

# Rare Disasters and Risk Sharing with Heterogeneous Beliefs

Hui Chen      Scott Joslin      Ngoc-Khanh Tran\*

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## Abstract

Although the threat of rare economic disasters can have large effect on asset prices, difficulty in inference regarding both their likelihood and severity provides the potential for disagreements among investors. Such disagreements lead investors to insure each other against the types of disasters each one fears the most. Due to the highly nonlinear relationship between consumption losses in a disaster and the risk premium, a small amount of risk sharing can significantly attenuate the effect that disaster risk has on the equity premium. The model has several predictions. First, it shows that time variation in the wealth distribution and the amount of disagreement across agents can significantly amplify the variation in disaster risk premium. Second, it provides caution to the approach of extracting disaster probabilities from asset prices, which can disproportionately reflect the beliefs of a small group of optimists and can either exaggerate or understate the variations in disaster probabilities over time. Third, the model predicts an inverse U-shaped relationship between the equity premium and the size of the disaster insurance market.

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\*Chen: MIT Sloan School of Management (huichen@mit.edu). Joslin: MIT Sloan School of Management (sjoslin@mit.edu). Tran: MIT Sloan School of Management (khanh@mit.edu). We thank David Bates, George Constantinides, Xavier Gabaix, Lars Hansen, Jakub Jurek, Leonid Kogan, Hanno Lustig, Rajnish Mehra, Jun Pan, Dimitris Papanikolaou, Monika Piazzesi, Bob Pindyck, Annette Vissing-Jorgensen, Martin Schneider, Jiang Wang, Ivo Welch, Amir Yaron, and seminar participants at Berkeley Haas, Northwestern Kellogg, MIT Sloan, Minnesota Carlson, the AEA Meeting in Atlanta, the Amsterdam Asset Pricing Retreat, Financial Economics in Rio, the NBER Asset Pricing Meeting in Chicago, the Stanford SITE conference, and the Tepper/LAEF Advances in Macro-Finance Conference for comments. All the remaining errors are our own.

# 1 Introduction

How likely is it that a severe economic disaster will occur in the next 100 years? With a relatively short sample of historical data, it is difficult to accurately estimate the likelihood of disasters or the size of their impact. For example, one cannot reject the hypothesis of a constant disaster intensity of 3% per year at the 5% significance level even after observing a 100 year sample without a disaster. This suggests that there is likely to be significant heterogeneity in the beliefs of market participants about disasters. In this paper, we show that such disagreements can generate strong risk sharing motives among investors and significantly affect asset prices.

We study an exchange economy with two types of agents. Markets are complete, so that the agents can trade contingent claims and achieve optimal risk sharing. Using the affine heterogeneous beliefs framework developed in [Chen, Joslin, and Tran \(2010\)](#), our model can capture very general forms of disagreements among the agents while maintaining tractability. For example, the agents can disagree about the intensity of disasters or the severity of disasters, and the amount of disagreements can fluctuate over time.

We find that having a second group of agents with different beliefs about disasters can cause the equity premium to drop substantially, even when the new agents only have a small amount of wealth. This result holds whether the disagreement is about the intensity or impact of disasters. In fact, the result can still be true even when the new agents are generally more pessimistic about disasters. We analytically characterize the sensitivity of risk premiums to the wealth distribution and derive its limit as the amount of disagreement increases. When we calibrate the beliefs of one agent using international macro data (from Barro (2006)) and the other using only consumption data from the US (where disasters have been relatively mild), raising the fraction of total wealth for the second agent from 0 to 10% lowers the equity premium from 4.4% to 2.0%. The decline in the equity premium becomes faster when the disagreement is larger, or when the new agents also have lower risk aversion.

The reasons behind this result are quite intuitive. First, the equity premium grows exponentially in the size of individual consumption losses during a disaster. Thus, removing just the “tail of the tail” from consumption losses can dramatically bring down the premium. For example, in a representative agent economy (with relative risk aversion  $\gamma = 4$ ), if the consumption loss in a disaster is reduced from 40% to 35%, the equity premium will fall by 40%! This non-linearity is an intrinsic property of disaster risk models, which generate high premium from rare events by making marginal utility in the disaster states rise substantially with the size of the consumption losses.

Second, in our economy, as is typical in models with moderate risk aversion and low volatility of consumption growth, the equity premium derives primarily from disaster risk, and the compensation for bearing disaster risk must be high. For example, if the equity premium due to disaster risk is 4% per year, and the market falls by 40% in a disaster, then a disaster insurance contract that pays \$1 when a disaster strikes within a year must cost at least 10 cents, regardless of the actual chance of payoff. Such a high premium for disaster risk provides strong incentive for risk sharing among investors with different beliefs about disasters. In a benchmark example of our model, the pessimists are willing to pay up to 13 cents per \$1 of disaster insurance, even though the payoff probability is only 1.7% under their own beliefs. The optimists, who believe the payoff probability is just 0.1%, underwrite insurance contracts with notional value up to 40% of their total wealth, despite the risk of losing 70% of their consumption if a disaster strikes.

Taken together, when we allocate a small amount of wealth to agents with heterogeneous beliefs, the risk sharing they provide will be enough to significantly reduce the equity premium in equilibrium. Importantly, the above mechanism does not require the new agents to be “globally” more optimistic about disasters than the existing ones. What is critical to the risk sharing mechanism is that the minority wealth holders assign a sufficiently small probability to the types of disasters the majority wealth holders fear the most. Although these minority wealth holders may fear other types of disasters (e.g., ones that are even more dev-

astating), they will still be willing to share the disaster risk that the majority wealth holders fear. Thus, heterogeneous beliefs can result in a low equity premium even if each investor individually would have demanded a high equity premium if other types of investors had not been present.

The model provides new insights on how disaster risk affects the dynamics of asset prices. In particular, the disaster risk premium crucially depends on the wealth distribution among investors with different beliefs, and it can become more or less volatile than in a model with homogeneous investors. On the one hand, when the wealth distribution is disperse, the disaster risk premium will remain low and smooth despite the fluctuations in the average belief of disaster risk in the market. This result is in sharp contrast to that of a time-varying disaster risk model with homogeneous agents. On the other hand, when the wealth distribution becomes more concentrated, the disaster risk premium will increase significantly, and become more sensitive to fluctuations in disaster risk going forward. This can happen when a disaster strike, which will make those optimists lose a large fraction of their wealth and greatly reduces their risk sharing capacity. It can also happen when the beliefs of investors converge, perhaps due to the updating following a relatively small market crash (an example is the recent financial crisis, which caused rising fear of a repeat of the Great Depression).

The above results also provide caution when we extract disaster probabilities from asset prices, for example, from equity index options. The link between the risk neutral and actual probabilities of disasters is simple and stable in a model with homogeneous agents, which makes it straightforward to estimate the actual probabilities from option prices. However, our model shows that the risk neutral probabilities of disasters (and therefore asset prices) tend to disproportionately reflect the beliefs of a small group of optimistic investors. Thus, we could substantially underestimate the probability of disasters without taking this effect into account. Moreover, the changes in wealth distribution can lead to substantial changes in the risk neutral probabilities of disasters independent of the actual probabilities, which

could cause us to overestimate the variations in actual disaster probabilities over time. Since the estimation of disaster probabilities has important policy implications, these potential biases due to heterogeneous beliefs should be treated carefully.

Another prediction of our model is an inverse U-shaped relationship between the equity premium and the size of the disaster insurance market. Empirical proxies for disaster insurances include any assets that are sensitive to an economic disaster, for example, out-of-the-money index put options, high-grade corporate bonds, or the senior tranche of collateralized debt obligations (CDOs). Our model predicts two distinct scenarios in which there will be little trading of these types of contracts: (i) when the market perceived disaster risk is low, or (ii) when investors all agree that disaster risk is high, so that no one is willing to provide the insurance. The disaster risk premium will be low in the first case, but high in the second case. Times when there is a lot of trading in disaster insurance are also times when there is significant heterogeneity across investors, which will keep the risk premium at low levels.

Finally, our model shows that in a model where the only heterogeneity across agents is their beliefs, the average belief is not sufficient to determine the equity premium. Instead, the amount of disagreement across agents represents another important source of fluctuations in the disaster risk premium. In contrast, when agents with the same preferences disagree about growth rates, the equity premium is determined completely by the average belief. Moreover, again in contrast to the case of disagreement about consumption growth rates, in our model agents with wrong beliefs are likely to survive for very long periods and even increase their wealth.

We discuss the robustness of our results to several considerations, including incomplete markets, the beliefs of the optimists, and learning. Since there is no default in equilibrium, all of the trading (including the disaster insurance) can be fully collateralized in our model. Moreover, we show that when disaster risks largely account for the equity premium, trading in the equity claim alone provides a close substitute to trading a disaster insurance contract. However, if a portion of the optimist agents' wealth is non-tradeable (such as human capital),

the risk sharing effect will be attenuated. We also show that, by the nature of rare disasters, learning occurs slowly and has little impact on asset prices.

This paper builds on the disaster risk model of [Rietz \(1988\)](#), [Longstaff and Piazzesi \(2004\)](#), and [Barro \(2006\)](#). [Barro \(2006\)](#) reinvigorated this literature by providing international evidence that disasters have been frequent and severe enough to generate a large equity premium.<sup>1</sup> The majority of these studies adopt a representative-agent framework. The two papers closest to ours are [Bates \(2008\)](#) and [Dieckmann \(2010\)](#). [Bates \(2008\)](#) studies investors with heterogeneous attitudes towards crash risk, which is isomorphic to heterogeneous beliefs of disaster risk. He focuses on small but frequent crashes and does not model intermediate consumption. [Dieckmann](#) considers only log utility. In these settings, risk sharing has limited effects on the equity premium. In addition, our model also captures more general disagreements about disasters, time-varying disaster intensities, and time-varying disagreement.

The paper also makes several contributions to the literature of heterogeneous beliefs and preferences.<sup>2</sup> We introduce a new framework to capture general forms of heterogeneity in beliefs about jumps (including the intensity and distribution of jump size), which is made tractable through the generalized transform method of [Chen and Joslin \(2010\)](#). We show that the effects of heterogeneous beliefs about disaster risk on asset prices are fundamentally different from those of disagreements about Gaussian risks. We also show that agents who are overly optimistic about disasters are likely to survive and even gain wealth for long periods of time. This is quite different from the case of disagreement about mean growth rates, where agents with wrong beliefs are likely to lose the majority of their wealth quickly.

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<sup>1</sup>A series of recent studies include [Liu, Pan, and Wang \(2005\)](#), [Weitzman \(2007\)](#), [Barro \(2009\)](#), [Gabaix \(2009\)](#), [Wachter \(2009\)](#), [Martin \(2008\)](#), [Farhi and Gabaix \(2009\)](#), [Backus, Chernov, and Martin \(2010\)](#), and others.

<sup>2</sup>See [Basak \(2005\)](#) for a survey on heterogeneous beliefs and asset pricing. Recent developments include [Kogan, Ross, Wang, and Westerfield \(2006\)](#), [Buraschi and Jiltsov \(2006\)](#), [Yan \(2008\)](#), [David \(2008\)](#), [Dumas, Kurshev, and Uppal \(2009\)](#), [Xiong and Yan \(2009\)](#), among others. Among the works on heterogeneous preferences are [Dumas \(1989\)](#), [Wang \(1996\)](#), [Chan and Kogan \(2002\)](#), and more recently [Longstaff and Wang \(2008\)](#).

## 2 Model Setup

We consider a continuous-time, pure exchange economy. There are two agents (A, B), each being the representative of her own class. Agent A believes that the aggregate endowment is  $C_t = e^{c_t^c + c_t^d}$ , where  $c_t^c$  is the diffusion component of log aggregate endowment, which follows

$$dc_t^c = \bar{g}dt + \sigma_c dW_t^c, \quad (1)$$

where  $\bar{g}$  and  $\sigma_c$  are the expected growth rate and volatility of consumption without jumps, and  $W_t^c$  is a standard Brownian motion under agent A's beliefs. The term  $c_t^d$  is a pure jump process whose jumps arrive with stochastic intensity  $\lambda_t$ ,

$$d\lambda_t = \kappa(\bar{\lambda}^A - \lambda_t)dt + \sigma_\lambda \sqrt{\lambda_t} dW_t^\lambda, \quad (2)$$

where  $\bar{\lambda}^A$  is the long-run average jump intensity under A's beliefs, and  $W_t^\lambda$  is a standard Brownian motion independent of  $W_t^c$ . The jumps  $\Delta c_t^d$  have time-invariant distribution  $\nu^A$ . We summarize agent A's beliefs with the probability measure  $\mathbb{P}_A$ .

Agent B believes that the probability measure is  $\mathbb{P}_B$ , which we shall suppose is equivalent to  $\mathbb{P}_A$ .<sup>3</sup> She may disagree about the growth rate of consumption without jumps, the likelihood of disasters or the distribution of the severity of disasters when they occur. We assume that the two agents are aware of each others' beliefs, but nonetheless “agree to disagree”.<sup>4</sup>

Specifically, as in [Chen, Joslin, and Tran \(2010\)](#), agent B's beliefs are characterized by

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<sup>3</sup>More precisely,  $\mathbb{P}^A$  and  $\mathbb{P}^B$  are equivalent when restricted to any  $\sigma$ -field  $\mathcal{F}_T = \sigma(\{c_t^c, c_t^d, \lambda_t\}_{0 \leq t \leq T})$ .

<sup>4</sup>We do not explicitly model learning about disasters. Given the nature of disasters, Bayesian updating of beliefs about disaster risk using realized consumption growth will likely be very slow, and the disagreements in the priors will persist for a long time. See also [Section 6](#).

the Radon-Nikodym derivative  $\eta_t \equiv (d\mathbb{P}_B/d\mathbb{P}_A)_t$ , which satisfies

$$\eta_t = e^{a_t + bc_t^c - I_t}, \quad (3)$$

$$I_t = \int_0^t \left( b\bar{g} + \frac{1}{2}b^2\sigma_c^2 + \lambda_s \left( \frac{\bar{\lambda}^B}{\bar{\lambda}^A} - 1 \right) \right) ds, \quad (4)$$

for some constants  $b$  and  $\bar{\lambda}^B > 0$ , and  $a_t$  is a pure jump process whose jumps are coincident with the jumps in  $c_t^d$  and have size

$$\Delta a_t = \log \left( \frac{\bar{\lambda}^B}{\bar{\lambda}^A} \frac{d\nu^B}{d\nu^A} \right), \quad (5)$$

where  $\frac{d\nu^B}{d\nu^A}$  is a function of the disaster size and reflects the disagreement about the distribution of disaster size (conditional on a disaster). It will be large (small) for the type of disasters that agent B thinks are relatively more (less) likely than agent A.

Intuitively,  $\eta_t$  expresses the differences in beliefs between the agents by letting agent B assign a higher probability to those states where  $\eta_t$  is large. The terms  $a_t$  and  $bc_t^c$  reflect B's potential disagreements regarding the likelihood of disasters and the growth rate of consumption, respectively. It follows from (3–5) that, under agent B's beliefs, the expected growth rate of consumption without jumps is  $\bar{g} + b\sigma_c^2$ , a disaster occurs with intensity  $\lambda_t \times \frac{\bar{\lambda}^B}{\bar{\lambda}^A}$  (with long run average intensity  $\bar{\lambda}^B$ ), and the disaster size distribution is  $\nu^B$  (which is equivalent to  $\nu^A$ ). The jumps in  $\eta_t$  specified in (5) are given by the log likelihood ratio for disasters of different sizes under the two agents' beliefs. Within this setup, agent B not only can disagree with A on the average frequency of disasters, but also the likelihoods for disasters of different magnitude. Moreover, this setup also has the advantage of remaining within the affine family as  $(c_t^c, c_t^d, \log \eta_t, \lambda_t)$  follows a jointly affine process, which makes it possible to compute the equilibrium analytically.

We assume that the agents are infinitely lived and have constant relative-risk aversion

(CRRA) utility over life time consumption:

$$U^i(C^i) = E_0^{\mathbb{P}^i} \left[ \int_0^\infty e^{-\rho_i t} \frac{(C_t^i)^{1-\gamma_i}}{1-\gamma_i} dt \right], \quad i = A, B. \quad (6)$$

We also assume that markets are complete and agents are endowed with some fixed share of aggregate consumption ( $\theta_A, \theta_B = 1 - \theta_A$ ).

The equilibrium allocations can be characterized as the solution of the following planner's problem, specified under the probability measure  $\mathbb{P}_A$ ,

$$\max_{C_t^A, C_t^B} E_0^{\mathbb{P}_A} \left[ \int_0^\infty e^{-\rho_A t} \frac{(C_t^A)^{1-\gamma_A}}{1-\gamma_A} + \tilde{\zeta}_t e^{-\rho_B t} \frac{(C_t^B)^{1-\gamma_B}}{1-\gamma_B} dt \right], \quad (7)$$

subject to the resource constraint  $C_t^A + C_t^B = C_t$ . Here,  $\tilde{\zeta}_t \equiv \zeta \eta_t$  is the belief-adjusted Pareto weight for agent B. From the first order condition and the resource constraint we obtain the equilibrium consumption allocations:  $C_t^A = f^A(\hat{\zeta}_t) C_t$  and  $C_t^B = (1 - f^A(\hat{\zeta}_t)) C_t$ , where  $\hat{\zeta}_t = e^{(\rho_A - \rho_B)t} C_t^{\gamma_A - \gamma_B} \tilde{\zeta}_t$ , and  $f^A$  is in general an implicit function.

The stochastic discount factor under A's beliefs,  $M_t^A$ , is given by

$$M_t^A = e^{-\rho_A t} (C_t^A)^{-\gamma_A} = e^{-\rho_A t} f^A(\hat{\zeta}_t)^{-\gamma_A} C_t^{-\gamma_A}. \quad (8)$$

Finally, we can solve for  $\zeta$  through the life-time budget constraint for one of the agents (see [Cox and Huang \(1989\)](#)), which is linked to the initial allocation of endowment.

Since our emphasis is on heterogeneous beliefs about disasters, for the remainder of this section we focus on the case where there is no disagreement about the distribution of Brownian shocks, and the two agents have the same preferences. In this case,  $b = 0$ ,  $\gamma_A = \gamma_B = \gamma$ ,  $\rho_A = \rho_B = \rho$ . The equilibrium consumption share then simplifies to

$$f^A(\tilde{\zeta}_t) = \frac{1}{1 + \tilde{\zeta}_t^{\frac{1}{\gamma}}}. \quad (9)$$

When a disaster of size  $d$  occurs,  $\tilde{\zeta}_t$  is multiplied by the likelihood ratio  $\frac{\bar{\lambda}^B}{\lambda^A} \frac{d\nu^B}{d\nu^A}(d)$  (see (5)). Thus, if agent B is more pessimistic about a particular type of disaster, she will have a higher weight in the planner's problem when such a disaster occurs, so that her consumption share increases.

The equilibrium allocations can be implemented through competitive trading in a sequential-trade economy. Extending the analysis of [Bates \(2008\)](#), we can consider three types of traded securities: (i) a risk-free money market account, (ii) a claim to aggregate consumption, and (iii) a series (or continuum) of disaster insurance contracts with 1 year maturity, which pay \$1 on the maturity date if a disaster of size  $d$  occurs within a year.

The instantaneous risk-free rate can be derived from the stochastic discount factor,

$$r_t = -\frac{\mathcal{D}^A M_t^A}{M_t^A} = \rho + \gamma \bar{g} - \frac{1}{2} \gamma^2 \sigma_c^2 - \lambda_t \left( \frac{E_t^{\Delta, A} [(C_t^A)^{-\gamma}]}{(C_t^A)^{-\gamma}} - 1 \right), \quad (10)$$

where  $\mathcal{D}^A$  denotes the infinitesimal generator under Agent A's beliefs of  $X_t = (c_t^c, c_t^d, \lambda_t, \eta_t)$  and we use the short-hand notation  $E_t^{\Delta, A}$  defined for any function  $f$  of  $X_t$  as

$$E_t^{\Delta, A}[f(X_t)] \equiv \int f \left( c_t^c, c_t^d + d, \lambda_t, \eta_t \times \frac{\bar{\lambda}^B}{\lambda^A} \frac{d\nu^B}{d\nu^A}(d) \right) d\nu^A(d). \quad (11)$$

The price of the aggregate endowment claim is

$$P_t = \int_0^\infty E_t^{\mathbb{P}^A} \left[ \frac{M_{t+\tau}^A}{M_t^A} C_{t+\tau} \right] d\tau = C_t h(\lambda_t, \tilde{\zeta}_t), \quad (12)$$

where the price/consumption ratio only depends on the disaster intensity  $\lambda_t$  and the stochastic weight  $\tilde{\zeta}_t$ . In the case where  $\lambda_t$  is constant, the price of the consumption claim is obtained in closed form. Similarly, we can compute the wealth of the individual agents as well as the prices of disaster insurance contracts using the stochastic discount factor (see [Appendix A](#) for details).

In order for prices of the aggregate endowment claim to be finite in the heterogeneous-agent economy, it is necessary and sufficient that prices are finite under each agent's beliefs in a single-agent economy (see [Appendix C](#) for a proof). As we show in the appendix, finite prices require that the following two inequalities hold:

$$0 < \kappa^2 - 2\sigma_\lambda^2(\phi^i(1 - \gamma) - 1), \quad (13a)$$

$$0 > \kappa \bar{\lambda}^i \frac{\kappa - \sqrt{\kappa^2 + 2\sigma_\lambda^2(1 - \phi^i(1 - \gamma))}}{\sigma_\lambda^2} - \rho + (1 - \gamma)\bar{g} + \frac{1}{2}(1 - \gamma)^2\sigma_c^2, \quad (13b)$$

where  $\phi^i$  is the moment generating function for the distribution of jumps in endowment  $\nu^i$  under measure  $\mathbb{P}_i$ . The first inequality reflects the fact that the volatility of the disaster intensity cannot be too large relative to the rate of mean reversion. It prevents the convexity effect induced by the potentially large intensity from dominating the discounting. The second inequality reflects the need for enough discounting to counteract the growth.

Additionally, the stochastic discount factor characterizes the unique risk neutral probability measure  $\mathbb{Q}$  (see, for example, Duffie (2001)), which facilitates the computation and interpretation of excess returns. The risk-neutral disaster intensity  $\lambda_t^{\mathbb{Q}} \equiv E_t^{\Delta, i}[M_t^i]/M_t^i \lambda_t^i$  is determined by the expected jump size of the stochastic discount factor at the time of a disaster. When the riskfree rate and disaster intensity are close to zero, the risk-neutral disaster intensity  $\lambda_t^{\mathbb{Q}}$  has the nice interpretation of (approximately) the value of a one-year disaster insurance contract that pays \$1 at  $t + 1$  when a disaster occurs between  $t$  and  $t + 1$ . The risk-neutral distribution of the disaster size is given by  $\frac{d\nu^{\mathbb{Q}}}{d\nu^i}(d) = M_t^{i, \Delta}(d)/E_t^{\Delta, i}[M_t^i]$ , where  $M_t^{i, \Delta}(d)$  denotes the pricing kernel when the state is  $(c_t^c, c_t^d + d, \lambda_t, \eta_t \times \frac{\bar{\lambda}^B}{\bar{\lambda}^A} \frac{d\nu^B}{d\nu^A}(d))$ . These risk adjustments are quite intuitive. The more the stochastic discount factor for agent  $i$  jumps up during a disaster, the larger is  $\lambda_t^{\mathbb{Q}}$  relative to  $\lambda_t^i$ , i.e. disasters occur more frequently under the risk-neutral measure. Thus, the ratio  $\lambda_t^{\mathbb{Q}}/\lambda_t^i$  is often referred to as the jump risk premium. Moreover, the risk-adjusted distribution of jump size conditional on a disaster slants the probabilities towards the types of disasters that lead to a bigger jump in the stochastic

discount factor, which generally makes severe disasters more likely under  $\mathbb{Q}$ .

Finally, the risk premium for any security under agent  $i$ 's beliefs is the difference between the expected return under  $\mathbb{P}_i$  and under the risk-neutral measure  $\mathbb{Q}$ . In the case of the aggregate endowment claim, the conditional equity premium, under agent  $i$ 's beliefs, which we denote by  $E_t^{\mathbb{P}_i}[R^e]$ , is

$$E_t^{\mathbb{P}_i}[R^e] = \gamma\sigma_c^2 + \lambda_t^i E_t^{\mathbb{P}_i}[\Delta R] - \lambda_t^{\mathbb{Q}} E_t^{\mathbb{Q}}[\Delta R], \quad i = A, B \quad (14)$$

where  $E_t^m[\Delta R] \equiv E_t^{\Delta, m}[P_t]/P_t - 1$  is the expected return on the endowment claim in a disaster under measure  $m$ .<sup>5</sup> The difference between the last two terms in (14) is the premium for bearing disaster risk. This premium is large if the jump risk premium is large, and/or the expected loss in return in a disaster is large (especially under the risk-neutral measure).

It follows that the difference in equity premium under the two agents' beliefs is

$$E_t^{\mathbb{P}_A}[R^e] - E_t^{\mathbb{P}_B}[R^e] = \lambda_t^A E_t^{\mathbb{P}_A}[\Delta R] - \lambda_t^B E_t^{\mathbb{P}_B}[\Delta R]. \quad (15)$$

This difference will be small relative to the size of the equity premium when the disaster intensity and expected loss under the risk-neutral measure are large relative to their values under actual beliefs. In the remainder of the paper, we report the equity premium relative to agent A's beliefs,  $\mathbb{P}_A$ . One interpretation for picking  $\mathbb{P}_A$  as the reference measure is that A has the correct beliefs, and we are studying the impact of the incorrect beliefs of agent B on asset prices.

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<sup>5</sup>To be concrete, we define the risk premium under measure  $i$  for any price process  $P(X_t, t)$  which pays dividends  $D(X_t, t)$  to be  $\mathcal{D}^i P_t/P_t - (r_t + D_t/P_t)$ .

### 3 Heterogeneous Beliefs and Risk Sharing

We start with a special case of the model where agents only disagree about the frequency of disasters. First, we analyze the impact of heterogeneous beliefs on asset prices and their implications for survival when the risk of disasters is constant, i.e.,  $\lambda_t = \bar{\lambda}^A$  (denoted as  $\lambda^A$  for simplicity). We then extend the analysis to the cases of time-varying disaster risk and time-varying disagreement.

#### 3.1 Disagreement about the Frequency of Disasters

In the simplest version of our model, the disaster size is deterministic,  $\Delta c_t^d = \bar{d}$ , and the two agents only disagree about the frequency of disasters ( $\lambda$ ). We set  $\bar{d} = -0.51$  so that the moment generating function (MGF)  $\phi^A(-\gamma)$  in this model matches the calibration of Barro (2006) for  $\gamma = 4$ . It implies that aggregate consumption falls by 40% when a disaster occurs. Agent A (pessimist) believes that disasters occur with intensity  $\lambda^A = 1.7\%$  (once every 60 years), which is also taken from Barro (2006). Agent B (optimist) believes that disasters are much less likely,  $\lambda^B = 0.1\%$  (once every 1000 years), but she agrees with A on the size of disasters as well as the Brownian risk in consumption. She also has the same preferences as agent A. The remaining parameters are  $\bar{g} = 2.5\%$ ,  $\sigma_c = 2\%$ , and  $\rho = 3\%$ .

Figure 1 shows the conditional equity premium and the jump risk premium under the pessimist's beliefs. If all the wealth is owned by the pessimist, the equity premium is 4.7%, and the riskfree rate is 1.3%. Since the optimist assigns very low probabilities to disasters, if she has all the wealth, the equity premium is only  $-0.21\%$  under the pessimist's beliefs,<sup>6</sup> which reflects the low compensation the optimist requires for bearing disaster risk. Thus, it is not surprising to see the premium falling when the optimist owns more wealth. However, the speed at which the premium declines in Panel A is impressive. When the optimistic agent

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<sup>6</sup>In this case, the equity premium under the optimist's own beliefs is still positive (0.43%). The difference in the premium under the two agents' beliefs is explained in Equation (15).

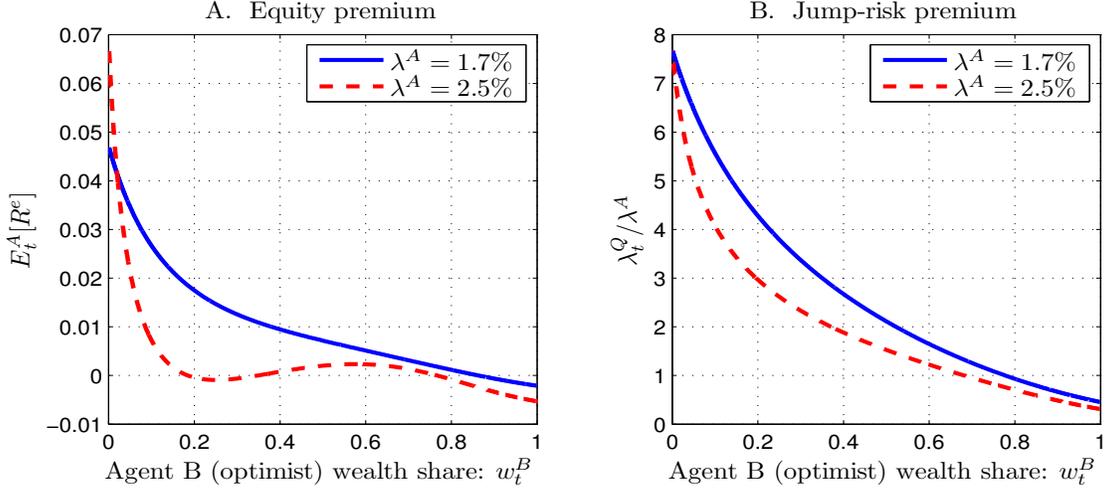


Figure 1: **Disagreement about the frequency of disasters.** Panel A plots the equity premium under the pessimist’s beliefs as a function of the wealth share of the optimist. Panel B plots the jump risk premium  $\lambda_t^Q/\lambda^A$  for the pessimist.

owns 10% of the total wealth, the equity premium has fallen from 4.7% to 2.7%. When the wealth of the optimist reaches 20%, the equity premium falls to just 1.7%.

We can derive the conditional equity premium as a special case of (14) using the assumption of constant disaster size,

$$E_t^{\mathbb{P}^A}[R^e] = \gamma\sigma_c^2 - \lambda^A \left( \frac{\lambda_t^Q}{\lambda^A} - 1 \right) \left( \frac{h(\tilde{\zeta}_t \frac{\lambda^B}{\lambda^A}) e^{\bar{d}}}{h(\tilde{\zeta}_t)} - 1 \right), \quad (16)$$

where  $h$  is the price-consumption ratio from (12), with  $\lambda_t$  being constant. The first term  $\gamma\sigma_c^2$  is the standard compensation for bearing Brownian risk. Heterogeneity has no effect on this term since the two agents agree about the Brownian risk. Given the value of risk aversion and consumption volatility, this term has negligible effect on the premium. The second term reflects the compensation for disaster risk. It can be further decomposed into three factors: (i) the disaster intensity  $\lambda^A$ , (ii) the jump risk premium  $\lambda_t^Q/\lambda^A$ , and (iii) the return of the consumption claim in a disaster.

How does the wealth distribution affect the jump risk premium? From the definition of the stochastic discount factor  $M_t^A$  and the risk-neutral intensity  $\lambda_t^{\mathbb{Q}}$ , it is easy to show

$$\lambda_t^{\mathbb{Q}}/\lambda^A = e^{-\gamma\Delta c_t^A}, \quad (17)$$

where  $\Delta c_t^A$  is the jump size of the equilibrium log consumption for agent A in a disaster. Without trading, the individual loss of consumption in a disaster will be equal to that of the endowment,  $\Delta c_t^A = \bar{d}$ , which generates a jump risk premium of  $\lambda_t^{\mathbb{Q}}/\lambda^A = 7.7$ . Since  $\lambda_t^{\mathbb{Q}}$  is approximately the premium of a one-year disaster insurance, before any trading the pessimist will be willing to pay an annual premium of about 13 cents for \$1 of protection against a disaster event that occurs with probability 1.7%.

The optimist views disasters as very unlikely events and is willing to trade away her claims in the future disaster states in exchange for higher consumption in normal times. Such trades help reduce the pessimist's consumption loss in a disaster  $\Delta c_t^A$ , which in turn lowers the jump risk premium. However, the optimist's capacity for underwriting disaster insurance is limited by her wealth, as she needs to ensure that her wealth is positive in all future states, including when a disaster occurs (no matter how unlikely such an event is). Thus, the more wealth the optimist has, the more disaster insurance she is able to sell.

The above mechanism can substantially reduce the disaster risk exposure of the pessimist in equilibrium. Panel B of Figure 1 shows that when the optimist owns 20% of total wealth, the jump risk premium drops from 7.7 to 4.2. According to equation (16), such a drop in the jump risk premium alone will cause the equity premium to fall by about half to 2.2%, which accounts for the majority of the change in the premium (from 4.7% to 1.7%).

Besides the jump risk premium, the equity premium also depends on the return of the consumption claim in a disaster, which in turn is determined by the consumption loss and changes in the price-consumption ratio. Following a disaster, the riskfree rate drops as the wealth share of the pessimist rises. With CRRA utility, the lower interest rate effect can

dominate that of the rise in the risk premium, leading to a higher price-consumption ratio.<sup>7</sup> Since a higher price-consumption ratio partially offsets the drop in aggregate consumption, it makes the return less sensitive to disasters, which will contribute to the drop in equity premium. However, our decomposition above shows that the reduction of the jump risk premium (due to reduced disaster risk exposure) is the main reason behind the fall in premium.

Can we “counteract” the effect of the optimistic agent and restore the high equity premium by making the pessimist even more pessimistic about disasters? The dashed lines in [Figure 1](#) plot the results when agent A believes that the disaster intensity is 2.5% ( $\lambda^A = 2.5\%$  and everything else equal). The results are striking. While the equity premium becomes significantly higher (6.8%) when the pessimist owns all the wealth, it falls to 4.1% with just 2% of total wealth allocated to the optimist (already lower than the previous case with  $\lambda^A = 1.7\%$ ), and is below 1% when the optimist’s wealth share exceeds 8.5%. As the wealth share of the optimist grows higher, the premium can even become negative, because the pessimist can acquire so much disaster insurance that her consumption actually rises in a disaster. The decline in the jump risk premium is still the main reason behind the lower equity premium. For example, when the optimist has 10% of total wealth, the jump risk premium falls to 4.0, which will drive the premium down to 3.1% (60% of the total fall). Thus, as the pessimist becomes more pessimistic, she seeks risk sharing more aggressively, which can quickly reverse the effect of her heightened fear of disasters on the equity premium.

To better illustrate the risk sharing mechanism, we compute the agents’ portfolio positions in the aggregate consumption claim, disaster insurance, and the money market account. Calculating these portfolio positions amounts to finding a replicating portfolio that matches the exposure to Brownian shocks and jumps in the individual agents’ wealth processes. [Appendix B](#) provides the details. The first thing to notice is that each agent will hold a constant proportion of the consumption claim. This is because they agree on the Brownian

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<sup>7</sup>Wachter (2009) also finds a positive relation between the price-consumption ratio and the equity premium in a representative agent rare disaster model with time-varying disaster probabilities and CRRA utility.

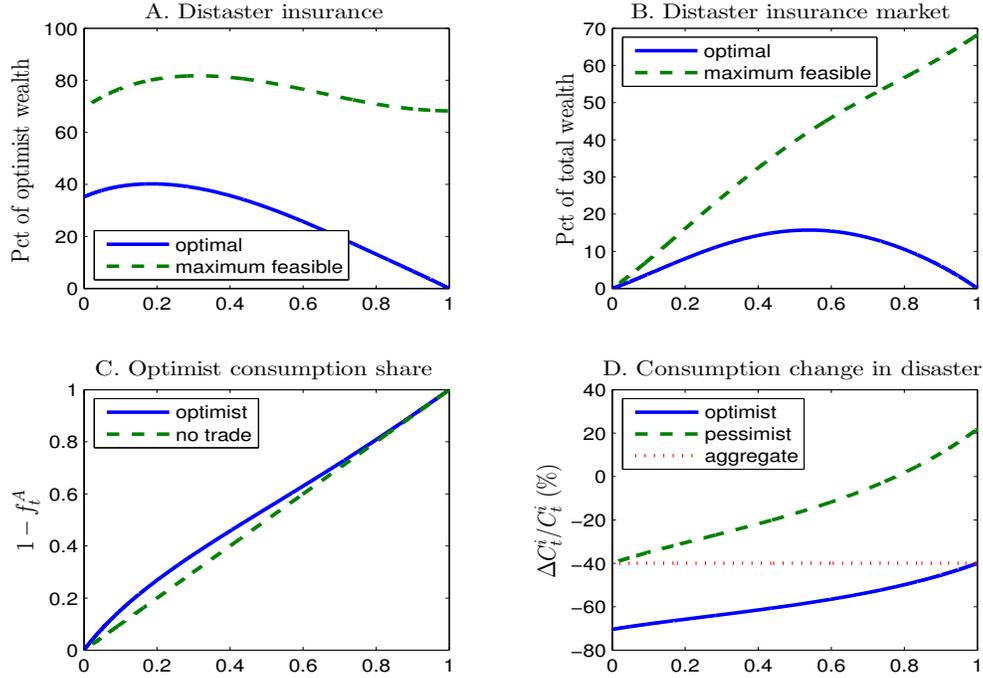


Figure 2: **Risk sharing.** Panel A and B plot the total notional value of disaster insurance relative to the wealth of the optimist and total wealth in the economy. Panel C plots the consumption share for the optimist in equilibrium. Panel D compares the two agents' consumption drops in a disaster with that of the aggregate endowment. These results are for the case  $\lambda^A = 1.7\%$ .

risk and share it proportionally. Disagreement over disaster risk is resolved through trading in the disaster insurance market, which is financed by the money market account.<sup>8</sup>

We first plot the notional value of the disaster insurance sold by the optimist as a fraction of her total wealth in Panel A of Figure 2. The dashed line is the maximum amount of disaster insurance the optimist can sell (as a fraction of her wealth) subject to her budget constraint. When the optimist has very little wealth, the notional value of the disaster insurance she sells is about 35% of her wealth. This value is initially high and then falls as the optimist gains more wealth. This is because when the optimist has little wealth, the pessimist has

<sup>8</sup>The implementation of the equilibrium is not unique. For example, instead of disaster insurance, we can use another contract that has exposure to both Brownian and jump risks, in which case the agents will also trade the consumption claim.

great demand for risk sharing and is willing to pay a higher premium, which induces the optimist to sell more insurance relative to her wealth. As the optimist gets more wealth, the premium on the disaster insurance falls, and so does the relative amount of insurance sold.

We can judge how extreme the risk sharing in equilibrium is by comparing the actual amount of trading to the maximum amount imposed by the budget constraint. At its peak, the amount of disaster insurance sold by the optimist is about half of the maximum amount that she can underwrite, which might appear reasonable. The caveat is that, in reality, underwriters of disaster insurance will likely be required to collateralize their promises to pay in the disaster states, which raises the costs of risk sharing. We will further investigate the feasibility of risk sharing and discuss an alternative implementation that do not require disaster insurance in [Section 6](#).

Panel B plots the size of the disaster insurance market (the total notional value normalized by total wealth). Naturally, the size of this market is zero when either agent has all the wealth, and the market is bigger when wealth is more evenly distributed. Notice that the model generates a non-monotonic relation between the size of the disaster insurance market and the equity premium. The premium is high when there is a lot of demand for disaster insurance but little supply, and is low when the opposite is true. In either case, the size of the disaster insurance market will be small.

Panel C plots the equilibrium consumption share for the optimist. The 45-degree line corresponds to the case of no trading. The optimist's consumption share is above the 45-degree line, more so when her wealth share is low. This is because the optimist is giving up consumption in future disaster states in exchange for higher consumption now.<sup>9</sup> Panel D shows that indeed the optimist does bear much greater losses in the event of a disaster. As for the pessimist, the less wealth she possesses, the more disaster insurance she is able to buy relative to her wealth, which lowers her disaster risk exposure and can eventually

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<sup>9</sup>This result is also due to the low elasticity of intertemporal substitution implied by the CRRA utility, which makes the optimists consume now instead of saving the insurance premium for the future.

turn the disaster insurance into a speculative position — her consumption can jump up in a disaster.

### 3.2 The Limiting Case for Risk Sharing

In the previous section we have numerically demonstrated the effects of risk sharing on asset prices. To highlight the key ingredients of the risk sharing mechanism, we now analytically characterize the equilibrium when a small fraction of wealth is controlled by an optimist who believes disasters are extremely unlikely.<sup>10</sup>

The intuition is as follows. Suppose the pessimist (agent A) consumes fraction  $f_{t-}^A$  of the aggregate endowment  $C_{t-}$  before a disaster at time  $t$ . Since the optimist (agent B) feels disasters are quite unlikely, she is willing to sell her entire share of endowment in the disaster state to the pessimist. Thus, when the disaster strikes, aggregate endowment drops to  $C_t = e^{\bar{d}}C_{t-}$ , but agent A now consumes essentially all the endowment ( $f_t^A \approx 1$ ). This argument implies that the jump in the marginal utility of agent A following a disaster, which is also the jump risk premium she demands, is equal to

$$\frac{\lambda_{t-}^Q}{\lambda^A} \approx \frac{\left(1 \times e^{\bar{d}}C_{t-}\right)^{-\gamma}}{\left(f_{t-}^A C_{t-}\right)^{-\gamma}} = \left(f_{t-}^A\right)^\gamma e^{-\gamma\bar{d}}. \quad (18)$$

For example, when the optimist has just 1% of the endowment before a disaster, the jump risk premium will be  $(.99)^\gamma e^{-\gamma\bar{d}}$ , or approximately a 4% drop from the jump risk premium in the case with only pessimists when  $\gamma = 4$ .

Formally, we show in [Appendix D](#) that the speed at which the jump risk premium changes with the optimist's consumption share is given by

$$\lim_{\lambda^B \rightarrow 0^+} \frac{\partial}{\partial f_t^B} \frac{\lambda_t^Q}{\lambda^A} \Big|_{f_t^B=0} = -\gamma e^{-\gamma\bar{d}}. \quad (19)$$

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<sup>10</sup>We thank Xavier Gabaix for suggesting this analysis.

We see that the effect of risk sharing (in terms of consumption share) becomes stronger with bigger disasters ( $|\bar{d}|$ ) and higher risk aversion ( $\gamma$ ).<sup>11</sup>

The above result only partially reflects the steep slope in the risk premium near  $w_t^B = 0$  we see in [Figure 1](#). If the optimist consumes a fraction  $f_t^B$  of the endowment at time  $t$ , his fraction of the aggregate wealth,  $w_t^B$ , will be *less* than  $f_t^B$ . This is because the optimist has sold his share of endowment in the disaster state in exchange and consumes more in normal times (see [Figure 2](#), Panel C). This effect implies that risk premium will decline even faster as a function of the wealth share of the optimist than the consumption share.

In [Appendix D](#), we show formally that

$$\lim_{\lambda^B \rightarrow 0^+} \left. \frac{\partial f_t^B}{\partial w_t^B} \right|_{f_t^B=0} = \frac{\rho + (\gamma - 1)\bar{g} - \frac{1}{2}\sigma_c^2(1 - \gamma)^2 + \frac{\gamma-1}{\gamma}\lambda^A}{\rho + (\gamma - 1)\bar{g} - \frac{1}{2}\sigma_c^2(1 - \gamma)^2 - \lambda^A(e^{(1-\gamma)\bar{d}} - 1)}. \quad (20)$$

This term reflects the ratio of the value of the entire consumption claim (the inverse of the denominator) to the value of the claim to consumption before any disasters occur (the inverse of the numerator). In [Appendix E](#) we show under very general conditions that a large equity premium due to disasters implies that this ratio will be large since the claim to consumption after disasters occur is very valuable.

Finally, we can summarize the effect of risk sharing on the equity premium via the jump risk premium from the decomposition in (14). Holding fixed the return on equity in a disaster,<sup>12</sup> the limiting differential effect of optimist on the equity premium is given by the following multiplier:

$$\lim_{\lambda^B \rightarrow 0^+} -\lambda^A(e^{\bar{d}} - 1) \left. \frac{\partial}{\partial w_t^B} \frac{\lambda_t^Q}{\lambda^A} \right|_{f_t^B=0} = - \left. \frac{\partial}{\partial f_t^B} \frac{\lambda_t^Q}{\lambda^A} \right|_{f_t^B=0} \times \left. \frac{\partial f_t^B}{\partial w_t^B} \right|_{f_t^B=0} \times (e^{\bar{d}} - 1) \times \lambda^A. \quad (21)$$

<sup>11</sup>We take limits since with  $\lambda^B = 0$ , the beliefs are not equivalent and there is no complete markets equilibrium.

<sup>12</sup>This assumption helps exclude the effect of changing price-consumption ratio on the premium, which we discussed in [Section 3.1](#).

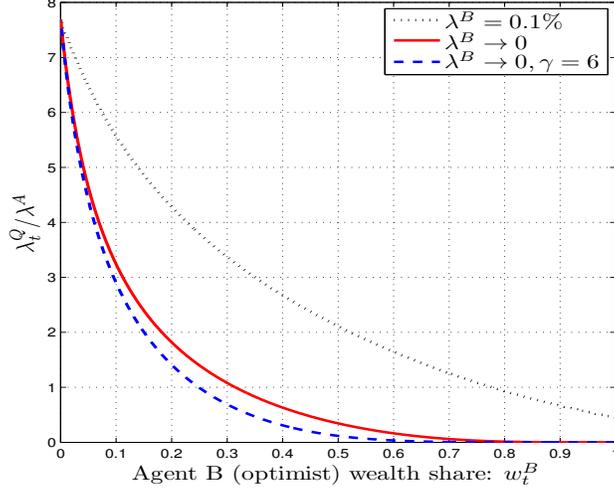


Figure 3: **Limiting Jump Risk Premia.** This figure plots the jump risk premium  $\lambda_t^Q/\lambda^A$  for the pessimist, where  $\lambda^A = 1.7\%$ . In the benchmark case,  $\gamma = 4$ , and  $\lambda^B = 0.1\%$ .

In the calibrated example, the multiplier (with  $\lambda^B = 0$ ) equals  $-0.581$ . Hence, due to the decline in the jump risk premium alone, allocating only 1% of the endowment to the extreme optimist results in a 58.1 basis points decline in the equity premium. In comparison, the benchmark case with  $\lambda^B = 0.1\%$  generates a multiplier of  $-0.19$ . When  $\lambda^A = 2.5\%$  and  $\lambda^B = 0$ , the multiplier is 294.3, which translates into a 2.94% drop in the equity premium when we introduce only 1% of extreme optimist into the economy!

Figure 3 compares the jump risk premium for several cases. First, the dotted line denotes the benchmark case from Section 3.1. We also plot the jump risk premium with the same parameters but for the limiting case where  $\lambda^B$  approaches zero. Additionally, we plot the case where we decrease the disaster size and increase the risk aversion to maintain the same jump risk premium for the single agent economy ( $\gamma = 6, \bar{d} = -0.34$ ). The graph shows that the marginal effect of a small amount of optimist with  $\lambda^B = 0.1\%$  on the jump risk premium is visibly smaller than in the limiting case of extreme optimism. Moreover, when we decrease the disaster size but increase risk aversion, the effects become more severe. This is because the larger risk sharing effect on the jump risk premium in (19) dominates the

smaller consumption-wealth effect in (20).

### 3.3 Survival

In models with heterogeneous agents, one type of agents often dominates in the long-run (a notable exception is Chan and Kogan (2002); see also Borovička (2010)). Our model also has the property that the agent with correct beliefs will dominate in the long run. For example, let's assume that agent A has the correct beliefs. The strong law of large numbers implies that  $\log \tilde{\zeta}_t \rightarrow -\infty$  almost surely. This implies that agent A will take over the economy with probability one. We now show that although agents with incorrect beliefs about disasters may not have permanent effects on asset prices, their effects may be long-lived in the sense that these agents can retain, and even build, wealth over long horizons.

With disaster intensity,  $\lambda_t$ , being constant, we need only consider the distribution of the stochastic Pareto weight,  $\tilde{\zeta}_t$ , to analyze the wealth distribution over time. From (3), we see that  $\tilde{\zeta}_t$  has a stochastic component, whereby the Pareto weight (and thus wealth) of the pessimistic agent will jump up when a disaster occurs. This is because the pessimist receives insurance payments from the optimist in a disaster. However, regardless of the occurrence of disasters, there is also a deterministic component in  $\tilde{\zeta}_t$ , whereby the optimist has a deterministic weight increase (and thus her relative wealth increases) which comes from collecting the disaster insurance premium. Thus, even when the pessimist has *correct* beliefs, her relative wealth will *decrease* outside of disasters. Since disasters are rare, it will be common to have extended periods without disasters, during which time an optimistic agent will gain relative wealth.

Table 1 presents a summary of the conditional distribution of wealth after 50 years for various initial wealth distributions. We report the results under the assumption that either the pessimist or the optimist has correct beliefs. If the number of disasters is either 0 or 1, the wealth of the agents remain relatively close to the original distribution. We see that the

Table 1: **Survival of Agents who Disagree about the Frequency of Disasters.** This table shows the redistribution of wealth over a 50 year horizon in the model of [Section 3.1](#). Future relative wealth only depends on the initial wealth, the time horizon, and the number of disasters that occur. The top panel provides the possible wealth redistributions throughout time. The bottom panel provides the probabilities of various number of disasters (under each agent’s beliefs).

Initial Wealth of B	Final Wealth of B after $N_d$ Disasters			
	$N_d = 0$	$N_d = 1$	$N_d = 2$	$N_d = 3$
1.0%	1.2%	0.6%	0.3%	0.1%
5.0%	6.1%	3.0%	1.5%	0.7%
10.0%	12.2%	6.0%	2.9%	1.4%
50.0%	55.7%	35.5%	19.3%	9.6%
99.0%	99.2%	98.3%	96.7%	93.5%
Probability under $\mathbb{P}^A$	42.7%	36.3%	15.4%	4.4%
Probability under $\mathbb{P}^B$	95.1%	4.8%	0.1%	0.0%

optimist is likely to retain wealth for long periods of time and will only be wiped out with the occurrence of several disasters, which is unlikely regardless of whose beliefs are correct.

The evolution of the wealth distribution over time also has important implications for the equity premium and other dynamic properties of asset prices. For example, when the initial wealth of agent  $B$  is 5% (10%), the equity premium will drop from 3.5%(2.7%) to 3.3% (2.4%) over 50 years if no disasters occurs. If after 120 years there are still no disasters, the equity premium would further drop to 2.9% (2.0%).

The survival results presented thus far stand in sharp contrast to survival in models of disagreement over Brownian risk in consumption growth. As discussed in [Section 2](#), it is possible to raise the equity premium under the true measure if there are agents who are pessimistic about the growth rate of consumption. For example, if the volatility of consumption is  $\sigma_c = 2.0\%$ , two types of agents have  $\gamma = 4$  and  $\rho = 3\%$ , one believing (correctly) consumption growth is 2.5%, the other believing it is 0% (no disasters in either

case), then the equity premium will be roughly 2.5% when the pessimist controls most of the wealth in the economy. However, even if the pessimist controls 99% of the wealth initially, her wealth share will be reduced to less than 1% after 50 years with a probability of 92.4%. Thus, even a very small amount of agents with correct beliefs will quickly dominate the economy in the Gaussian setting.

### 3.4 Time-varying Disaster Risk

In the previous sections we have analyzed in depth the impact of heterogeneous beliefs when disaster intensity is constant. Now we extend the analysis to allow the risk of disasters and the amount of disagreements about disasters to vary over time, which not only makes the model more realistic, but also has important implications for the dynamics of asset prices. As in [Gabaix \(2009\)](#) and [Wachter \(2009\)](#), time-varying disaster intensity serves to drive both asset prices and expected excess returns. We now demonstrate that within our framework, wealth distribution becomes an important factor that drives asset price dynamics through the risk sharing mechanism. In particular, it affects how sensitive the conditional risk premium will be to time variation in disaster risk.

Our calibration of the intensity process  $\lambda_t$  in equation (2) is as follows. First, the long-run mean intensity of disasters under the two agents' beliefs are  $\bar{\lambda}^A = 1.7\%$  and  $\bar{\lambda}^B = 0.1\%$ . Next, following [Wachter \(2009\)](#), we set the speed of mean reversion  $\kappa = 0.142$  (with a half life of 4.9 years). The volatility parameter is  $\sigma_\lambda = 0.05$ , so that the Feller condition is satisfied.<sup>13</sup> For simplicity, we assume that the size of disasters is constant,  $\bar{d} = -0.51$ , as in Section 3.1. The remaining preference parameters are also the same as in the constant disaster risk case.

[Figure 4](#) plots the conditional equity premium and the jump risk premium under agent A's beliefs as functions of agent B's wealth share  $w_t^B$  and the disaster intensity  $\lambda_t$ . First, in Panel A, holding  $\lambda_t$  fixed, the equity premium drops quickly as the wealth share of the

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<sup>13</sup>The Feller condition,  $2\kappa\bar{\lambda}^A > \sigma_\lambda^2$ , ensures that  $\lambda_t$  will remain strictly positive under agent A's beliefs.

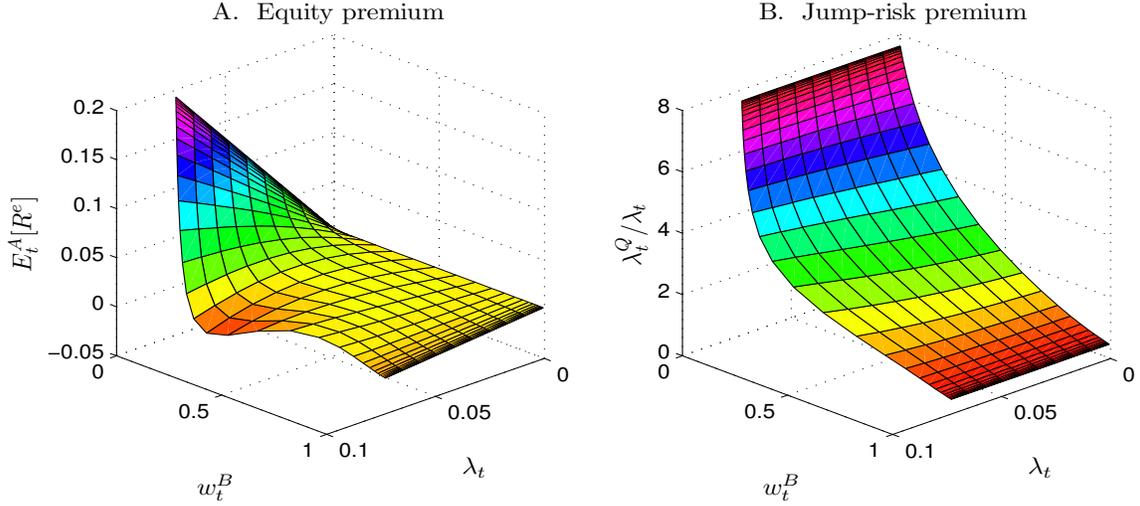


Figure 4: **Time-varying Disaster Risk.** Panel A plots the equity premium under agent A’s beliefs as a function of agent B’s wealth share ( $w_t^B$ ) and the disaster intensity under A’s beliefs ( $\lambda_t$ ). Panel B plots the jump risk premium  $\lambda_t^Q/\lambda_t$  for agent A.

optimistic agent rises from zero, which is consistent with the results from the case with constant disaster risk. Moreover, this decline is particularly fast when  $\lambda_t$  is large, suggesting that the agents engage in more risk sharing when disaster risk is high. Indeed, the jump risk premium in Panel B also declines faster when  $\lambda_t$  is large, which is the result of agent A reducing her consumption loss in a disaster more aggressively at such times.

Next, we see that the sensitivity of the equity premium to disaster intensity can be very different depending on the wealth distribution. The sensitivity is largest when the pessimist has all the wealth, but it becomes smaller as the wealth of the optimist increases. When the optimist’s wealth share becomes sufficiently high, the equity premium becomes essentially flat as  $\lambda_t$  varies. This result has important implications for the time series properties of the equity premium. It suggests that when  $\lambda_t$  fluctuates over time, the equity premium can either be volatile or smooth, depending on the wealth distribution.

We can understand the above results through the equity premium formula,

$$E_t^A[R^e] = \gamma\sigma_c^2 - E_t[\Delta R] \left( \frac{\lambda_t^Q}{\lambda_t} - 1 \right) \lambda_t, \quad (22)$$

where now the return conditional on a disaster occurring,  $E_t[\Delta R]$ , does not depend on the probability measure since there is a single disaster type. Variations in the wealth distribution drive  $\lambda_t^Q/\lambda_t$  and  $E_t[\Delta R]$ . Due to increased risk sharing, the jump risk premium declines with greater fraction of wealth controlled by the optimistic agent. As a result, the premium becomes less sensitive to variations in  $\lambda_t$ . Moreover, we see in Panel B of [Figure 4](#) that the effect of wealth on the jump risk premium depends on the disaster intensity – when the disaster intensity is high, the risk sharing motives are very strong, resulting in larger effect on the jump risk premium when the optimistic agent controls even a small amount of wealth. Finally, the returns in disasters also vary somewhat with the wealth distribution as the price-consumption ratio changes after a disaster.

These results also suggest that some care must be taken in extracting investors' perception of the likelihood of disaster from asset prices. As [Figure 4](#) indicates, a given risk neutral probability of disasters could be associated with a wide range of beliefs depending on the wealth distribution. For example, [Backus, Chernov, and Martin \(2010\)](#) show that option prices imply smaller probabilities of disasters than estimated from international macroeconomic data, which is consistent with our main result that adding a small group of agents with different beliefs can dramatically lower the risk neutral probabilities of disasters. [Collin-Dufresne, Goldstein, and Yang \(2010\)](#) extract risk neutral probabilities of extreme events from the prices of CDX tranches. They find that the risk neutral probabilities of large losses are less than 1% per year. Based on our model, their findings can be either due to a very low true probability (since the actual probability will typically be much less than the risk neutral probability) or a sufficient amount of disagreement among investors about the likelihood of disasters.

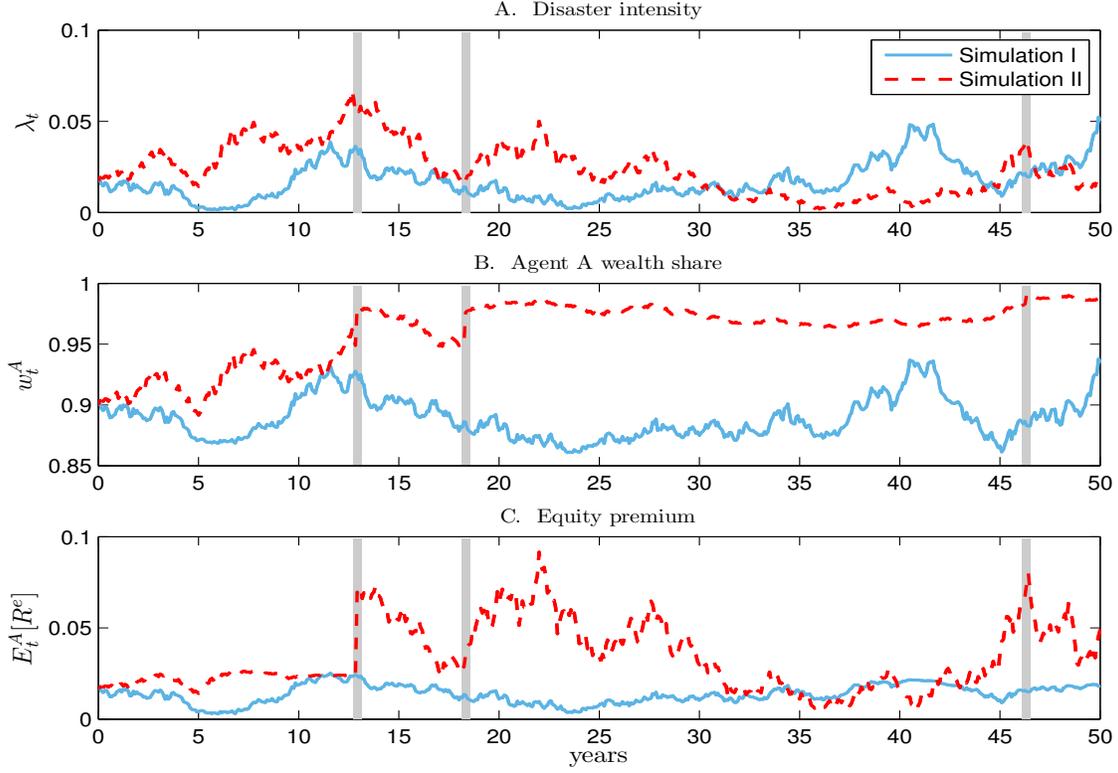


Figure 5: **Simulation with Time-varying Disaster Risk.** The results are from two simulations of the model with time-varying disaster risk under agent A’s beliefs. Panel A plots the simulated paths of disaster intensity. Panel B and C plot the corresponding wealth share of agent A and the conditional equity premium she demands. The shaded areas denote the timing of disasters in Simulation II. There are no disasters in Simulation I.

To further investigate the time series properties of the model, we simulate the disaster intensity  $\lambda_t$  and the jump component of aggregate endowment  $c_t^d$  under agent A’s beliefs, which jointly determine the evolution of the stochastic Pareto weight  $\tilde{\zeta}_t$ . Then, along the simulated paths, we compute the equilibrium wealth fraction of agent A,  $w_t^A$ , and the conditional equity premium under A’s beliefs,  $E_t^A[R^e]$ . In each simulation we start with  $\lambda_0 = 1.7\%$  and set the initial wealth share of agent A to  $w_0^A = 90\%$ . The results from two of the simulations are reported in [Figure 5](#).

Panel A plots the paths of  $\lambda_t$  from the simulations. The disaster intensities from both

simulations are fairly persistent, and show similar amount of variation over time. In Simulation I, there are no disasters. In Simulation II, disasters occur three times within the first 50 years, around year 13, 18, and 46, indicated by grey bars in the figure.

What determines the evolution of the wealth distribution? When there are no disasters, holding  $\lambda_t$  fixed, agent A is losing wealth share to B as she pays B the premium for disaster insurance. This effect is captured by the negative drift in the Radon-Nikodym derivative  $\eta_t$  (see equation (3)), and is stronger when  $\lambda_t^A$  is larger. In addition, as  $\lambda_t$  falls (rises), the value of the disaster insurance that agent A owns falls (rises), causing her wealth to fall (rise) relative to agent B, who is short the disaster insurance. As Panel B shows, the second effect appears to be the main force driving the wealth distribution in Simulation I.

When a disaster strikes, the wealth distribution can change dramatically. In Simulation II, the wealth share of agent A jumps up each time a disaster strikes. This is because the disaster insurance that A (pessimist) purchases from B (optimist) pays off at such times, causing the wealth of A to increase relative to B. The size of the jump in  $w_t^A$  is bigger in the first two disasters, which is mainly because agent B has relatively more wealth going into the first two disasters, so that he is able to provide more disaster insurance, but also loses more wealth in the two disasters.

Panel C shows the joint effect of the disaster intensity and wealth distribution on the equity premium. In Simulation I (no disasters), despite the fact that the optimistic agent never owns more than 15% of total wealth and that disaster intensity  $\lambda_t$  shows considerable variation over the period, the equity premium is below 2% nearly 90% of the time. This result confirms our finding in Figure 4 that risk sharing between the agents keeps the premium low and smooth when the wealth share of agent B is not too small. It also illustrates that if we were to extract disaster probabilities from asset prices without taking into account the effect of heterogeneous beliefs, we will dramatically understate both the level and variation in the disaster probabilities.

In contrast, the equity premium in Simulation II shows large variation, ranging from

0.5% to 9.2%. Following the first disaster in year 13, the premium jumps from 2.4% to 7.0%, and becomes significantly more sensitive to fluctuations in  $\lambda_t$  and the wealth distribution afterwards. Since the wealth share of agent B drops in a disaster, her risk sharing capacity is reduced, which drives up both the level and volatility of the equity premium. As shown in [Figure 4](#), this effect is stronger when  $\lambda_t$  is high, which is why the jump in premium is the most visible after the first disaster. In this case, without considering heterogeneous beliefs, our estimates of disaster probabilities from asset prices can have too much variation relative to the actual values.

### 3.5 Time-varying Disagreement

One implication of our analysis is that risk sharing induces non-linearity in the relationship between the wealth-weighted average belief about the likelihood of disasters and the jump risk premium. This suggests that the average belief about the likelihood of disasters is not a sufficient statistic to determine the equity premium, but rather the amount of disagreement among agents is also important. In this section, we extend the model from [Section 3.1](#) to capture the effect of time variation in disagreement.

We assume the economy can be in one of two states,  $s_t = L, H$ . In state  $L$ , the two agents' perceived disaster intensity are  $\lambda_L^A$  and  $\lambda_L^B$ , while in state  $H$ , they become  $\lambda_H^A$  and  $\lambda_H^B$ . The transitions between the two states are governed by a continuous-time Markov chain, with the generator matrix

$$\mathbf{\Lambda} = \begin{bmatrix} -\delta_L & \delta_L \\ \delta_H & -\delta_H \end{bmatrix}.$$

For example, the probability of the economy moving from state  $L$  to state  $H$  over a short period  $\Delta t$  will be approximately  $\delta_L \Delta t$ . We assume that the agents agree on the transition probabilities of the Markov chain. Moreover, they agree on the size of disasters (which is constant) as well as the Brownian risk, and have the same preference parameters as in

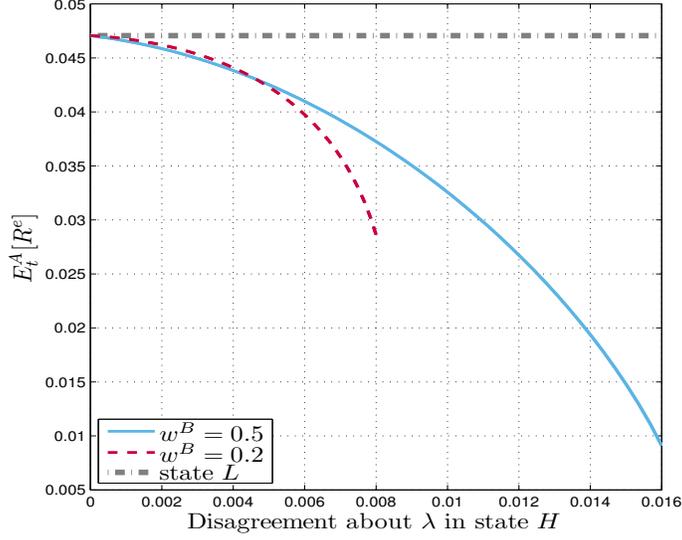


Figure 6: **Time-varying Disagreement.** Panel A plots the equity premium in the case where beliefs converge in the state with higher disaster risk. Panel B plots the premium as a function of the amount of disagreement for given wealth distribution.

Section 3.1. The model can again be solved analytically using results on the occupation time of continuous-time Markov chains (see e.g., Darroch and Morris (1968)). The full details are in Appendix F.

We focus our analysis on the following case. We assume that there is no disagreement between the two agents in state  $L$ , so that  $\lambda_L^A = \lambda_L^B = 1.7\%$ . There is disagreement in state  $H$ . In order to isolate the effect of disagreement, we consider different combinations of beliefs in state  $H$  ( $\lambda_H^A > \lambda_H^B$ ) such that the wealth-weighted average belief for a given wealth distribution is the same as in state  $L$ , i.e.,  $(1 - w^B)\lambda_H^A + w^B\lambda_H^B = 1.7\%$ , where  $w^B$  is the wealth share of agent B. We measure the amount of disagreement using the wealth-weighted standard deviation in beliefs,

$$\text{Disagreement Measure} = \sqrt{(1 - w^B)(\lambda_H^A - 1.7\%)^2 + w^B(\lambda_H^B - 1.7\%)^2}.$$

Finally, we set the transition probabilities of the Markov chain to be  $\delta_L = 0.1$  and  $\delta_H = 0.5$ .

As [Figure 6](#) shows, holding the average belief constant, the premium can fall substantially as the amount of disagreement increases. As a benchmark, the dash-dotted line gives the equity premium (under agent A's beliefs) in state  $L$ . Since the agents have the same beliefs in that state, the premium remains at 4.7% as the amount of disagreement increases in state  $H$ . The solid line plots the equity premium in state  $H$  when the two agents have equal share of total wealth. The premium falls from 4.7% to 0.9% when  $\lambda_H^B$  drops from 1.7% to 0.1% (where the disagreement measure is 1.6%). When agent B has just 20% of total wealth, the premium falls by a smaller amount to 2.9% (when the disagreement measure reaches 0.8%). An interesting implication of this graph is that the premium can actually be *decreasing* while the average belief of disaster risk *increases*, provided that there is enough increase in the amount of disagreement at the same time.

In summary, besides the variation in disaster risk and wealth distribution across agents with heterogeneous beliefs, time variation in the amount of disagreement across agents can be another importance source of fluctuations in the disaster risk premium. In particular, we see that the average belief is not sufficient to determine the equity premium. Fixing the average belief, a larger amount of disagreement can substantially reduce the equity premium due to the non-linearity in the risk sharing mechanism.

## 4 Calibrating Disagreement: Is the US Special?

Having considered a series of special examples of heterogeneous beliefs, we now extend the analysis to a less stylized model of beliefs on disasters. We calibrate the beliefs of the two types of agents as follows. Agent A believes that the US is no different from the rest of the world in its disaster risk exposure. Hence her beliefs are calibrated using cross-country consumption data. Agent B, on the other hand, believes that the US is special. She forms her beliefs on disaster risk using only the US consumption data.

An important contribution of Barro (2006) is to provide detailed accounts of the major

consumption declines cross 35 countries in the twentieth century. Rather than directly using the empirical distribution from Barro (2006), we estimate a truncated Gamma distribution for the log jump size from Barro’s data using maximum likelihood (MLE).<sup>14</sup> Our estimation is based on the assumption that all the disasters in the sample were independent, and that the consumption declines occurred instantly.<sup>15</sup> We also bound the jump size between  $-5\%$  and  $-75\%$ . In comparison, the smallest and largest declines in per capita GDP in Barro’s sample are  $15\%$  and  $64\%$ , respectively. Choosing a more negative lower bound than  $-75\%$  will have little effect on our results, because even though the larger disasters can imply a higher premium in a single-agent economy, they can also raise the risk sharing motives in our model. The disaster intensity under A’s beliefs is still  $\lambda^A = 1.7\%$ . The remaining parameters are: the mean growth rate and volatility of consumption without a disaster,  $\bar{g} = 2.5\%$  and  $\sigma_c = 2\%$ , which are consistent with the US consumption data post WWII.

As for agent B, we assume that she agrees with the values of  $\bar{g}$  and  $\sigma_c$ , but we estimate the truncated Gamma distribution of disaster size using MLE from annual per-capita consumption data in the US 1890-2008.<sup>16</sup> Over the sample of 119 years, there are three years where consumption falls by over  $5\%$ . Thus, we set  $\lambda^B = 3/119 = 2.5\%$ . Alternatively, we can also jointly estimate  $\lambda^B$  and the jump size distribution.

Panel A of [Figure 7](#) plots the probability density functions of the log jump size distributions for the two agents, which are very different from each other. The solid line is the distribution fitted to the international data on disasters. The average log drop is 0.36, which is equivalent to  $30\%$  drop in the level of consumption. In the US data, the average drop in log consumption is only 0.075, or  $7.3\%$  in level. In addition, agent A’s distribution has a much

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<sup>14</sup>The truncated Gamma distribution has probability density function (PDF)  $f(d; \alpha, \beta | d_{min}, d_{max}) = f(d; \alpha, \beta) / (F(d_{max}; \alpha, \beta) - F(d_{min}; \alpha, \beta))$ , where  $f(x; \alpha, \beta)$  and  $F(x; \alpha, \beta)$  are the PDF and CDF of the standard Gamma distribution with shape parameter  $\alpha$  and scale parameter  $\beta$ .

<sup>15</sup>These assumptions are debatable. For example, many of the major declines cross European countries are in WWI and WWII. Moreover, many of the declines spanned several years. See [Barro and Ursúa \(2008\)](#), [Donaldson and Mehra \(2008\)](#), and [Constantinides \(2008\)](#) for more discussions on the measurement of historical disasters.

<sup>16</sup>The data is taken from Robert Shiller’s web site <http://www.econ.yale.edu/~shiller/data.htm>

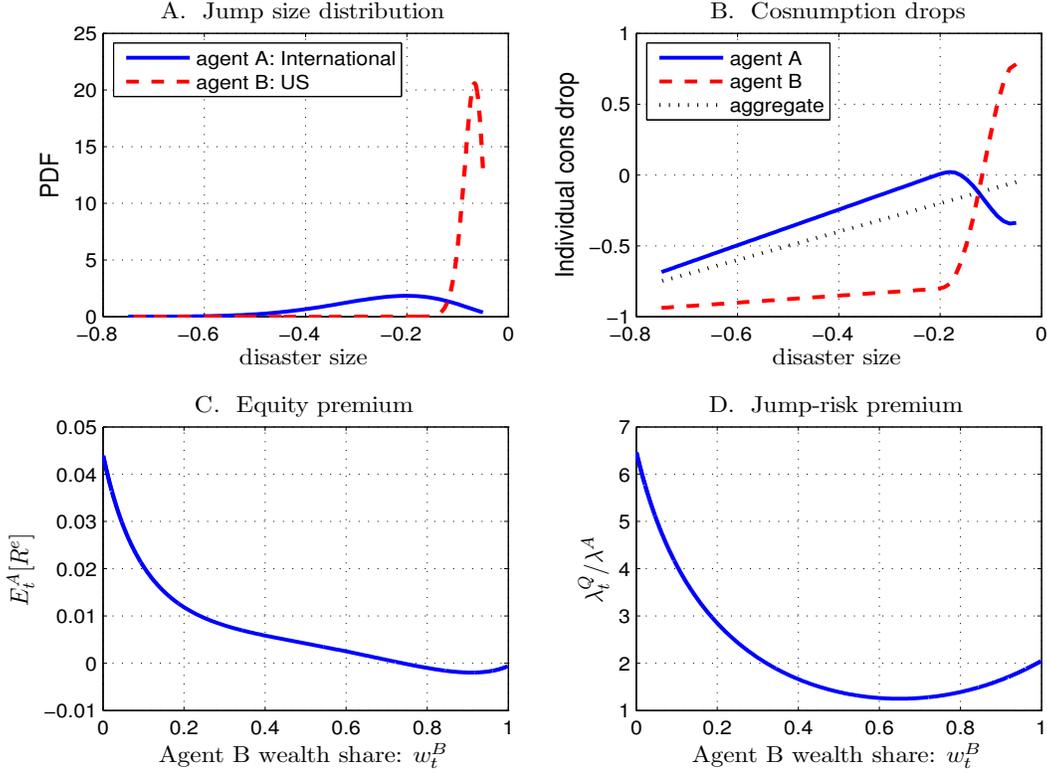


Figure 7: **Calibrated Disagreements: International vs US Experiences.** Panel A plots the truncated Gamma distribution of disaster size for the two agents. Panel B plots the equilibrium consumption drops for the two agents given the size of the disaster. Panel C and D plot the equity premium and jump risk premium under A’s beliefs.

fatter left tail than B. Thus, while A assigns significantly higher probabilities than B to large disasters (where consumption drops by 15% or more), agent B assigns more probabilities to small disasters, especially those ranging from 5 to 12%. In fact, agent B’s beliefs are close to the calibration adopted by Longstaff and Piazzesi (2004), who assume that the jump in aggregate consumption during a disaster is 10%.

The differences in beliefs lead the two agents to insure each other against the types of disasters they fear more, and the trading can be implemented using a continuum of disaster insurance contracts with coverage specific to the various disaster sizes. Panel B plots drops in the equilibrium consumption (level) for the two agents when disasters of different sizes occur,

assuming that agent B owns 10% of total wealth. The graph shows that through disaster insurances, agent A is able to reduce her consumption loss in large disasters (comparing the solid line to the dotted line). For example, her own consumption will only fall by 24% in a disaster where aggregate consumption falls by 40%, a sizable reduction especially considering the small amount of wealth that agent B has. At the same time, she also provides insurances to B on smaller disasters, which increases her consumption losses when such disasters strike. Agent B's consumption changes are close to a mirror image of agent A's. However, the changes are magnified both for large and small disasters due to her small wealth share.

Panel C shows the by-now familiar exponential drop in the equity premium as the wealth share of agent B increases. The equity premium is 4.4% when all the wealth is owned by the agents who form their beliefs about disasters based on international data, but drops to 2.0% when just 10% of total wealth is allocated to the agents who form their beliefs using only the US data. The main reason for the lower equity premium is again due to the decrease of the jump risk premium (Panel D), which falls from 6.5 to 4.0 when agent B's wealth share rises to 10%. This effect alone drives the equity premium down to 2.4%. Notice that the jump risk premium is no longer monotonic in the wealth share of agent B. This is because when agent A has little wealth, she would be betting against small disasters so aggressively that the big losses for her during small disasters can cause the jump risk premium to rise again.

## 5 Comparison with Other Forms of Heterogeneity

Many studies on heterogeneous beliefs focus on disagreement about Brownian risks as opposed to jump risks. In this section, we compare these two forms of disagreements to highlight their different impacts on asset prices, in particular, the prices of Brownian and jump risk. In addition, we also compare our results to a model of heterogeneous risk aversion.

## 5.1 Disagreement about Brownian risk

As a special case of the model presented in [Section 2](#), we can remove the jump component in endowment,  $c_t^d$ , and assume that agents A and B only disagree about the growth rate of endowment. We again assume that agent A has the correct beliefs, who thinks the growth rate of endowment is  $\bar{g} = 2.5\%$ , while agent B thinks the growth rate is  $\bar{g} + b\sigma_c^2$ . From the stochastic discount factor  $M_t^A$ , one can show that the price of Brownian risk (which is also the Sharpe ratio of the market portfolio) under A's beliefs is a linear function of her consumption share:

$$SR_t^A = \gamma\sigma_c - (1 - f_t^A) b\sigma_c. \quad (23)$$

Thus, if A has all the wealth in the economy, the price of Brownian risk will be  $\gamma\sigma_c$ , which is small for moderate risk aversion  $\gamma$  and low consumption volatility  $\sigma_c$ . As we allocate more wealth and hence higher consumption share to a pessimistic agent B, the price of equity will fall and the expected return under agent A's beliefs will rise, which leads to a higher Sharpe ratio under the correct beliefs.

In the case of disagreement about jump risk, the price of jump risk under agent A's beliefs can also be expressed explicitly as function of her consumption share,

$$\frac{\lambda_t^Q}{\lambda^A} = \frac{1}{\lambda^A} \left( f_t^A (\lambda^A)^{\frac{1}{\gamma}} + (1 - f_t^A) (\lambda^B)^{\frac{1}{\gamma}} \right)^\gamma e^{-\gamma\bar{d}}, \quad (24)$$

which converges to  $e^{-\gamma\bar{d}}$  when A's consumption share goes to 1. However, unlike the price of Brownian risk, the price of jump risk changes nonlinearly with the consumption share. This difference is clearly illustrated in [Figure 8](#), where the price of jump risk initially declines quickly when agent B consumes a small share of aggregate endowment, but the decline slows down later on.

The nonlinearity in the price of jump risk has important asset pricing consequences. We

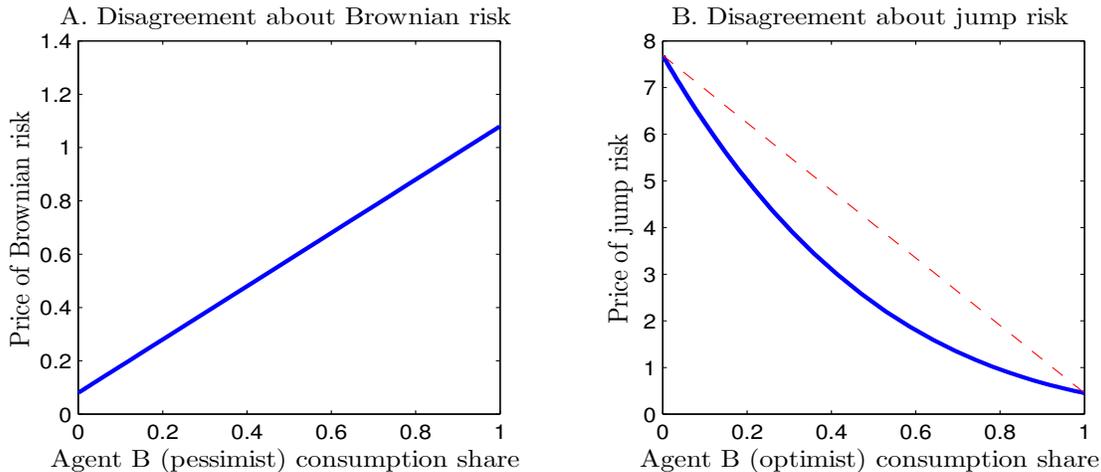


Figure 8: **Disagreement about Brownian risk versus jump risk.** Panel A plots the price of Brownian risk (market Sharpe ratio) under the beliefs of agent A (with perceived consumption growth 2.5%) as a function of the consumption share of agent B (with perceived consumption growth 0.5%). Panel B plots the price of jump risk ( $\lambda_t^Q/\lambda^A$ ) for agent A ( $\lambda^A = 1.7\%$ ) as a function of the consumption share of B ( $\lambda^B = 0.1\%$ ).

see the important point that the average belief is not a sufficient statistic for determining the jump risk premium. Comparing Panel A and Panel B in Figure 8, we see that this is in contrast to the case of diffusive risks where the (wealth-weighted) average belief completely determines the risk premium and the dispersion in beliefs is irrelevant. With disasters, asset prices disproportionately reflect the optimist’s beliefs about the jump risk. This means that when we extract the disaster probability or its expected impact from asset prices, the presence of just a small group of optimists can lead to substantial downward biases.

## 5.2 Heterogeneous risk aversion

Intuitively, besides heterogeneous beliefs, heterogeneity in risk aversion should also be able to induce risk sharing among agents and reduce the equity premium in equilibrium. Recall that the jump risk premium is  $\lambda_t^Q/\lambda_t^i = e^{-\gamma_i \Delta c_t^i}$ , which is not only sensitive to changes in individual consumption loss  $\Delta c_t^i$ , but also to the relative risk aversion  $\gamma_i$ . Thus, we

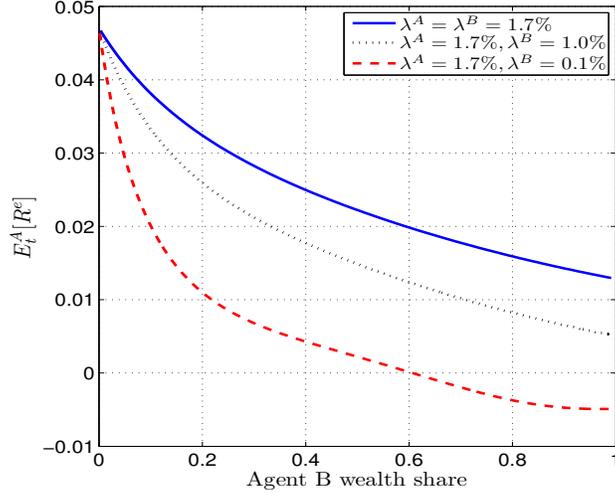


Figure 9: **The effects of heterogeneous risk aversion.** This graph plots the equity premium when the two agents have different risk aversion:  $\gamma_A = 4, \gamma_B = 2$ . Their beliefs about disasters are specified in the legend. Disaster size is constant.

expect that heterogeneous risk aversion can have similar effects on the equity premium as heterogeneous beliefs about disasters.

To check this intuition, we consider the following special case of the model. Agent A is the same as in the example of [Section 3.1](#):  $\lambda^A = 1.7\%$ ,  $\gamma_A = 4$ . Agent B has identical beliefs about disasters but is less risk averse:  $\lambda^B = 1.7\%$ ,  $\gamma_B < \gamma_A$ . We then solve the model using the technique in [Chen and Joslin \(2010\)](#). [Figure 9](#) plots the equity premium as a function of agent B's wealth share for  $\gamma_B = 2$ . The equity premium does decline as agent B's wealth share rises. However, the decline is slow and closer to being linear. In order for the equity premium to fall below 2%, the wealth share of the less risk-averse agent needs to rise to 60%. The decline in the equity premium becomes faster as we further reduce the risk aversion of agent B (not reported here), but the non-linearity is still less pronounced than in the cases with heterogeneous beliefs.

Combining heterogeneous beliefs about disasters and different risk aversion can amplify risk sharing and accelerate the decline in the equity premium. As shown in the figure, if

agent B believes disasters are less likely than does agent A, and she happens to be less risk averse, the equity premium falls faster. Consider the case where agent B believes disasters only occur once every hundred years ( $\lambda^B = 1.0\%$ ). With 20% of total wealth, she drives the equity premium down by almost a half to 2.5%. If  $\lambda^B = 0.1\%$ , the decline in the equity premium will be even more dramatic.

## 6 Robustness

We have made a number of simplifying assumptions in this paper, including complete markets and dogmatic beliefs. In this section, we discuss the impact of relaxing some of these assumptions and the implications for a wider class models.

*How important is the assumption of complete markets for the results?*

In our main analysis, we consider completing the market with a disaster insurance contract which pays off with certainty exactly when a disaster occurred. One concern is that such a contract might be difficult to implement since the timing of payment can lead to substantial counterparty risk. Within the model, because the marginal utility of the optimist is unbounded as consumption drops to 0, she will never “over-promise” on the amount of disaster insurance she can provide. Hence, there is no counterparty risk in the model. In fact, we can impose the requirement that disaster insurance be fully collateralized (either by stocks or other real assets), in which case the optimist will have more than enough wealth to post collateral, and the equilibrium outcome will not change.

Still, there could be other practical reasons for why disaster insurances might be difficult to implement. Our model suggests that any two securities with differential exposure to the Brownian and jump risks would complete the market. For example, high grade corporate bonds, senior CDX tranches, and put options on the market index can all be used to trade disaster risk. However, even if none of these contracts exist, investors will still be able to

effectively share disaster risks by trading the stock. This is because in our model, the risk of holding equity is primarily the exposure to disaster risk (which is bundled with a small amount of Brownian risk that has little effect on the premium). Following this intuition, we consider a variation of the benchmark model by turning off Brownian risk. Then markets will be dynamically complete via the trading of the aggregate stock and riskless bonds.

**Figure 10** plots the equity premium and portfolio positions for both agents. In Panel A, the equity premium in the model with only disaster risk is nearly identical to the benchmark case with Brownian risk. The difference between the two equity premiums is roughly constant and equal to  $\gamma\sigma_c^2 = 16$  basis points. Panel B shows that the agents now trade disaster risk using the stock market. The pessimist sells part of her equity claim to the optimist and invests the proceeds in riskless bonds. From the perspective of the optimist, the equity claim offers a high premium due to disaster risk, which she believes rarely occur. Her capacity to share risk with the pessimist is limited by her wealth. Because of their budget constraint and the Inada condition, their leverage is in fact fairly modest.

Although solving the incomplete markets model where there are both diffusive and jumps risks is considerably more complicated, the intuition we get from the above example is likely to hold: in equilibrium the optimist will (moderately) lever up in equity and the equity premium will be fairly close to the complete markets case, provided that disaster risk is main force behind the equity premium relative to the diffusive risk.<sup>17</sup>

Another important consideration is that a big part of total wealth is human capital, which may not be tradable. In that case, the amount of insurance that the optimist can provide will be reduced, and so will the effect of heterogeneous beliefs on the disaster risk premium. For example, in Panel D of **Figure 2**, the optimist loses up to 70% of her consumption in a disaster when her wealth share is low. Such an allocation might no longer be feasible if a big part of her wealth is non-tradable. Similarly, since only the tradable wealth can be used as

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<sup>17</sup>In the case of log utility, **Dieckmann (2010)** finds that introducing incomplete markets actually raises the risk premium, which would imply a steeper slope on the left side of Panel A of **Figure 10**.

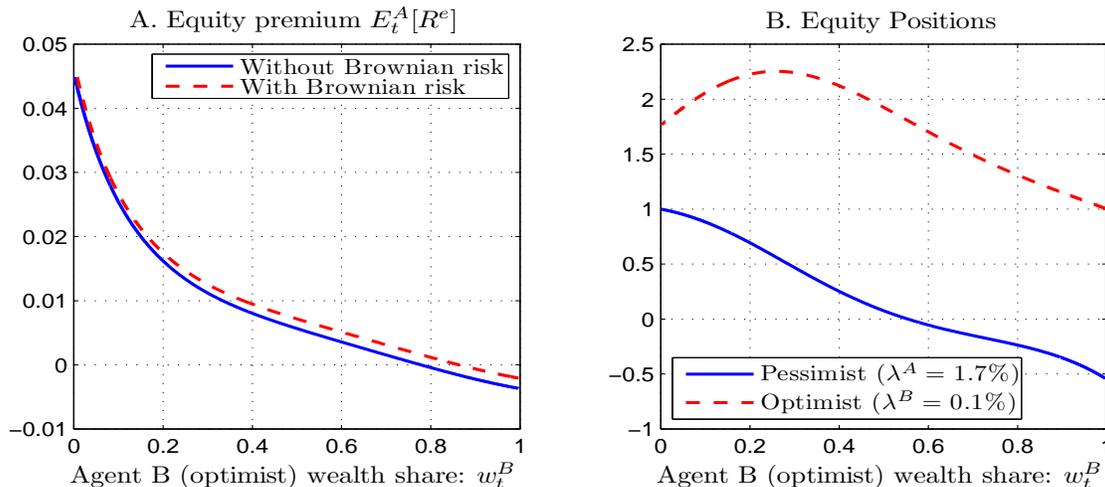


Figure 10: **The relative impact of disaster and Brownian risks.** Panel A plots the equity premium under the pessimist’s beliefs as a function of the wealth share of the optimist assuming either that the conditional volatility of consumption is  $\sigma_c = 2\%$ . Panel B plots the position of the relatively optimistic agent in the equity claim when there is only disaster risk.

collateral, the optimist will no longer be able to sell as much insurance if collateral constraint is imposed. In practice, the investors that are selling index put options and buying senior CDX tranches tend to be institutional investors or high wealth individuals, whose wealth are mostly tradable. Still, it would be important to carefully study the effect of non-tradable wealth on risk sharing.

*Are the beliefs of the optimists too optimistic?*

In the simple version of our model (Section 3.1), the optimist believes that the disaster intensity is only 0.1% per year. How reasonable this belief is depends on the size of the disaster, which we assume to be 40% in the example. Based on a century of U.S. data, aggregate consumption has never fallen more than 15% in a given year. The maximum cumulative consumption drop over any consecutive number of years is 23%, which occurred during the Great Depression. In addition, [Malmendier and Nagel \(2010\)](#) argue that individual experiences of macroeconomic outcomes can have long-term effects on their preferences

and beliefs. Thus, it is possible that an investor born in the U.S. who did not experience the Great Depression assigns close to zero probability to a 40% drop of aggregate consumption. Moreover, our analysis focuses on the case where only a small fraction of agents are optimistic, which also makes these beliefs easier to justify. That is, one could think of the optimist in the model as representing a small group of “irrationally optimistic” investors.

As we show in [Section 4](#), the optimism about disaster risk can also show up in the belief about the distribution of disaster size. In that example we calibrate the belief of the optimist to the U.S. aggregate consumption data in the last 120 years. The data suggest that smaller jumps in aggregate consumption are relatively more likely, but these jumps have rather limited effect on the equity premium.

*What are the effects of learning on our results?*

We do not model learning about disaster risk explicitly in this paper. In principle, investors will update their beliefs about disasters using the data, and the beliefs of those that are overly optimistic or pessimistic about disasters will eventually converge to the correct one. However, due to the nature of disaster risk, learning about either the intensity or size of disasters will be very slow.<sup>18</sup> Thus, it is unlikely that such learning mechanism will have significant effects on our results.

For example, suppose agents think the disaster intensity  $\lambda$  is either 1.7% or 0.1%, and the pessimist (optimist) has a strong prior in that she assigns 99% probability to the higher (lower) value. Then the average belief of  $\lambda$  for the two agents are 1.68% and 0.12%. [Table 2](#) reports the two agents’ average beliefs of  $\lambda$  after 50 years using the Bayes’ Rule. For the pessimist, any realization of disasters over 50 years essentially convinces her that  $\lambda = 1.7\%$ . For the optimist, without a disaster, her belief is essentially unchanged, whereas with one disaster, her perceived  $\lambda$  only rises to 0.21%. Only with two or more disasters do we see the optimist’s belief converging significantly to that of the pessimist, but these are unlikely

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<sup>18</sup>Here we think about learning about the likelihood of disasters during normal periods; the model in [Section 3.5](#) considers when agent update their beliefs during periods where disasters are more likely.

Table 2: **Learning about Disaster Intensity.** This table shows the updating of the average belief of disaster intensity  $\lambda$  after observing the number of disasters  $N_d$  over 50 years. We assume that the pessimist's prior belief is to assign 99% (1%) probability to the high (low)  $\lambda$ , and the opposite for the optimist.

Agent	$N_d = 0$	$N_d = 1$	$N_d = 2$	$N_d = 3$
Pessimist	1.66%	1.70%	1.70%	1.70%
Optimist	0.11%	0.21%	1.01%	1.63%
Probability when $\lambda = 1.7\%$	42.7%	36.3%	15.4%	4.4%
Probability when $\lambda = 0.1\%$	95.1%	4.8%	0.1%	0.0%

events even under the pessimist's belief.

To study the potential effects of learning on asset pricing, we consider an extension of the model that captures the essence of the learning effect. We assume that the optimist maintains her belief until a disaster occurs, and from that point onward she fully agrees with the pessimist about the likelihood of disasters. That is,

$$\lambda_t^B = \lambda^B 1_{\{N_t=0\}} + \lambda^A 1_{\{N_t \geq 1\}}.$$

Figure 11 plots the equity risk premium and jump risk premium in this case. Both the equity premium and jump risk premium are slightly higher in the case where beliefs converge after the first disaster, which is consistent with the intuition that learning can reduce risk sharing in the long run. However, the quantitative effect of learning on pricing is very small.

## 7 Concluding Remarks

We demonstrate the equilibrium effects of heterogeneous beliefs about disasters on risk premia and trading activities. When agents disagree about disaster risk, they will insure each other against the types of disasters they fear most. Because of the highly nonlinear effect

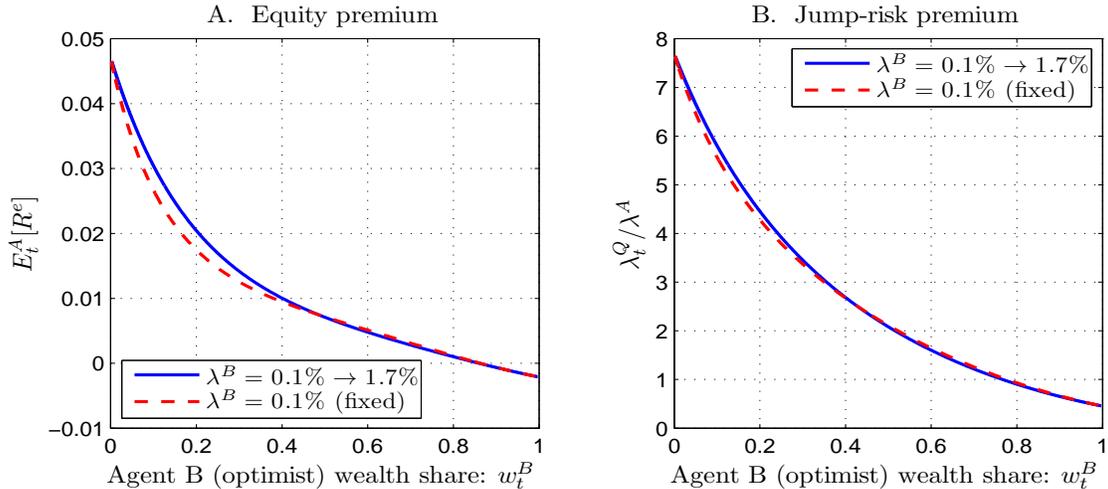


Figure 11: **Learning through disasters.** Panel A plots the equity premium under the pessimist’s beliefs as a function of the wealth share of the optimist assuming that either the optimist holds his beliefs fixed or that the optimist updates his beliefs to agree with the pessimist after a disaster occurs. Panel B plots the jump-risk premium  $\lambda_t^Q/\lambda^A$  for the pessimist.

of disaster size on risk premia, the risk sharing provided by a small amount of agents with heterogeneous beliefs can significantly attenuate the effect of disasters on the equity premium. The model also has important implications for how disaster risks affect the dynamics of asset prices, the potential bias of estimating disaster probabilities from prices, and the link between the size of disaster insurance market and equity premium.

Our results also suggest a few directions for future research on disaster risk. The effectiveness of the risk sharing mechanism has significant impact on how disaster risk affects asset prices in the equilibrium. It would be useful to study what happens to asset prices when we limit the risk sharing among investors with heterogeneous beliefs about disasters, perhaps by imposing transaction costs, borrowing constraints, and short-sales constraints as in [Heaton and Lucas \(1996\)](#). Another interesting consideration is ambiguity aversion. As [Hansen \(2007\)](#) and [Hansen and Sargent \(2010\)](#) show, if investors are ambiguity averse, they deal with model/parameter uncertainty by slanting their beliefs pessimistically. In the case with disaster risk, ambiguity averse investors will behave as if they believe the disaster prob-

abilities are high, even though their actual priors might suggest otherwise. This mechanism could also limit the effects of risk sharing. We leave these questions to future research.

# Appendix

## A Securities' prices and portfolio positions

In this appendix we compute the prices of the claim on aggregate endowment (stock), the claim on individual agents' consumption streams (agents' personal wealth), disaster insurance, and the equilibrium portfolio positions. We begin with the general setting of time-varying disaster intensity. To concentrate on the effects of heterogeneous beliefs, we assume that the two agents have the same relative risk aversion  $\gamma$ .

### A.1 Aggregate and individual consumption claim prices: general setting

The price of the aggregate endowment claim is

$$P_t = \int_0^\infty E_t^{\mathbb{P}^A} \left[ \frac{M_{t+T}^A}{M_t^A} C_{t+T} \right] dT, \quad (\text{A.1})$$

where  $M_t^A$  is the stochastic discount factor

$$M_t^A = e^{-\rho t} C_t^{-\gamma} \left( 1 + (\zeta_0 e^{\log \eta_t})^{\frac{1}{\gamma}} \right)^\gamma. \quad (\text{A.2})$$

This price can be viewed as a portfolio of zero coupon aggregate consumption claims

$$\begin{aligned} M_t^A P_t^{t+T} &= E_t^{\mathbb{P}^A} [M_{t+T}^A C_{t+T}] \\ &= e^{-\rho(t+T)} e^{T[\bar{g}(1-\gamma) + \frac{1}{2}\sigma_c^2(1-\gamma)^2]} e^{(1-\gamma)c_t} \times E_t^{\mathbb{P}^A} \left[ e^{(1-\gamma)c_{t+T}^d} \left( 1 + (\zeta_0 e^{\log \eta_{t+T}})^{\frac{1}{\gamma}} \right)^\gamma \right]. \end{aligned}$$

Under our assumption of integer  $\gamma$ , the final term will be a sum of expectations of the form

$$E_t^{\mathbb{P}^A} [e^{(1-\gamma)c_{t+T}^d + \beta_i \log \eta_{t+T}}] = e^{A_i(T) + (1-\gamma)c_t^d + \beta_i \log \eta_t + B_i(T)\lambda_t}, \quad (\text{A.3})$$

where  $(A_i, B_i)$  satisfy a simplified version of the familiar Riccati differential equations

$$\dot{B}_i = -\frac{\bar{\lambda}^B}{\bar{\lambda}^A} \beta_i - \kappa B_i + \frac{\sigma_\lambda^2}{2} B_i^2 + (\phi(\langle 1 - \gamma, \beta_i \rangle) - 1), \quad B_0(0) = 0, \quad (\text{A.4a})$$

$$\dot{A}_i = \kappa \theta B_i, \quad A_i(0) = 0, \quad (\text{A.4b})$$

where  $\phi$  is the moment generating function of jumps in  $\langle c_t^d, a_t \rangle$ .

It follows that price/consumption ratio of the zero-coupon equity varies only with the stochastic weight  $\tilde{\zeta}_t$  and the disaster intensity:

$$P_t^{t+T} = C_t h^T(\lambda_t, \tilde{\zeta}_t). \quad (\text{A.5})$$

Next, agent A's wealth  $P_t^A = \int_0^\infty E_t^{\mathbb{P}^A} \left[ \frac{M_{t+T}^A}{M_t^A} C_{t+T}^A \right] dT$  at time  $t$  is a portfolio of her zero coupon consumption claims

$$\begin{aligned} M_t^A P_t^{A,t+T} &= E_t^{\mathbb{P}^A} [M_{t+T}^A C_{t+T}^A] \\ &= e^{-\rho(t+T)} e^{T[\bar{g}(1-\gamma) + \frac{1}{2}\sigma_c^2(1-\gamma)^2]} e^{(1-\gamma)c_t} \times E_t^{\mathbb{P}^A} \left[ e^{(1-\gamma)c_{t+T}^d} \left( 1 + (\zeta_0 e^{\log \eta_{t+T}})^{\frac{1}{\gamma}} \right)^{\gamma-1} \right]. \end{aligned}$$

We can compute agent A's wealth process by making a similar binomial expansion as in the case of  $P_t$ , and then computing the expectation concerning the same affine jump diffusion process. Finally, the wealth process of agent B is simply  $P_t^B = P_t - P_t^A$ .

## A.2 Special case: constant disaster risk

Closed form expressions can now be obtained in the special case of constant disaster intensity and constant disaster size. Let's denote  $\tilde{\zeta}_t \equiv \zeta_0 e^{\log \eta_t}$ . Again by expanding the binomial for the cases with integer  $\gamma$ ,

$$\begin{aligned} E_t^{\mathbb{P}^A} [M_{t+T}^A C_{t+T}^A] &= e^{-\rho(t+T)} E_t^{\mathbb{P}^A} \left[ \left( 1 + (\tilde{\zeta}_{t+T})^{1/\gamma} \right)^\gamma C_{t+T}^{1-\gamma} \right] \\ &= e^{-\rho(t+T)} C_t^{1-\gamma} \sum_{k=0}^{\gamma} \binom{\gamma}{k} E_t^{\mathbb{P}^A} \left[ \frac{(\tilde{\zeta}_{t+T})^{k/\gamma} C_{t+T}^{1-\gamma}}{C_t^{1-\gamma}} \right]. \end{aligned}$$

Plugging in the explicit expressions for aggregate consumption  $C_t$ , the stochastic discount factor  $M_t^A$ , and performing the simple affine jump diffusion expectation we obtain

$$P_t^{t+T} = C_t \sum_{k=0}^{\gamma} \alpha_{k,t} e^{-\beta_k T}, \quad (\text{A.6})$$

with

$$\alpha_{k,t} \equiv \binom{\gamma}{k} \frac{(\tilde{\zeta}_t)^{k/\gamma}}{(1 + (\tilde{\zeta}_t)^{1/\gamma})^\gamma}, \quad (\text{A.7a})$$

$$\beta_k \equiv \rho + (\gamma - 1)\bar{g} - \frac{1}{2}\sigma_c^2(\gamma - 1)^2 - \bar{\lambda}(e^{(\gamma-1)\bar{d} + \frac{k\Delta a}{\gamma}} - 1) + \frac{\bar{\lambda}k}{\gamma}(e^{\Delta a} - 1), \quad (\text{A.7b})$$

where  $\Delta a$  is given in (5).

Finally, integrating over time  $T$  yields the explicit price of aggregate endowment claim

$$P_t = \int_0^\infty P_t^{t+T} dT = C_t \sum_{k=0}^{\gamma} \frac{\alpha_{k,t}}{\beta_k}. \quad (\text{A.8})$$

The restriction  $\beta_k^A > 0$  is needed to ensure finite value for  $P_t$ . We will come back to this type of restriction below.

By identical approach, we obtain the price of agent A's consumption claim (i.e. her wealth process)

$$P_t^A = \int_0^\infty P_t^{A,t+T} dT = C_t \sum_{k=0}^{\gamma-1} \frac{\alpha_{k,t}^A}{\beta_k}, \quad (\text{A.9})$$

where  $\beta_k$  remains the same as above and

$$\alpha_{k,t}^A \equiv \binom{\gamma-1}{k} \frac{(\tilde{\zeta}_t)^{k/\gamma}}{(1 + (\tilde{\zeta}_t)^{1/\gamma})^\gamma}. \quad (\text{A.10})$$

### Price of disaster insurance

Let  $P_{t,t+T}^{DI}$  denotes the price of disaster insurance which pays \$1 at maturity time  $t+T$  if there was at least one disaster taking place in the time interval  $(t, t+T)$ . In the main text we consider disaster insurance  $P_t^{DI}$  of maturity  $T=1$  in particular.

$$\begin{aligned} P_{t,t+T}^{DI} &= E_t^{\mathbb{P}^A} \left[ \frac{M_{t+T}^A}{M_t^A} \mathbf{1}_{(N_{t+T} > N_t)} \right] \\ &= \frac{e^{-\rho T}}{(C_t^A)^{-\gamma}} E_t^{\mathbb{P}^A} \left[ (C_T^A)^{-\gamma} \mathbf{1}_{(N_{t+T} > N_t)} \right] \\ &= \frac{e^{(-\rho - \gamma \bar{g} + \frac{1}{2} \gamma^2 \sigma_c^2) T}}{(1 + (\tilde{\zeta}_t)^{1/\gamma})^\gamma} E_t^{\mathbb{P}^A} \left[ e^{\gamma \bar{d} \Delta N_T} (1 + (\tilde{\zeta}_{t+T})^{1/\gamma} e^{(\Delta a \Delta N_T - \bar{\lambda} T (e^{\Delta a} - 1)) / \gamma})^\gamma \mathbf{1}_{(\Delta N_T > 0)} \right] \\ &= \frac{e^{(-\rho - \gamma \bar{g} + \frac{1}{2} \gamma^2 \sigma_c^2) T}}{(1 + (\tilde{\zeta}_t)^{1/\gamma})^\gamma} \left\{ E_t^{\mathbb{P}^A} \left[ e^{\gamma \bar{d} \Delta N_T} (1 + (\tilde{\zeta}_{t+T})^{1/\gamma} e^{(\Delta a \Delta N_T - \bar{\lambda} T (e^{\Delta a} - 1)) / \gamma})^\gamma \right] \right. \\ &\quad \left. - (1 + (\tilde{\zeta}_t)^{1/\gamma} e^{-\bar{\lambda} T (e^{\Delta a} - 1) / \gamma})^\gamma \mathbb{P}_A(\Delta N_T = 0) \right\}, \end{aligned}$$

where  $\Delta N_T \equiv N_{t+T} - N_t$  is number of disasters taking place in  $[t, t+T]$ , and  $\mathbb{P}_A(\Delta N_T = 0) = e^{-\bar{\lambda} T}$  is the probability that no such disaster did happen. Again by expanding the binomial  $(1 + (\tilde{\zeta}_{t+T})^{1/\gamma} e^{(\Delta a \Delta N_T - \bar{\lambda} T (e^{\Delta a} - 1)) / \gamma})^\gamma$ , and then computing the expectation of each resulting

term, we obtain

$$P_{t,t+T}^{DI} = \frac{a_T}{(1 + (\tilde{\zeta}_t)^{1/\gamma})^\gamma} \left\{ \left[ \sum_{k=0}^{\gamma} b_{k,T} (\tilde{\zeta}_t)^{k/\gamma} \right] - e^{-\bar{\lambda}T} (1 + (\tilde{\zeta}_t)^{1/\gamma}) e^{-\bar{\lambda}T(e^{\Delta a} - 1)/\gamma} \right\}, \quad (\text{A.11})$$

where

$$a_T = e^{(-\rho - \gamma \bar{g} + \frac{1}{2} \gamma^2 \sigma_c^2)T}, \quad (\text{A.12a})$$

$$b_{k,T} = \binom{\gamma}{k} e^{-\bar{\lambda}kT(e^{\Delta a} - 1)/\gamma} e^{\bar{\lambda}T[e^{(\gamma d + \frac{\Delta a k}{\gamma})} - 1]}. \quad (\text{A.12b})$$

## B Equilibrium portfolio positions

In the current case of constant jump size with two dimensions of uncertainties (Brownian motion and disaster jump), the market is complete when agents are allowed to trade contingent claims on aggregate consumption (stock)  $P_t$ , money market account  $RFB_t$  and disaster insurance  $P_t^{DI}$ . We can use generalized Ito lemma on jump-diffusion (see, for example, [Protter \(2003\)](#)) to determine the price processes for each asset. Portfolio positions are then determined by equating the exposures to the Brownian and jump risks of each agents consumption claim to a portfolio of the aggregate claim and disaster insurance, which are then financed with the risk free bond.

## C Boundedness of prices

This appendix discusses the boundedness of securities prices in general heterogeneous-agent economy. As claimed in the main text, as long as agents have different but equivalent beliefs, necessary and sufficient condition for finite price of a security in heterogeneous-agent economy is that this price be finite under each agent's beliefs in a single-agent economy. This is easy to see since

$$\max(f_{A,0}^\gamma, f_{B,0}^\gamma \eta_t) \leq M_t^A \leq (2f_{A,0}^\gamma) + (2f_{B,0}^\gamma) \eta_t \quad (\text{C.1})$$

Conditions for the finiteness of prices in the single agent economy can be found by studying the fixed points of the equations [\(A.4a\)](#). Setting  $dB/dt = 0$ , we find the fixed point of this differential equation is

$$B^* = \frac{\kappa - \sqrt{\kappa^2 + 2\sigma_\lambda^2(1 - \phi^{\mathbb{P}^i}(1 - \gamma^i))}}{\sigma_\lambda^2}, \quad (\text{C.2})$$

provided that [\(13a\)](#) holds. Otherwise there is no fixed point and  $B \rightarrow \infty$  implying infinite prices. Furthermore, it is easily seen that the initial condition  $B(0) = 0$  is in the domain of

attraction. For equity price to be finite, it is easy to see that the limiting exponent in (A.3) must be negative, or

$$-\rho + (1 - \gamma^i)\bar{g} + \frac{1}{2}(\gamma^i - 1)^2\sigma_c^2 + \kappa\bar{\lambda}^i B^* < 0, \quad (\text{C.3})$$

for both  $i = 1, 2$ . This is (13b) after we plug in the above expression for  $B^*$ .

## D Proofs from Section 3.2

In this section, we provide the proofs for (19) and (20). It is useful to rewrite expression for the consumption fractions in terms of the initial consumption sharing rule  $(f_0^A, f_0^B)$  and the Radon-Nikodym derivative  $(\eta_t)$ . In these terms,

$$\begin{aligned} f_t^A &= \frac{f_0^A}{f_0^A + f_0^B \eta_t^{\frac{1}{\gamma}}}, \\ M_t^A / M_0^A &= \left( f_0^A + f_0^B \eta_t^{\frac{1}{\gamma}} \right)^\gamma C_t^{-\gamma} / C_0^{-\gamma}, \\ \lambda_t^{\mathbb{Q}} &= \lambda_A e^{-\gamma d} \left( f_t^A + f_t^B \left( \frac{\lambda_B}{\lambda_A} \right)^{\frac{1}{\gamma}} \right)^\gamma. \end{aligned}$$

Additionally, for ease of notation, we set  $N_0 = 0$  and  $C_0 = 1$  which results in the expressions being fractions of the initial endowment.

Taking derivatives, we find

$$\frac{\partial \lambda^{\mathbb{Q}}}{\partial f_0^A} = \lambda_A e^{-\gamma d} \gamma \left( f_0^A + f_0^B \left( \frac{\lambda_B}{\lambda_A} \right)^{\frac{1}{\gamma}} \right)^{\gamma-1} \left( 1 - \left( \frac{\lambda_B}{\lambda_A} \right)^{\frac{1}{\gamma}} \right).$$

Setting  $f_0^A = 1$  and taking the limit  $\lambda^B \rightarrow 0^+$ , we obtain (19).

In order to compute the derivative of the wealth fraction of Agent B with respect to  $f_0^B$ , we first compute the derivative of the value of his claim, call it  $P^B$ , with respect to  $f_0^B$ . Since

$$P^B = \int_0^\infty E_0^{\mathbb{P}^A} [(f_0^A + f_0^B \eta_t^{\frac{1}{\gamma}})^{\gamma-1} f_0^B \eta_t^{\frac{1}{\gamma}} C_t^{1-\gamma}] e^{-\rho t} dt,$$

we have that

$$\begin{aligned} \frac{\partial P^B}{\partial f_0^A} &= \int_0^\infty (\gamma - 1) E_0^{\mathbb{P}^A} [(f_0^A + f_0^B \eta_t^{\frac{1}{\gamma}})^{\gamma-2} (1 - \eta_t^{1/\gamma}) f_0^B \eta_t^{\frac{1}{\gamma}} C_t^{1-\gamma}] e^{-\rho t} dt \\ &\quad - \int_0^\infty E_0^{\mathbb{P}^A} [(f_0^A + f_0^B \eta_t^{\frac{1}{\gamma}})^{\gamma-1} \eta_t^{\frac{1}{\gamma}} C_t^{1-\gamma}] e^{-\rho t} dt. \end{aligned}$$

From which it follows

$$\begin{aligned} \left. \frac{\partial P^B}{\partial f_0^A} \right|_{f_0^A=1} &= - \int_0^\infty E_0^{\mathbb{P}^A} [\eta_t^{1/\gamma} C_t^{1-\gamma}] e^{-\rho t} dt \\ &= - \frac{1}{\rho + (\gamma - 1)\bar{g} - \frac{1}{2}\sigma_c^2(1 - \gamma)^2 + \frac{1}{\gamma}(\lambda_B - \lambda_A) - \lambda_A(e^{(1-\gamma)\bar{d} + \frac{1}{\gamma}\log(\frac{\lambda_B}{\lambda_A})} - 1)}. \end{aligned}$$

And so

$$\left. \frac{\partial P^B}{\partial f_0^A} \right|_{f_0^A=1} \rightarrow - \frac{1}{\rho + (\gamma - 1)\bar{g} - \frac{1}{2}\sigma_c^2(1 - \gamma)^2 + \frac{\gamma-1}{\gamma}\lambda_A} \quad \text{as } \lambda_B \rightarrow 0^+.$$

Now, it is easy to see that the derivative of the value of the claim to the entire endowment is bounded and since  $P_B = 0$  when  $f_0^A = 1$ , the derivative  $\partial w_0^B / \partial f_0^A$  is simply  $\frac{\partial P^B}{\partial f_0^A}$  divided by the value of the claim to the entire endowment. This proves (20).

## E General valuation of disaster states

In [Section 3.2](#), we demonstrated that within a simple calibration a large fraction of the the value of the endowment claim arises from the disaster states, even though these states are very rare. Here we demonstrate that in fact this property is a feature of a broad class of models. Specifically, suppose that the model is such that the dynamics of aggregate consumption under the actual measure, as well as the risk-neutral measure, follow the dynamics in [1](#) and that the risk-free rate is constant. This is true in our model with CRRA preferences and remains true with Epstein-Zin preferences (cf. [Wachter \(2009\)](#).) In particular, this reduced form setting removes the link between risk aversion and elasticity of intertemporal substitution.

Within this setting, let  $\bar{g}^{\mathbb{Q}}$  denote the growth rate of consumption under the risk neutral measure. The fractional value of consumption in the non-disaster states is then

$$\frac{\int_0^\infty E_0^{\mathbb{Q}} [e^{-rt} C_t \times 1_{\{N_t=0\}}]}{\int_0^\infty E_0^{\mathbb{Q}} [e^{-rt} C_t]} = \frac{r - \bar{g}^{\mathbb{Q}} - .5\sigma_c^2 - \lambda^{\mathbb{Q}}(e^{\bar{d}} - 1)}{r - \bar{g}^{\mathbb{Q}} - .5\sigma_c^2 + \lambda^{\mathbb{Q}}}$$

The difference between the numerator and denominator is  $\lambda^{\mathbb{Q}}e^{\bar{d}}$ . In order for disasters to account for a substantial risk premium, this term should be sizeable (it is 6% in the example of [Section 3.1](#).) Moreover, it is reasonable to expect the price-consumption ratio (the inverse of the denominator) should not be too small. Setting these to 4% and 10 gives a fraction 4/14 due to disaster states. Setting them to 6% and 20 give a fraction of 6/11 to the disaster states. In summary, under these very general reduced form assumptions on the endowment and preferences along with the assumptions that (i) disasters account for a significant risk premium and (ii) the price-consumption ratio is not too small, the fraction of wealth due to non-disaster states is significant.<sup>19</sup>

## F Time-varying Disagreement

The model solution is generally analogous to the case without Markov regime-switching, so we sketch the major differences between the models.

The Radon-Nikodym derivative  $\eta_t$  now reflects the change of state  $s_t$ ,

$$\eta_t = e^{\sum_{i \in \{L, H\}} (\Delta a_i N_t^i - \lambda_i^A T_t^i (e^{a_i} - 1))}, \quad (\text{F.1})$$

where

$$\Delta a_i = \log \left( \frac{\lambda_i^B}{\lambda_i^A} \right), \quad (\text{F.2})$$

$$T_t^i = \int_0^t 1_{\{s_\tau = i\}} d\tau, \quad (\text{F.3})$$

and  $N_t^i$  counts the number of disasters that have occurred up to time  $t$  while the state is  $s_t = i$ .

The key expectations to compute are of the form

$$E_0^{\mathbb{P}^A} [e^{aN_t^L + bN_t^H + cT_t^L + dT_t^H}], \quad (\text{F.4})$$

where  $N_t^i$  is the number of disasters that occur in state  $i$  and  $T_t^i$  is the occupation time in state  $i$  defined in [\(F.3\)](#). These expectations can be computed by first conditioning on the path of the Markov state and using the conditional independence of the Poisson process in

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<sup>19</sup>In the CRRA version of this equation,  $r = \rho + \gamma \bar{g} - .5\sigma_c^2 \gamma^2 - (\lambda^{\mathbb{Q}} - \lambda^{\mathbb{P}})$ . This causes increasing  $\lambda^{\mathbb{P}}$  (and thus  $\lambda^{\mathbb{Q}}$ ) to increase the price-consumption ratio. In the general formula if we fix  $r$  and increase  $\lambda^{\mathbb{Q}}$  independently this *decreases* P/C so clearly the generic form dont have EIS-risk aversion link problems.

each state:

$$E_0^{\mathbb{P}^A} [e^{a_L N_t^L + b_H N_t^H + c T_t^L + d T_t^H}] = E_0^{\mathbb{P}^A} \left[ E_0^{\mathbb{P}^A} [e^{a_L N_t^L + b_H N_t^H + c T_t^L + d T_t^H} | \{S_\tau\}_{\tau=0}^t] \right] \quad (\text{F.5})$$

$$= E_0^{\mathbb{P}^A} \left[ e^{(\lambda_L^A (e^a - 1) + c) T_t^L + (\lambda_H^A (e^b - 1) + d) T_t^H} \right] \quad (\text{F.6})$$

This reduces the problem to computing the joint moment-generating function of the occupation times  $(T_t^L, T_t^H)$ . Darroch and Morris (1968) show that this expectation reduces to

$$E_0^{\mathbb{P}^A} [e^{\alpha T_t^L + \beta T_t^H}] = \pi_0' \exp(A t) \vec{1}, \quad \text{where } A = \Lambda + \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad (\text{F.7})$$

and  $\pi_0$  is either  $(1, 0)'$  or  $(0, 1)'$ , as the initial state is  $L$  or  $H$ .

The price of consumption claims involve sums of integrals of such expectations. These integral can be computed in closed form by diagonalizing  $A$  to deliver closed form expressions for the prices of interest.

## G General Forms of Disagreements

The affine heterogeneous beliefs framework in Section 2 can capture other forms of heterogeneous beliefs besides disagreement about disaster intensity. In this section, we first show that disagreement about the size of disasters has similar impact on the risk premium as disagreement about the frequency of disasters. We then provide an example with strong effects of risk sharing even when both agents are pessimistic about disasters.

### G.1 Disagreement about the Size of Disasters

For simplicity, let's assume that the drop in aggregate consumption in a disaster follows a binomial distribution, with the possible drops being 10% and 40%. Both agents agree on the intensity of a disaster ( $\lambda = 1.7\%$ ). Agent A (pessimist) assigns a 99% probability to a 40% drop in aggregate consumption, thus having essentially the same beliefs as in the previous example. On the contrary, agent B (optimist) only assigns 1% probability to a 40% drop, but 99% probability to a 10% drop. The rest of the parameter values are the same as in the first example.

Figure 12 (solid lines) plots the conditional equity premium and jump risk premium under the pessimist's beliefs. When the pessimist has all the wealth, the equity premium is 4.6% (almost the same as in the first example). Again, the equity premium falls rapidly as we starts to shift wealth to the optimist. The premium falls by almost half to 2.4% when the optimist owns just 5% of total wealth, and becomes 1.4% when the optimist's share of

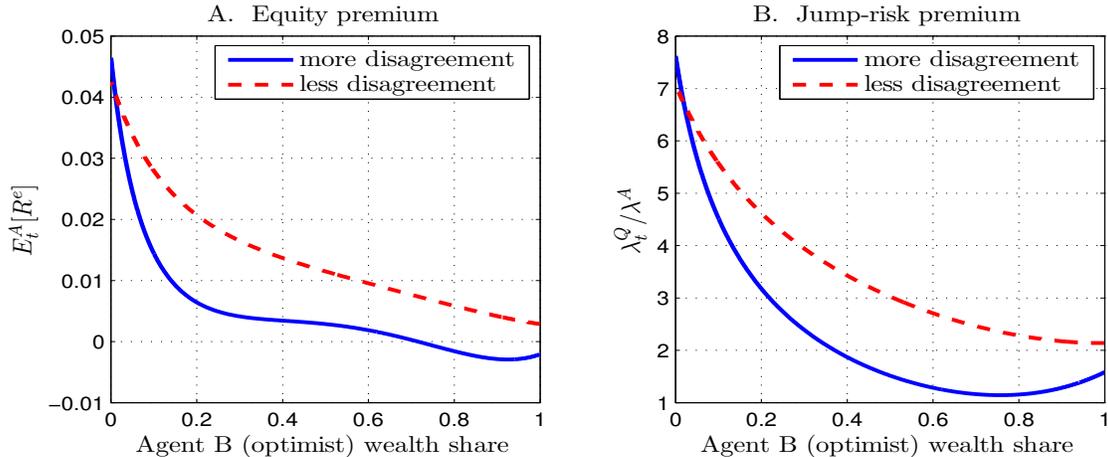


Figure 12: **Disagreement about the size of disasters.** The left panel plots the equity premium under the pessimist’s beliefs. The right panel plots the jump risk premium for the pessimist. In the case with “more disagreement”, the pessimist (optimist) assigns 99% probability to the big (small) disaster, conditional on a disaster occurring. With “less disagreement”, the probability assigned to big (small) disaster drops to 90%.

total wealth grows to 10%. Similarly, the jump risk premium falls from 7.6 to 4.5 with the optimist’s wealth share reaching 10%, which by itself will lower the premium to 2.4%.

These results show that, in terms of asset pricing, introducing an agent who disagrees about the severity of disasters is similar to having one who disagrees about the frequency of disasters. Even though the two agents agree on the intensity of disasters in general, they actually strongly disagree about the intensity of disasters of a specific magnitude. For example, under A’s beliefs, the intensity of a big disaster is  $1.7\% \times 99\% = 1.68\%$ , which is 99 times the intensity of such a disaster under B’s beliefs. The opposite is true for small disasters. Thus, B will aggressively insure A against big disasters, while A insures B against small disasters. For agent A, the effect of the reduction in consumption loss in a big disaster dominates that of the increased loss in a small disaster, which drives down the equity premium exponentially. Such trading can also become speculative when B has most of the wealth: agent A will take on so much loss in a small disaster that the jump risk premium rises up again.

Naturally, we expect that the agents will be less aggressive in trading disaster insurances when there is less disagreement on the size of disasters, and that the effect of risk sharing on the risk premium will become smaller. The case of “less disagreement” in Figure 12 confirms this intuition. In this case, we assume that the two agents assign 90% probability (as opposed to 99%) to one of the two disaster sizes. While the equity premium still falls rapidly near the left boundary, the pace is slower than in the previous case. Similarly, we

see a slower decline in the jump risk premium.

## G.2 When Two Pessimists Meet

The examples we have considered so far have one common feature: the new agent we are bringing into the economy has more optimistic beliefs about disaster risk, in the sense that the distribution of consumption growth under her beliefs first-order stochastically dominates that of the other's, and that the equity premium is significantly lower when she owns all the wealth. However, the key to generating aggressive risk sharing is not that the new agent demands a lower equity premium, but that she is willing to insure the majority wealth holders against the types of disasters that they fear most.

In order to highlight this insight, we consider the following example, which combines disagreements about disaster intensity as well as disaster size. Both agents believe that disaster risk accounts for the majority of the equity premium. The key difference in their beliefs is that one agent believes that disasters are rare but big, while the other thinks disasters are more frequent but less severe. Specifically, we assume that disasters can cause aggregate consumption drops of a 30% or 40%. Agent A believes that  $\lambda^A = 1.7\%$ , and assigns 99% probability to the bigger disaster. B believes that  $\lambda^B = 4.2\%$ , and assigns 99% probability to the smaller disaster.

By themselves, the two agents both demand high equity premium. We have chosen  $\lambda^B$  so that, under the beliefs of agent A, the equity premium is 4.6% whether A or B has all the wealth. However, they have significant disagreement on the exact magnitude of the disaster. Such disagreement generates a lot of demand for risk sharing. As we see in Panel A of [Figure 13](#), the conditional equity premium falls rapidly as the wealth share of agent B moves away from the two boundaries. In fact, the premium will be below 2% when B owns between 9% and 99% of total wealth. In Panel B, the jump risk premium also falls by half from 7.6 and 10 on the two boundaries when B's wealth share moves from 0% to 25% and from 100% to 91%, respectively.

To get more information on the risk sharing mechanism, in Panel C and D we examine the equilibrium consumption changes for the individual agents during a small or big disaster. Since agent A assigns a low probability to the small disaster, she insures agent B against this type of disasters. As a result, her consumption loss in such a disaster exceeds that of the aggregate endowment (-30%), and it increases with the wealth share of agent B. When B has almost all the wealth in the economy, agent A sells so much small disaster insurance to B that her own consumption can fall by as much as 82% when such a disaster occurs. As a result, agent B is able to reduce her risk exposure to small disasters significantly. In fact, her consumption actually jumps up in a small disaster when she owns less than 75% of total wealth, sometimes by over 100% (when her wealth share is small).

The opposite is true in Panel D. As agent B insures A against big disasters, she expe-

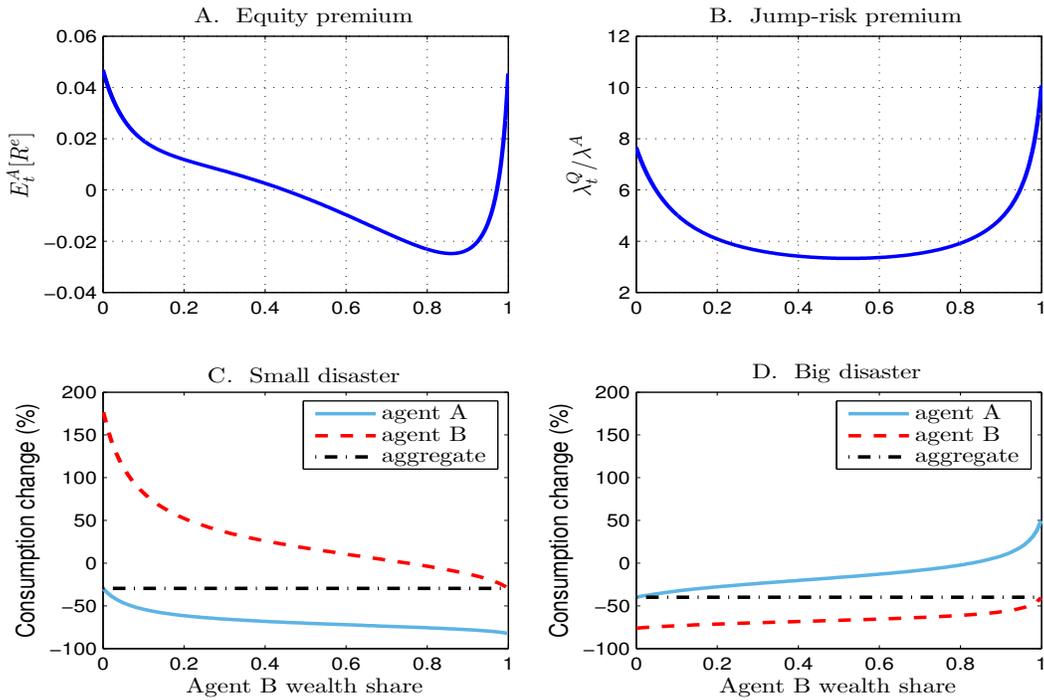


Figure 13: **When Two Pessimists Meet.** Panel A and B plot the equity premium and jump risk premium under agent A's beliefs. Panel C and D plot the individual consumption changes in small and big disasters.

riences bigger consumption losses in such a disaster than the aggregate endowment (-40%). The equilibrium consumption changes of the two agents are less extreme compared to the case of small disasters, which is due to two reasons. First, the relative disagreement on big disasters is smaller than on small disasters. Second, the insurance against larger disasters is more expensive, so that agent A's ability to purchase disaster insurance is more constrained by her wealth.

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