{y}_T > y_T$ such that the agent's IC constraint continues to hold if he produces $\tilde{y}_T$. Increasing y_T would increase both total surplus and the principal's surplus because $y_T < y^{FB}$, and all other constraints in period T onwards would continue to be satisfied. So total surplus is strictly increasing in the principal's payoff.

Suppose $b_t + r_t < y_t$ and the agent's continuation utility is $U_{t+1} > 0$ for some period t. Consider the following perturbation: increase b_t by $\delta\epsilon$, increase $c(y_{t+1})$ by ϵ, and do not change any other variables. Because the agent's IC constraint is slack in period $t + 1$, play from period $t + 1$ onward remains an equilibrium. The principal is willing to pay $b_t + \delta\epsilon$ because she receives at least ϵ more surplus in period $t + 1$ due to higher output. So the principal's DE constraint holds. Feasibility is satisfied by assumption because $b_t + r_t < y_t$. So this perturbation is an equilibrium that strictly dominates the original; contradiction.

(iv) Suppose agent IC in period $t = 0$ is slack. Analogous to the argument in (iii), the agent is willing to produce strictly higher output y_0. Higher y_0 continues to satisfy agent IC in $t = 0$ by assumption, strictly increases the principal's payoff, and leaves all other constraints unchanged; contradiction

(v) Consider a period $t > t^*$ in which the agent's continuation utility is $U_t = 0$. Because $t > t^*$, we know that $r_{t'} = 0$ for all $t' > t$. Suppose towards contradiction that $U_{t+1} > 0$. Then $b_t < c(y_t)$, since otherwise $U_t > 0$. But then $b_t < y_t$ as well. By claim (iii), $b_t < y_t$ is only optimal if $U_{t+1} = 0$; contradiction. We conclude that if $U_t = 0$ for some $t > t^*$, then $U_{t'} = 0$ for all $t' > t$. This sequence of payoffs can be implemented by a stationary equilibrium. ∎

7.4 Proof of Proposition 3

Consider the optimal program, and define

$$\Pi_t = \sum_{t'=t}^{\infty} \delta^{t'-t}(y_{t'} - b_{t'} - r_{t'})$$
$$U_t = \sum_{t'=t}^{\infty} \delta^{t'-t}(b_{t'} - c(y_{t'}))$$

.

Let T be the final period before the steady-state. We argue that Lemma 1, $U_T \geq 0$, and $U_0 \geq 0$ together imply the IC constraints in every other period are satisfied in an optimal equilibrium. Let t^* be the final period in which $r_{t^*} > 0$. By Lemma 1, for $t < t^*$, $r_t = y_t$ and $b_t = 0$. Therefore, the IC constraint in periods $0 \leq t < t^*$ may be written

$$U_t = -c(y_t) + \delta U_{t+1} \geq 0.$$

But $-c(y_t) \leq 0$. So $U_0 \geq 0$ implies that $U_1 = \frac{c(y_0)+U_0}{\delta} > 0$, and similarly $U_{t+1} = \frac{c(y_t)+U_t}{\delta} > 0$ by induction. So $U_t \geq 0$ for each $t \leq t^*$. For $T > t > t^*$, Lemma 1 implies that $b_t = y_t$. So for each $t < T$,

$$U_t = \sum_{t'=t}^{T-1} \delta^{t'-t}(y_t - c(y_t)) + \delta^{T-t}U_T.$$

In an optimal equilibrium, $y_t \leq y^{FB}$ so $y_t - c(y_t) \geq 0$. Hence, $U_T \geq 0$ implies that $U_t \geq 0$ for all $t^* < t \leq T$. Together, these arguments prove that we can ignore all IC constraints except at $t = 0$

and $t = T$, as desired.

Next, Lemma 1 implies that dynamic enforcement binds whenever $y_t < y^{FB}$, so $b_t + r_t = \delta\Pi_{t+1}$ in these periods. The upper bound of Limited Liability also binds if $t < T$, so $\delta\Pi_{t+1} = b_t + r_t = y_t$ for $t < T$. Furthermore, for every $t \leq T$,

$$\Pi_t = \delta^{T-t}\Pi_T = \delta^{T-t}(y_T - b_T - r_T + \delta\Pi_{T+1}) = \delta^{T-t}y_T$$

where the first equality holds because $b_t + r_t = y_t$ for all $t < T$, the second is by definition of Π_T, and the final follows because $y_T < y^{FB}$ and so the dynamic enforcement constraint binds. Because output equals y^* for every $t > T$, we conclude that output in each period depends entirely on the choice variables T and y_T.

Given these simplifications and the result that $\sum_{t=0}^{T}\delta^t r_t = K$, the optimal program may be written

$$\max_{T, y_T, \{b_t\}, \{r_t\}} \Pi_0 = \delta^T y_T$$

subject to:

1. **Incentive compatibility:** $U_0 \geq 0$ and $U_T \geq 0$.

2. **Dynamic enforcement**: $\forall t \geq 0, b_t + r_t = \delta\Pi_{t+1}$.

3. **Loan Repayment:** $\sum_{t=0}^{T}\delta^t r_t = K$.

4. **Limited liability**. $\forall t \geq 0$,
$$b_t \geq 0$$
$$r_t \geq 0 \qquad .$$
$$b_t + r_t \leq y_t$$

Lemma 1 implies that $U_0 = 0$ in the optimal equilibrium, so

$$\sum_{t=0}^{T} -\delta^t c(y_t) + \sum_{t=t'}^{T} \delta^t b_t + \delta^{T+1}U^* = 0$$

where U^* is the agent's steady-state payoff. Using the fact that DE binds and $b_t + r_t = y_t$ in every

$t < T$, we may rewrite

$$\sum_{t=0}^{T-1} \delta^t(y_t - c(y_t)) + \delta^T(b_T + r_T) - \delta^T c(y_T) + \delta^{T+1} U^* = \sum_{t=0}^{T} \delta^t r_t = K.$$

Plugging in $b_T + r_T = \delta\Pi^* =$ and our solved output sequence, we conclude that

$$\sum_{t=0}^{T-1} \delta^t(\delta^{T-t} y_T - c(\delta^{T-t} y_t)) + \delta^T \left(\delta S^* - c(y_T)\right) = K$$

In this optimal equilibrium. This constraint may be rearranged to give the first constraint of Proposition 3.

Finally, we argue that $U_T \geq 0$ if and only if $y_T \leq y^*$. Since $U_{T+1} = U^* = 0$, $U_T = b_T - c(y_T)$. But $b_T \leq \delta\Pi^* = c(y^*)$ by Dynamic Enforcement and the definition of y^*, so $c(y_T) \leq c(y^*)$ or $y_T \leq y^*$ is clearly a necessary condition. To show that it is sufficient, suppose first that $r_T = 0$. Then $b_T = \delta\Pi^*$ because Dynamic Enforcement binds, proving the claim. Suppose instead that $r_T > 0$. Then $b_t = 0$ for all $t < T$, so

$$U_0 = \delta^T(b_T - c(y_T)) - \sum_{t=0}^{T-1} \delta^t c(y_t) \leq b_T - c(y_T).$$

Therefore, $U_0 \geq 0$ implies $U_T \geq 0$, and so $y_T \leq y^*$ is (vacuously) sufficient in that case as well. So we can replace $U_T \geq 0$ with $y_T \leq y^*$.

After these derivations, the only variables that enter the problem are y_T and T. Hence, we can ignore all constraints that do not involve these variables. We conclude that the optimal program is equivalent to the simplified program given in Proposition 3. ∎

7.5   Proof of Corollary 1

Proof of part (i). To see $y_T < y^*$, recall that T is the first period following which the agent earns a continuation payoff of 0. The agent's incentive constraint in period T can then be written as $c(y_T) \leq b_T \leq \delta S^* = c(y^*)$, where the second inequality follows from the dynamic enforcement constraint. This proves that $y_T \leq y^*$. Now if $y_T = y^*$, then $c(y_T) = b_T = \delta S^* = c(y^*)$. Therefore, the agent's payoff in period T is also 0. This implies that the agent earns a continuation payoff of

0 following $T-1$, contradicting the definition of T.

To see that $y_T \geq \delta y^*$, note that we have $y_T \geq b_T + r_T = \delta S^* \geq \delta y^*$, where the first inequality follows from the feasibility constraint at T, the first equality follows from that the dynamic enforceability constraint is binding at T, and the second inequality follows from the dynamic enforceability constraint at the steady state.

Before proving part (ii) and (iii), it is useful to introduce two auxiliary functions. First, let

$$f(t,y) \equiv \sum_{\tau=0}^{t-1} \delta^\tau \left(\delta^{t-\tau} y - c \left(\delta^{t-\tau} y \right) \right) + \delta^t \left(\delta S^* - c(y) \right).$$

Note that $f(T, y_T) = K$ by Proposition 3. In addition, $f(0, y^*) = \delta S^* - c(y^*) = 0$ by definition. Moreover, $f(t, y^*) = f(t-1, \delta y^*)$ for all $t \geq 1$. To see this, note that

$$f(t-1, \delta y) = \sum_{\tau=0}^{t-2} \delta^\tau \left(\delta^{t-\tau} y - c \left(\delta^{t-\tau} y \right) \right) + \delta^{t-1} \left(\delta S^* - c(\delta y) \right).$$

It then follows that

$$f(t-1, \delta y) - f(t, y) = \delta^t \left(-(y - c(y)) + (1 - \delta) S^* \right).$$

Since S^* is the steady state surplus, we have $(1 - \delta) S^* = y^* - c(y^*)$, and this gives that $f(t-1, \delta y^*) - f(t, y^*) = 0$.

Note that $f(t, y^*)$ is defined only for integer-valued t. We now consider an extension of $f(t, y^*)$ to the realm of all real $t \geq 0$. Let $\lfloor t \rfloor$ be the floor function. Define

$$g(t) \equiv f\left(\lfloor t \rfloor, y^* \delta^{t - \lfloor t \rfloor} \right).$$

Notice that $g(t) = f(t, y^*)$ when t is an integer and that $g(0) = 0$. In addition, $G(t)$ is continuous in t. This follows because $f(\lfloor t \rfloor, \delta y^*) = f(\lfloor t \rfloor + 1, y^*)$.

Also define $y(t) = \delta^{t - \lceil t \rceil + 1} y^*$, where $\lceil t \rceil$ is the ceiling function t. Take any integer n. If $t \in (n, n+1]$, we then have $g(t) = f(n, y(t))$. To see this, notice that if $t \in (n, n+1)$, $\lfloor t \rfloor = n$ and $y(t) = y^* \delta^{t - \lfloor t \rfloor}$, so $g(t) = f\left(n, y^* \delta^{t - \lfloor t \rfloor} \right) = f(n, y(t))$. If $t = n+1$, notice that $y(n+1) = \delta y^*$.

Therefore,

$$g(n+1) = f(n+1, y^*) = f(n, \delta y^*) = f(n, y(n+1))).$$

Conversely, for all $y \in [\delta y^*, y^*)$ and an integer n, there exists a well defined function $\tau(y, n) \in (n, n+1]$ such that $f(n, y) = g(\tau(y, n))$. This follows because $y(t) \in [\delta y^*, y^*)$ and is strictly decreasing in t. The function g allows us to establish the following result, which leads to part (ii) and (iii).

Result: Let T^* be the smallest value with $g(T^*) = K$. We have $T = \lceil T^* \rceil - 1$ and $y_T = y(T^*) = \delta^{T^* - \lceil T^* \rceil + 1} y^*$.

To see this, suppose the optimal equilibrium specifies (T, y_T). Recall that $y_T \in [\delta y^*, y^*)$ and that $f(T, y_T) = K$. The properties of g then implies that $g(\tau(y_T, T)) = f(T, y_T) = K$. Therefore, $T^* \le \tau(y_T, T)$. We now claim that $g(t) < K$ for all $t < \tau(y_T, T)$, so $T^* = \tau(y_T, T)$.

Notice that $\tau(y_T, T) \in (T, T+1]$ and the principal's expected profit is therefore given by $\delta^T y_T$. Now suppose to the contrary that $g(t') = K$ for some $t' < \tau(y_T, T)$. Now either $t' \in (T, \tau(y_T, T)]$ or $t' \in (n, n+1]$ for some integer $n < T$. When $t' \in (T, \tau(y_T, T)]$, Proposition 3 implies that one feasible payoff of the principal is $\delta^T y(t')$. But since $y(t)$ is decreasing in t, it follows that $\delta^T y(t') > \delta^T y_T$. This contradicts the optimality of (T, y_T).

When $t' \in (n, n+1]$, the principal's payoff is $\delta^n y(t') \ge \delta^{n+1} y^* > \delta^T y_T$, where the first inequality follows because $y(t') \ge \delta y^*$ and the second inequality follows because $y_T < y^*$. This again contradicts the optimality of (T, y_T). This proves the result.

Proof of part (ii). Notice that $g(t)$ is continuous in t and that $g(0) = f(0, y^*) = 0$. The result above then implies that T^*, and therefore T, increases with K.

Proof of part (iii). Suppose $T^*(K) \in (n, n+1]$, where recall that $T^*(K)$ is smallest t such that $g(T^*) = K$. Now consider two cases. First, $T^*(K) = n+1$, so $g(n+1) = K$ and $g(t) < K$ for all $t < n+1$. The result before part (ii) then implies that $y_T(K) = y_n(K) = \delta y^*$. Now for any $K' > K$, we then have $T^*(K') > n+1$, so $T(K) \ge n+1$. It then follows that $y_n(K') = \delta y_{n+1}(K') \le \delta y_T(K') < \delta y^*$, where the last inequality follows from part (i). This proves that $y_n(K') < y_n(K)$, and Proposition 3 then implies that $y_t(K') \le y_t(K)$ for all t, and the inequality is strict for all $t \le T(K')$.

Second, $T^* \in (n, n+1)$, $T^*(K)$ is not an integer, In this case, $T = n$. Now consider a small increase in K to K'. If $T^*(K') > n+1$, then the argument in the first case can be applied again to obtain $y_n(K') < y_t(K)$. If $T^*(K') = n$, notice that $T^*(K') > T^*(K)$ by the definition of T^*. The result before part (ii) then implies that $y_T(K) = \delta^{T^*(K)-n}y^* > \delta^{T^*(K')-n}y^* = y_T(K')$.

Combining the two cases, we obtain that y_t strictly decreases in K for all $t \leq T(K)$, and this proves part (iii). ∎

7.6 Proof of Corollary 2

Proof of part (i). This follows directly from part (iii) from Corollary 1 because y_t decreases in K.

Proof of part (ii). Define $\tilde{K} = \max_t g(t)$. Notice that $\tilde{K}$ is well defined because $g(t)$ is continuous and is bounded from above. The upper bound exists because $g(t) = f(t, y^*)$ for integer-valued t, and it can be checked that $f(t, y^*)$ goes to 0 as t goes to infinity. The result before part (ii) in Corollary 1 then implies that the project is funded if and only if $K \leq \tilde{K}$. Finally, $S^*(\tilde{K}) > \tilde{K}$ follows because the principal's profit is strictly positive at $\tilde{K}$ and the loan is paid back. ∎

7.7 Proof of Proposition 4

Proof of part (1). Consider a scale investment, and ignore the creditor IR constraint. We claim that the profit-maximizing equilibrium is stationary and entails output y^* that is independent of K. The contract is stationary by the same proof as in Lemma 1. Therefore, $b^* = f(K)c(y^*)$ in each period. If first-best output is not attainable, then the dynamic enforcement constraint binds and so y^* is the unique output that solves

$$f(K)c(y^*) = \frac{\delta}{1-\delta}(f(K)y^* - f(K)c(y^*)).$$

The term $f(K)$ cancels from both sides of this expression, so y^* is independent of K. If first-best y^{FB} is attainable, then $y^* = y^{FB}$ maximizes $f(K)(y - c(y))$. So y^{FB} is likewise independent of K. We conclude that steady-state output is independent of K.

Given this result and assuming $K > 0$, we can replicate the steps in Lemma 1 and Proposition 3 to derive the relaxed program

$$f(K) * \max_{T, y_T} \delta^T y_T \tag{12}$$

subject to

$$\sum_{t=0}^{T-1} \delta^t(\delta^{T-t}y_T - c(\delta^{T-t}y_T)) + \delta^T \left(\delta\tilde{S} - c(y_T)\right) = \frac{K}{f(K)}, \tag{13}$$

$$y_T \leq y^*. \tag{14}$$

where $\tilde{S} = \frac{1}{1-\delta}(y^* - c(y^*))$ equals steady-state output divided by $f(K)$. This optimal program is identical to the optimal program in Proposition 3, except the right-hand side of the first constraint is $\frac{K}{f(K)}$, which is strictly increasing in K because $f(\cdot)$ is strictly concave.

Because the agent earns 0 payoff and the bank earns exactly K, we can rewrite the principal's payoff as $f(K)\tilde{S}(K) - K$. Corollary 1 immediately implies that $\tilde{S}(K)$ is strictly decreasing in the debt level K. So $K \leq K^{scale}(\tilde{S}(K))$ in equilibrium.

Suppose $K^* = K^{scale}(\tilde{S}(K))$, and consider an alternative scale $\tilde{K} = K^{scale}(\tilde{S}(K)) - \epsilon$. For $K = K^{FB}$ to be optimal, it must be that

$$f(K^*)\tilde{S}(K^*) - K^* \geq f(\tilde{K})\tilde{S}(\tilde{K}) - K^* + \epsilon.$$

Since $\tilde{S}(K)$ is strictly decreasing in K, the inequality must hold strictly if we replace $\tilde{S}(\tilde{K})$ with $\tilde{S}(K^*)$. Dividing by ϵ and taking $\epsilon \to 0$ then yields

$$f'(K^{scale}(\tilde{S}(K^*))) > \frac{1}{\tilde{S}(K^*)}.$$

This contradicts the definition of $K^{scale}(S)$.

Proof of part (2). As in part (1), if we ignore the creditor IR constraint than the profit-maximizing equilibrium is stationary. In it, either first-best output is attainable ($y^* = y^{FB}$) or

output is chosen so that the dynamic enforcement constraint binds:

$$c(y^*(K)) = \frac{\delta}{1-\delta}(y^*(K) + f(K) - c(y^*(K))).$$

Note that $y^*(K)$ is strictly increasing in K and $y^*(K) \leq y^{FB}$, so $\tilde{S}(K) \equiv \frac{1}{1-\delta}(y^*(K) - c(y^*(K)))$ is strictly increasing in K.

Replicating the steps from Lemma 1 and Proposition 3, we derive the relaxed program

$$\max_{T, y_T} \delta^T (y_T + f(K)) \tag{15}$$

subject to

$$\sum_{t=0}^{T-1} \delta^t (\delta^{T-t} y_T + f(K) - c(\delta^{T-t} y_T)) + \delta^T \left(\delta \tilde{S}(K) + \delta f(K) - c(y_T) \right) = K, \tag{16}$$

$$y_T \leq y^*. \tag{17}$$

Constraint (16) may be rewritten as (11).

Suppose that δ satisfies $c(y^{FB}) = \frac{\delta}{1-\delta}(y^{FB} - c(y^{FB})$, so first-best output is barely attainable in steady-state with $K = 0$. Then $\tilde{S}(K) = y^{FB} - c(y^{FB}) \equiv \tilde{S}^{FB}$ is constant in K. Assume $f(\cdot)$ is such that $\frac{\delta}{1-\delta} f(K^{xp}) - K^{xp} < 0$. (Note that this parameter restriction is consistent with the expansion having strictly positive net present value $\frac{1}{1-\delta} f(K^{xp}) - K^{xp} > 0$.)

Consider the right-hand side of (11):

$$K + \delta^T f(K) - \frac{1}{1-\delta} f(K). \tag{18}$$

We first claim that this expression is strictly positive at K^{xp}. Indeed, suppose not. Then (11) is slack and hence $T = 0$ in equilibrium. But if $T = 0$, then (18) is strictly positive because $K^{xp} > \frac{\delta}{1-\delta} f(K^{xp})$. Contradiction; so (18) is strictly positive.

Next, we argue that (18) is strictly increasing in K in a neighborhood about K^{xp}. Towards contradiction, suppose this expression is weakly decreasing in K. By an argument analogous to Corollary 1, it must be that T is then weakly decreasing in K. But then (18) is bounded below by

33

the identical expression holding T fixed on this neighborhood. The derivative of this lower bound is

$$1 + \delta^{\tilde{T}} f'(K^{xp}) - \frac{1}{1-\delta} f'(K^{xp}) = \delta^{\tilde{T}} f'(K^{xp}) > 0$$

by definition of K^{xp}. Hence, (18) is bounded below by a strictly increasing function, and the two functions are equal at K^{FB}. So (18) is strictly increasing; contradiction.

As in part (i), the principal's payoff may be written

$$\sum_{t=0}^{\infty} \delta^t (y_t - c(y_t)) + \frac{1}{1-\delta} f(K) - K.$$

Because (18) is strictly increasing in K, Corollary 1 implies that the first term in this expression is strictly decreasing in K. Decreasing K at K^{xp} creates a first-order gain in the second term because $y_t < y^{FB}$. However, $\frac{1}{1-\delta} f'(K^{xp}) - K^{xp} = 0$, so decreasing K at K^{xp} entails a second-order loss in the second term. Thus, $K < K^{xp}$ dominates K^{xp}. It also dominates $K > K^{xp}$ because (18) tightens as K increases.

Suppose instead that $\frac{\delta}{1-\delta} f(K^{xp}) - K^{xp} < 0$ and $c(y^{FB}) > \frac{\delta}{1-\delta}(y^{FB} - c(y^{FB}))$. Then (18) is negative and so (11) is vacuously satisfied. Hence, $T = 0$ and $y_T = y^*$ is optimal. As established above, y^* is strictly increasing in K for $y^* < y^{FB}$. The principal's payoff is therefore strictly increasing in K at K^{xp}, so $K > K^{xp}$ is strictly optimal. ∎