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# Portfolio Rebalancing in General Equilibrium[*]

Miles S. Kimball
Matthew D. Shapiro
Tyler Shumway
Jing Zhang

University of Michigan

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### Abstract

Standard portfolio advice is that agents should hold a constant share of risky assets. All agents cannot, however, follow this advice because supply and demand of risky assets must be equal. To study equilibrium rebalancing, the paper develops an overlapping generations model in which agents differ both in age and risk tolerance. Equilibrium rebalancing is driven by a leverage effect that affects levered and unlevered agents in opposite directions, an aggregate risk tolerance effect which depends on the distribution of wealth, and an intertemporal hedging effect. Optimal equilibrium portfolio rebalancing departs significantly from the standard advice.

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# 1 Introduction

Standard rebalancing advice is that households should maintain a fixed share of accumulated financial wealth in risky assets. Maintaining a constant share requires selling risky assets when the market goes up and buying them when the market goes down. The large price movements experienced recently in financial markets have highlighted a fairly obvious problem with this advice. It is impossible for everyone to simultaneously sell or for everyone to simultaneously buy, since someone must be on the other side of every transaction. It is clear that in general equilibrium how a household should react to market returns must depart from the standard rebalancing advice.

The advice to rebalance to a constant asset allocation is motivated by partial equilibrium application of the standard model of portfolio choice. In Merton (1971), the optimal share of an investor's portfolio that consists of the risky asset, $\theta$, is equal to the investor's risk tolerance coefficient, $\tau$, times the Sharpe ratio of the asset, $\mathcal{S}$, divided by the volatility of the asset, $\sigma$, so that $\theta = \tau \mathcal{S}/\sigma$. Assuming that neither risk tolerance nor the Sharpe ratio of the market vary with time, standard rebalancing advice is that as realized market returns cause the actual risky asset share to deviate from its optimal level it is important to trade to bring the risky asset share back to the constant share.

If one takes general equilibrium seriously in a setting where there is heterogeneity that leads to trade, the portfolio behavior of agents will depart from this standard advice. While this has been recognized by economists, including Cochrane (2008) and Sharpe (2010), the implications of general equilibrium for asset allocation and rebalancing have not been explored completely. This paper specifies a simple, overlapping-generations general equilibrium model with a minimum amount of heterogeneity necessary to study rebalancing in a meaningful way. Specifically, the model features investor heterogeneity across both age and risk aversion, with three generations of investors (young, middle-aged, and old) and two sets of preferences (daring and cautious). The three periods are the minimum necessary to allow for the possibility of rebalancing. The heterogeneity in risk preferences and age generates trade in inelastically supplied risky assets and an endogenously supplied risk-free asset.

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The model lets us ask who should buy and who should sell when the market fluctuates. It also helps us understand what considerations are the most relevant when agents decide how to rebalance. It goes beyond the observation that the Sharpe ratio might be time-varying and thus portfolio shares should shift over time. The general equilibrium effects we find are not driven mainly by the time-varying expected returns, but rather by the need for households to rebalance consistently when the distribution of wealth changes endogenously.

Section 2 describes the key findings of our model. We also motivate our findings with an example of a market crash similar to the crash of 2008/9 that illustrates the general equilibrium logic without solving a full model. In Section 3 we describe the overlapping generations model in detail. The model has a partial closed form solution, which makes the intuition more transparent and also eases the solution of the model. Section 4 presents our quantitative solution of the model. Section 5 discusses how our research relates to other research. Section 6 presents our conclusions.

## 2    Preview

Optimal portfolio rebalancing deviates from the standard advice for at least three reasons. First, there is a *leverage effect*. Since the risk-free asset is in zero net supply in our model, there will always be some investors that hold a levered position in the risky asset. For those with a levered position, their risky asset share actually declines as the market rises, and vice versa. The levered investors need to buy when the market rises and sell when it falls, which is the opposite of standard rebalancing. Second, there is an *aggregate risk tolerance effect*. As the wealth of different types of agents shifts, aggregate risk tolerance also shifts, which in turn affects equilibrium prices and expected returns. Movements in aggregate risk tolerance induce people to trade in response to market movements instead of maintaining a constant share. In our specification, the aggregate risk tolerance effect is substantial. Third, there is an *intertemporal hedging effect*. Since future expected returns are time-varying, young agents have an incentive to hedge changes in their investment opportunities when they are middle aged. In principle, intertemporal hedging can have a significant effect on asset demand, but

for reasons we will explain, in our specification it is actually quite difficult to generate enough mean reversion in asset returns to generate a substantial intertemporal hedging effect.

In Section 2.1, we give a numerical example to illustrate the leverage effect. In Section 2.2, we give the simple analytics of the aggregate risk tolerance effect. The intertemporal hedging effect is more complicated. We defer discussing it until Section 3 after we present the formal model. Finally, we give a simple example motivated by the recent stock market crash in Section 2.3 that illustrates how the leverage effect and the aggregate risk tolerance effect operate in general equilibrium.

## 2.1  Leverage Effect

A simple example can help clarify the leverage effect. Suppose an investor wants to maintain a levered portfolio with a share of 200% in the risky asset. With a $100 stake to begin with, the investor borrows $100 to add to that stake and buys $200 worth of the risky asset. If the risky asset suddenly doubles in value, the investor then has $400 worth of the risky asset and an unchanged $100 of debt, for a net financial wealth of $300. The standard advice would be to sell the risky asset since it has risen in value. However, after this increase in the price of the risky asset, the share of the investor's net financial wealth in the risky asset is only $400/300 = 133\%$, since the investor's net financial wealth has risen even more than the value of the risky asset. Therefore, if the goal is to maintain a 200% risky asset share, the investor needs to borrow more to buy additional shares. Hence, levered investors rebalance in a direction that is the opposite of the standard advice.

## 2.2  Aggregate Risk Tolerance Effect

The aggregate risk tolerance effect is a consequence of the general equilibrium adding up. It arises because the share of the risky asset held by an individual is a function of that individual's risk tolerance relative to aggregate risk tolerance and aggregate risk tolerance must move in general equilibrium when there are shocks that affect the distribution of wealth. We will now show that this feature of the general equilibrium makes it impossible for all

investors to maintain a constant portfolio share. Returns adjust to make people willing to adjust their portfolio shares in a way that allows the adding-up constraint to be satisfied.

To see this, consider the aggregation of an economy characterized by individuals who had constant portfolio shares as in Merton (1971). That is, an individual $i$'s share of investable wealth $\mathcal{W}_i$ held in the risky asset would be

$$\theta_i \equiv \frac{PS_i}{\mathcal{W}_i} = \tau_i \frac{\mathcal{S}}{\sigma}, \tag{1}$$

where $P$ is the price of the risky asset and $S_i$ is the amount of the risky asset. The only heterogeneity across individuals comes from individual risk tolerance $\tau_i$ and the only hetero-geneity over time comes from a time-varying expected Sharpe ratio $\mathcal{S}$ or volatility $\sigma$. Recall that the risk free asset is in zero net supply, so the total value of all investable wealth must equal the total value of the risky asset. Multiplying (1) through by each type's wealth and aggregating across types yields

$$\sum_i \theta_i \mathcal{W}_i = \sum_i PS_i = \left[ \sum_i \mathcal{W}_i \tau_i \right] \frac{\mathcal{S}}{\sigma} = \mathcal{W}. \tag{2}$$

Dividing equation (2) through by aggregate wealth and comparing the resulting expression to equation (1) allow us to derive an expression for the risky asset share that does not depend on the distribution of the asset's return:

$$\theta_i = \tau_i \left[ \sum_i \frac{\mathcal{W}_i \tau_i}{\mathcal{W}} \right]^{-1} = \frac{\tau_i}{\mathcal{T}}, \tag{3}$$

where $\mathcal{T} = \sum_i \frac{\mathcal{W}_i \tau_i}{\mathcal{W}}$ is aggregate risk tolerance.

Now suppose that there is a positive shock to the value of the risky asset $P$, but the expected return going forward remains unchanged. In the Merton world, the portfolio share as described in equation (1) remains unchanged. But note that the shock to $P$ will affect different investors differently. Those with high risk tolerance who held a disproportionate share of the risky asset prior to the shock earn a higher total return than those with low risk

tolerance. Thus, after a positive shock, a higher fraction of all wealth will be held by those who have high risk tolerance and therefore a high share of the risky asset in their portfolios. If each type continued to hold the same share in risky assets, the overall share of risky assets held would have to increase as a greater fraction of all wealth is held by those with a high risky asset share. But this is impossible, since with the risky asset representing a share in all the capital income of the economy, the overall share of risky assets must be 1. Thus, the portfolio shares of each type cannot remain unchanged.

For the simple Merton model to yield market-clearing behavior, the aggregate risk tolerance $\mathcal{T}$ would have to be constant. But aggregate risk tolerance cannot be constant. After a positive shock to $P$, the high risk tolerance investors who invest disproportionately in the risky asset will have increased wealth weight, so aggregate risk tolerance increases. Our model, described in the next section, will make precise how the aggregate risk tolerance and individual portfolio shares move together consistently in general equilibrium.

This illustration of the failure of the constant portfolio share model does not assume time-variation in asset returns. It indicates instead that general equilibrium considerations stemming from a fixed asset supply will force the necessary time-variation in asset returns to counterbalance the effects of changes in aggregate risk tolerance on asset demand. That is, even without knowing what is happening to the asset return distribution, one can predict that the change in aggregate risk tolerance will affect portfolio decisions going forward. We illustrate this point with the following example.

## 2.3   Rebalancing After a Stock Market Crash

Many of our key results can be illustrated with a simple numerical example that resembles the stock market crash of 2008/9. Suppose we have an economy with equal masses of two types of agents, daring and cautious investors. Let the daring investors have risk aversion coefficients of 2 and an initial equity share of 0.90 and let the cautious have corresponding values of 6 and 0.30. Suppose further that the leverage in the economy is such that equity represents 60% of the value of all the productive assets in positive net supply in the economy,

or the "tree." Hence, the daring agents' exposure to the tree is actually 150% ($= 0.90/0.60$).[1]
With these initial values summarized in column (1) of Table 1, consider the effect of a stock
return of $-35\%$, which is approximately the return realized from mid-2007 to mid-2009.

Table 1: Market Crash Example

| | Pre-Crash (1) | Before Rebalancing (2) | Rebalancing Strategy | | |
| | | | Constant Stock Share (3) | Constant Tree Share (4) | Equilibrium (5) |
|---|---|---|---|---|---|
| Daring Stock Share | 0.90 | 0.85 | 0.90 | 0.74 | 0.79 |
| Daring Bond Share | 0.10 | 0.15 | 0.10 | 0.26 | 0.21 |
| Daring Tree Share | 1.50 | 1.73 | 1.82 | 1.50 | 1.61 |
| Cautious Stock Share | 0.30 | 0.22 | 0.30 | 0.25 | 0.26 |
| Cautious Bond Share | 0.70 | 0.78 | 0.70 | 0.75 | 0.74 |
| Cautious Tree Share | 0.50 | 0.44 | 0.61 | 0.50 | 0.54 |
| Total Stock Share | 0.60 | 0.49 | 0.56 | 0.46 | 0.49 |
| Total Bond Share | 0.40 | 0.51 | 0.44 | 0.54 | 0.51 |
| Total Tree Share | 1.00 | 1.00 | 1.13 | 0.93 | 1.00 |
| Total Stock Value | 60 | 39 | 44 | 36 | 39 |
| Total Bond Value | 40 | 40 | 35 | 43 | 40 |
| Total Tree Value | 100 | 79 | 79 | 79 | 79 |

Column (2) of the table describes the effect of the market crash before any rebalancing.
The stock return of $-35\%$ causes the value of the stock to drop from 60 to 39, and the stock
share of the economy drops to 49%.[2] The stock share of the daring investors drops from 0.9
to $0.85 = (0.90 \times 0.65)/(0.90 \times 0.65 + 0.10)$, and the stock share of the cautious agents drops
similarly. Importantly, while the market crash causes the daring investors' stock share to
drop, it has the opposite effect on their tree share, which rises from 1.5 to $1.73 = 0.85/0.49$
due to the leverage effect described above. The tree share for the cautious investors drops to
$0.44 = 0.22/0.49$ along with the stock share. The wealth weights of the daring and cautious
agents move from 0.50 and 0.50 to 0.43 and 0.57, respectively. The allocation remains feasible
because the investors have not traded after the stock market crash.

Now we consider three possible rebalancing strategies that agents might engage in. The
first strategy is to follow the standard advice that each agent returns to his or her initial

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[1]In our model the focuses is on the "tree" as the risky asset. We thus set aside the corporate finance issue
of the degree of leverage underlying stocks. At the end of Section 5, we return to how our analysis relates
to portfolio advice in terms of stock share rather than risky-asset share.

[2]The value of the tree falls 21%. The value of the stock falls by more because the stock is a levered claim.

stock share. Since the stock market has fallen in value, this strategy requires that both the cautious and daring agents buy more stocks. As the calculations in column (3) of the table show, this strategy leads to 13% excess demand for both the stock and the tree. The standard rebalancing advice is infeasible in aggregate. The second rebalancing strategy we consider is to maintain a constant portfolio share of the tree, as illustrated in column (4) of the table. Since the daring investors' exposure to the tree has risen, they need to sell stock to maintain a constant tree share. The cautious investors need to buy stock. This strategy leads to 7% excess supply of the stock and the tree.

Both the constant stock share and constant tree share strategies ignore how changes in aggregate risk tolerance should affect the tree share. Initial aggregate risk tolerance is $0.333 = 0.5 \times 1/2 + 0.5 \times 1/6$, corresponding to aggregate risk aversion of 3. After the stock market crash, aggregate risk tolerance moves to $0.311 = 0.433 \times 1/2 + 0.567 \times 1/6$, corresponding to a risk aversion of 3.22. Since the market crash was a negative shock, the cautious investors have more of the total wealth in the economy, and thus aggregate risk tolerance falls.

The third rebalancing strategy we consider, which is described in column (5) of the table, is equilibrium rebalancing. This calculation accounts for both the leverage and aggregate risk tolerance effect by rescaling the daring and cautious tree shares ($1.61 = 1.50/0.93$ and $0.54 = 0.50/0.93$) so that they sum to one when weighted by wealth ($1.61 \times 0.43 + 0.54 \times 0.57 = 1$). These tree exposures then imply stock shares, bond shares, stock values and bond values that are consistent with adding up constraints. Comparing column (1) and (4) shows that both the daring and cautious investors rebalance in the same direction. The daring investors move their tree share from 1.5 to 1.61 and the cautious investors move theirs from 0.5 to 0.54. Given what happens passively to the tree shares after the negative stock return, the daring investors achieve their new lower share by selling (taking their tree share from 1.73 to 1.61) and the cautious investors achieve it by buying (taking their share from 0.44 to 0.54).

The example in Table 1 can be easily adapted to an increase in asset prices. If prices rise 35% instead of dropping, the daring agents' stock share passively goes up from 0.9 to

1.22 and after equilibrium rebalancing it becomes 1.27, again with the daring agents trading in the opposite direction of standard rebalancing. The cautious agents' stock share goes up from 0.3 to 0.41 passively and after equilibrium rebalancing it is 0.36.

The equilibrium strategy described here is quite close to the optimal strategy that results from our modeling exercise. While we have not solved for full optimizing behavior in the example, the main findings of the model are reflected in it. We now turn to the model to make precise the implications delivered by this example.

# 3 Model

The model economy has an infinite horizon and overlapping generations. There are two types of agents with different degrees of risk tolerance. The level of risk tolerance is randomly assigned at the beginning of the agent's life and remains unchanged throughout the life. Each agent lives for three equal periods: young ($Y$), middle-aged ($M$) and old ($O$). Agents work in a representative firm and receive labor income when they are young, and live off their savings when they are middle-aged and old. All agents start their life with zero wealth, and do not leave assets to future generations; that is, they have no bequest motive. There is no population growth, so there is an equal mass across the three cohorts. In this section, we provide the details of how we model and specify the technology, the asset markets, and the agents' problem.

## 3.1 Technology and Production

Our setup has a character very similar to that of an endowment economy. There is a fixed supply of productive assets or "trees." Hence, we have a Lucas (1978) endowment economy, though unlike Lucas, we will have trade because not all agents are identical. The young supply labor inelastically. Trees, combined with labor, produce output that is divided into labor income and dividend income. The labor income stream serves to provide resources to each generation. The dividend income provides a return to risky saving. This setup allows

us to provide resources to each generation without inheritance. By providing labor income only at the start of the life, we intentionally abstract from the interaction of human capital with the demand for risky assets. While the effect of human capital on portfolio choice can be important, it is not the topic of this paper.

The production function is Cobb-Douglas $Y_t = \frac{1}{\alpha} D_t K_t^\alpha L_t^{1-\alpha}$, where $K_t$ is the fixed quantity of trees, $L_t$ is labor, $\alpha$ is capital's share in production, and $D_t$ is the stochastic shock to technology. We normalize the quantity of trees $K_t$ to be one. For simplicity, all labor will be supplied by young agents. The quantity of labor is normalized to be one. Therefore, total output is $Y_t = \frac{D_t}{\alpha}$, total labor income is $\frac{(1-\alpha)D_t}{\alpha}$, and total dividend income is $D_t$. Given our normalization, technology shocks will also be stochastic dividends. All labor income goes to the young. Dividend income will be distributed to the middle-aged and old agents depending on their holdings of the risky tree.

In order not to build in mean reversion mechanically, in our benchmark case we assume the growth rate $d_{t+1} = D_{t+1}/D_t$ is serially uncorrelated. In our calculations, we parameterize the dividend growth as having finite support $\{G_n\}_{n=1}^N$, where $G_n$ occurs with probability $\pi_n$ each period. The dividend growth shocks are realized at the beginning of each period before any decisions are made.

## 3.2 Assets

Before turning to agents' problem, it is helpful to define assets and returns. In our economy, there are two assets available to agents: one risk-free discount bond and one risky tree security. The risk-free bond has a net zero supply, and the risky tree has a fixed supply of one. In our benchmark case, we parameterize the dividend growth process to have two states. These two assets provide a complete market.

One unit of a bond purchased at period $t$ pays 1 unit of consumption in period $t+1$ regardless of the state of the economy then. A bond purchased at time $t$ has a price of $1/R_t$, where $R_t$ is the gross risk-free interest rate. One unit of the tree has a price $P_t$ in period $t$ and pays $D_{t+1} + P_{t+1}$ in period $t+1$. Thus, the gross tree return $R_t^{\text{Tree}}$ is $\frac{P_{t+1}+D_{t+1}}{P_t}$, and the

excess tree return is $Z_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t R_t}$.

## 3.3 Agents' Problem

Since this paper focuses on asset allocation, we specify preferences so that there is meaningful heterogeneity in risk tolerance, but the consumption-saving tradeoff is simple. We use the Epstein-Zin-Weil preferences with a unit elasticity of intertemporal substitution (Epstein and Zin 1989, Weil 1990). The utility of agents born in date $t$ with risk tolerance $\tau_j = \frac{1}{\gamma_j}$ is given by

$$\rho_Y \ln(C_{Yt}^j) + \frac{1 - \rho_Y}{1 - \gamma_j} \ln E_t \exp\left((1 - \gamma_j)\rho_M \ln(C_{Mt+1}^j) + (1 - \rho_M)\ln\left(E_{t+1}(C_{Ot+2}^j)^{1-\gamma_j}\right)\right), \quad (4)$$

where $C_{Yt}^j$, $C_{Mt+1}^j$ and $C_{Ot+2}^j$ denote consumption at young, middle and old ages, and $\rho_Y$ and $\rho_M$ govern time preference when young and when middle-aged. These parameters $\rho_Y$ and $\rho_M$ will turn out to be the average propensities to consume. With an underlying discount factor equal to $\delta$, we have $\rho_Y = \frac{1}{1+\delta+\delta^2}$ and $\rho_M = \frac{1}{1+\delta}$. The degree of risk tolerance can be either high ($j = H$) or low ($j = L$). Because this utility has a recursive form, we work backwards.

*Old Agents*

The problem for old agents at date $t$ is trivial. They simply consume all of their wealth, so total consumption of the old $C_{Ot}$ equals the total wealth of the old $W_{Ot}$.

*Middle-aged Agents*

A middle-aged agent at date $t$ chooses consumption when middle-aged, expected consumption when old, bond holding $B_{Mt}$ and tree holding $S_{Mt}$ to maximize intertemporal utility

$$\rho_M \ln(C_{Mt}^j) + (1 - \rho_M)\ln\left(E_t(C_{Ot+1}^j)^{1-\gamma_j}\right)^{\frac{1}{1-\gamma_j}}, \quad (5)$$

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subject to the budget constraints

$$C_{Mt}^j + B_{Mt}^j/R_t + P_t S_{Mt}^j = W_{Mt}^j, \tag{6}$$

$$C_{Ot+1}^j = W_{Ot+1}^j = B_{Mt}^j + S_{Mt}^j(P_{t+1} + D_{t+1}). \tag{7}$$

Recall that the middle-aged agents have no labor income, so they must live off of their wealth $W_{Mt}^j$.

Consider the portfolio allocation of this middle-aged agent. It is convenient to define the *certainty equivalent total return*

$$\phi_{Mt}^j = \max_{\theta \in \Theta} \left\{ E_t \left\{ (1 - \theta)R_t + \theta R_t Z_{t+1} \right\}^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}}, \tag{8}$$

where $\theta$ is the share of invested wealth in the risky asset. Given the homogeneity of preferences and that there is no labor income after the young period, the optimal portfolio share $\theta_{Mt}^j$ that yields $\phi_{Mt}^j$ also maximizes expected utility from consumption when old.

Thus, the middle-aged agent equivalently solves the following decision problem:

$$\max_{C_{Mt}^j, \theta \in \Theta} \rho_M \ln(C_{Mt}^j) \quad + \quad (1 - \rho_M) \ln(W_{Mt}^j - C_{Mt}^j)$$

$$+ \quad (1 - \rho_M) \ln \left\{ E_t \left\{ (1 - \theta)R_t + \theta R_t Z_{t+1} \right\}^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}}.$$

Since only the first two terms depend on $C_{Mt}^j$, the first order condition for $C_{Mt}^j$ is

$$\frac{\rho_M}{C_{Mt}^j} = \frac{1 - \rho_M}{W_{Mt}^j - C_{Mt}^j},$$

which has the solution

$$C_{Mt}^j = \rho_M W_{Mt}^j.$$

The simple consumption function follows from log utility in the intertemporal dimension, which makes the saving and portfolio choices independent. Thus, the maximized utility of

the middle aged is given by

$$V_{Mt}^j(W_{Mt}^j) = \rho_M \ln(\rho_M W_{Mt}^j) + (1 - \rho_M) \ln\left((1 - \rho_M)W_{Mt}^j \phi_{Mt}^j\right). \tag{9}$$

Bond holdings $B_{Mt}^j$ are given by $(1 - \theta_{Mt}^j)(1 - \rho_M)W_{Mt}^j R_t$ and tree holdings $S_{Mt}^j$ are given by $\theta_{Mt}^j(1 - \rho_M)W_{Mt}^j/P_t$.

*Young Agents*

With no initial wealth, the young have only their labor income for consumption and saving. Labor income of a young agent in period $t$ is $W_{Yt} = (1 - \alpha)D_t/\alpha$. A young agent of type $j$ decides on consumption, bond and tree holdings $(B_{Yt}^j, S_{Yt}^j)$ (that together determine middle-aged wealth $W_{Mt+1}^j$) to maximize the utility

$$\rho_Y \ln(C_{Yt}^j) + (1 - \rho_Y) \ln\left(E_t \exp((1 - \gamma_j)V_{Mt+1}^j(W_{Mt+1}^j))\right)^{\frac{1}{1-\gamma_j}} \tag{10}$$

subject to the budget constraints

$$C_{Yt}^j + B_{Yt}^j/R_t + P_t S_{Yt}^j = W_{Yt}^j \tag{11}$$

and

$$W_{Mt+1}^j = B_{Yt}^j + S_{Yt}^j(P_{t+1} + D_{t+1}). \tag{12}$$

Substituting the maximized middle-aged utility given by equation (9) into (10) yields the following decision problem for a young agent:

$$\max_{C_{Yt}^j, \theta \in \Theta} \quad \rho_Y \ln(C_{Yt}^j) + (1 - \rho_Y) \ln(W_{Yt}^j - C_{Yt}^j) \tag{13}$$

$$+ (1 - \rho_Y) \ln\left\{ \left[E_t\left((1 - \theta)R_t + \theta R_t Z_{t+1})(\phi_{Mt+1}^j)^{1-\rho_M}\right)^{1-\gamma_j}\right]^{\frac{1}{1-\gamma_j}} \right\}$$

$$+ (1 - \rho_Y)\left[\rho_M \ln(\rho_M) + (1 - \rho_M)\ln(1 - \rho_M)\right].$$

Only the first two terms depend on $C_{Yt}^j$. Again because of log utility, optimal consumption

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is given by

$$C_{Yt}^j = \rho_Y W_{Yt}^j.$$

Since only the third term of equation (4) depends on $\theta$, the optimal tree portfolio share $\theta_{Yt}^j$ for the young maximizes the certainty equivalent, time-aggregated total return function for the young $\phi_{Yt}^j$:

$$\phi_{Yt}^j = \max_{\theta \in \Theta} \left\{ E_t \left\{ [(1-\theta)R_t + \theta R_t Z_{t+1}](\phi_{Mt+1}^j)^{1-\rho_M} \right\}^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}}. \tag{14}$$

Comparing the middle-aged portfolio problem in (8) with the young portfolio problem in (14), we find that the young solve a more complicated problem given their longer life span. Middle-aged agents with only one more period life simply maximize the certainty equivalent return of their portfolio investment next period, as in equation (8). Young agents with two more periods to live care about the certainty equivalent returns of their portfolio investment when they are middle-aged and old, and the covariance between these two returns.

*Intertemporal Hedging and the Target Variable Annuity*

Since future expected returns are time-varying, the multi-period decision problem of the young creates an opportunity for intertemporal hedging (Merton 1973). To sharpen the interpretation of intertemporal hedging, define

$$Q_{Mt+1}^j = \frac{1}{\rho_M^{\rho_M}(1-\rho_M)^{1-\rho_M}(\phi_{Mt+1}^j)^{1-\rho_M}}, \tag{15}$$

which prices the target variable annuity that yields—at lowest cost—a lifetime utility equal to the lifetime utility from one unit of consumption at period $t+1$ and one unit of consumption at period $t+2$. That is, the agent will be indifferent between being limited to consuming one unit of consumption at both middle and old age and having wealth of $Q_{Mt+1}^j$ to deploy optimally. To see this, note that by (5), consumption of 1 at both middle and old age yields a lifetime utility of 0 for the middle-aged agent. Middle-aged wealth of $Q_{Mt+1}$ must therefore make the middle-aged lifetime utility given by (9) equal to zero once the optimal

consumption and portfolio share are substituted in:

$$\rho_M \ln(\rho_M Q^j_{Mt+1}) + (1 - \rho_M) \ln((1 - \rho_M) Q^j_{Mt+1}) + (1 - \rho_M) \ln(\phi^j_{Mt+1}) = 0. \qquad (16)$$

The value of $Q^j_{Mt+1}$ given in (15) is the solution to (16). The target variable annuity itself implicitly replicates the optimal portfolio strategy for the agent. Given this definition of the target annuity price, equation (14) can be rewritten as follows:

$$\phi^j_{Yt} = \rho_M^{-\rho_M} (1 - \rho_M)^{\rho_M - 1} \max_{\theta \in \Theta} \left\{ E_t \left\{ \frac{(1 - \theta) R_t + \theta R_t Z_{t+1}}{Q^j_{Mt+1}} \right\}^{1-\gamma_j} \right\}^{\frac{1}{1-\gamma_j}}. \qquad (17)$$

Thus, a young agent maximizes the simple certainty equivalent of his or her portfolio's return relative to the return on the target variable annuity. In other words, the target variable annuity serves as a numeraire for returns. Because the target variable annuity is by its nature an asset as long-lived as the agent, equation (17) embodies the intuition that portfolio returns should be judged relative to an appropriate long-lived asset rather than (as is often done in practice) relative to short-term assets such as Treasury Bills.

Note that, because of its definition as the amount of wealth yielding lifetime utility equal to the lifetime utility from consumption constant at one, equation (9) can be rewritten

$$V^j_{Mt}(W^j_{Mt}) = \ln(W^j_{Mt}) - \ln(Q^j_{Mt}). \qquad (18)$$

Defining the corresponding price of a target variable annuity in youth that can give lifetime utility equivalent to the lifetime utility of one unit of consumption in each of the three periods of life, $[Q^j_{Yt}]^{-1} = \rho_Y^{\rho_Y} (1 - \rho_Y)^{1-\rho_Y} [\rho_M^{\rho_M} (1 - \rho_M)^{1-\rho_M}]^{1-\rho_Y} (\phi^j_{Yt})^{1-\rho_Y}$, the maximized utility for a young agent that results from substituting in the optimal decisions can be similarly written as

$$V^j_{Yt}(W^j_{Yt}) = \ln(W^j_{Yt}) - \ln(Q^j_{Yt}).$$

Thus, the price of the target variable annuity fully captures the dependence of the value

function on the investment opportunity set. Note that, unless $\gamma = 1$, wealth and $Q$ are no longer additively separable after the application of the appropriate curvature for risk preferences.

## 3.4 Market Clearing

In our economy, the safe discount bonds are in zero net supply, and the risky tree has an inelastic supply normalized to one. We now substitute the asset demand we derived in the previous subsection into the asset market clearing conditions. Recall that the asset demands of the young are given by $S_{Yt}^j = (1 - \rho_Y)W_{Yt}^j \theta_{Yt}^j / P_t$ and $B_{Yt}^j = (1 - \rho_Y)W_{Yt}^j R_t (1 - \theta_{Yt}^j)$. The asset demands of the middle-aged are given by $S_{Mt}^j = (1 - \rho_M)W_{Mt}^j \theta_{Mt}^j / P_t$ and $B_{Mt}^j = (1 - \rho_M)W_{Mt}^j (1 - \theta_{Mt}^j)R_t$. The old have no asset demand. Summing over the type $j$'s and the generations, the asset market clearing conditions can be written as

$$1 = \frac{1}{P_t} \sum_{j=H,L} \psi_j \left[ (1 - \rho_M)W_{Mt}^j \theta_{Mt}^j + (1 - \rho_Y)W_{Yt}^j \theta_{Yt}^j \right] \tag{19}$$

and

$$0 = R_t \sum_{j=H,L} \psi_j \left[ (1 - \rho_M)W_{Mt}^j (1 - \theta_{Mt}^j) + (1 - \rho_Y)W_{Yt}^j (1 - \theta_{Yt}^j) \right], \tag{20}$$

where $\psi_j$ is the constant fraction of labor income going to young agents of type $j$.

In addition to the asset market clearing conditions, the goods market clearing condition is also instructive. It is

$$\frac{D_t}{\alpha} = \frac{\rho_Y(1-\alpha)D_t}{\alpha} + \rho_M(\psi_H W_{Mt}^H + \psi_L W_{Mt}^L) + W_{Ot}, \tag{21}$$

where the wealth level of the old $W_{Ot}$ is given by the sum across the two types. Note that the propensity to consume out of wealth is 1 for the old, so there is no parameter multiplying $W_{Ot}$. Since the asset demands for old agents are equal to zero regardless of the level of risk aversion, we do not need to keep track separately of the wealth of each old group.

The total financial wealth of the economy is the sum of the value of the trees $P_t$ and the

dividend $D_t$, since the risk-free asset is in net zero supply. At the beginning of the period the total financial wealth is held by the middle-aged and the old. Hence, the total wealth of the old is $W_{Ot} = P_t + D_t - \psi_H W_{Mt}^H - \psi_L W_{Mt}^L$. Substituting this expression into equation (21) and solving for $P_t$ gives rise to an equation for $P_t$ based on only contemporaneous variables:

$$P_t = (1 - \rho_Y)\frac{1-\alpha}{\alpha}D_t + (1 - \rho_M)(\psi_H W_{Mt}^H + \psi_L W_{Mt}^L). \tag{22}$$

The right-hand side of (22) can be interpreted simply as the sum of saving supplies—and therefore total asset demands—of the young and middle aged, determined by their propensities to save and initial levels of wealth.

Normalizing all the variables in the above equation by dividend $D_t$ gives the expression for the price-dividend ratio $p_t$:

$$p_t = (1 - \rho_Y)\frac{1-\alpha}{\alpha} + (1 - \rho_M)(\psi_H w_{Mt}^H + \psi_L w_{Mt}^L), \tag{23}$$

where lower case letters denote the corresponding variables divided by the dividend. Thus, the price-dividend ratio $p_t$ is linear in the total (per-dividend) middle-aged wealth $\psi_H w_{Mt}^H + \psi_L w_{Mt}^L$. The first term on the right-hand side of (23) corresponds to the demand for saving of the young, which is a constant share of dividends and therefore a constant in the per-dividend expression.

Having an analytic expression for the price-dividend ratio both adds transparency to our analysis and simplifies the solution of the model. Analytic expressions for portfolio shares and expected returns for the risky and risk-free assets are not available. In the next section, we will turn to numerical solutions of the model. Unlike the price-dividend ratio, which only depends on total middle-aged wealth, portfolio shares and expected returns will depend on the distribution of wealth among the middle-aged agents. Intuitively, this distribution depends on the history of shocks. For example, as we will discuss explicitly in the next section, a good dividend growth shock raises the share of wealth held by the middle-aged risk-tolerant agents who invested heavily in the risky asset when young. This increased

wealth share of the risk-tolerant agents in turn affects asset demands and expected returns.

## 3.5   Aggregate Risk Tolerance Revisited

Recall that in Merton's partial equilibrium environment the portfolio share depends analytically on the ratio of individual to aggregate risk tolerance as in equation (3). In terms of our model, aggregate risk tolerance is

$$\mathcal{T}_t = \frac{\mathcal{W}_{Yt}^H \tau_H + \mathcal{W}_{Mt}^H \tau_H + \mathcal{W}_{Yt}^L \tau_L + \mathcal{W}_{Mt}^L \tau_L}{\mathcal{W}_t}, \tag{24}$$

where the invested wealth of each type $j = \{H, L\}$ and $a = \{Y, M\}$ is $\mathcal{W}_{at}^j = (1 - \rho_a)\psi_j w_{at}^j D_t$ and the total invested wealth across all types is $\mathcal{W}_t = \mathcal{W}_{Yt}^H + \mathcal{W}_{Mt}^H + \mathcal{W}_{Yt}^L + \mathcal{W}_{Mt}^L$.[3] There is no analytic expression for portfolio shares corresponding to equation (3) in our model. Nonetheless, changes in aggregate risk tolerance have similar effects on portfolio shares. For example, when aggregate risk tolerance is high, there is a tendency for all agents to reduce their share of the risky asset. In the next section, we show that aggregate risk tolerance remains a useful construct for understanding the demand for the risky asset and for the determination of expected returns in our model. In particular, the combination of the price-dividend ratio and aggregate risk tolerance is sufficient to fully characterize the state of the model economy at any point in time.

# 4   Quantitative Results

In this section, we examine numerical solutions of a particular parameterization of our model. We first show that the model generates mean reversion in asset prices under i.i.d. dividend growth rate shocks as a result of general equilibrium. The aggregate risk tolerance in the economy varies over time as the wealth distribution changes in response to the dividend growth rate shocks.

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[3]Note that the relevant wealth for this aggregation in our model is invested wealth, i.e., wealth after consumption. In the continuous-time Merton model, there is no consumption during the investment period, so the distinction between before- and after-consumption wealth is absent.

We then examine the optimal portfolio allocations and rebalancing behavior of households in general equilibrium. Portfolios vary across agents with different risk tolerance. In particular, the optimal portfolio share is larger than the market portfolio share (100%) for the high-risk-tolerance agents, and lower for the low-risk-tolerant agents. As in our market crash example in Section 2.3, the portfolio tree shares of all types comove in response to the dividend growth rate shock: declining after a good shock and rising after a bad shock. The underlying wealth dynamics ensure that such comovements in the portfolio shares are consistent with general equilibrium.

## 4.1  Parameterization and Solution

Table 2 summarizes the parameter values used in the quantitative analysis. A model period corresponds to 20 years in the data. The 20-year discount rate of 0.67 corresponds to an annual discount rate of 0.98. Consequently, $\rho_M$ is 0.75 and $\rho_Y$ is 0.69. Capital's share is 0.33. The cross-sectional heterogeneity in risk tolerance is set to match the findings in Kimball, Sahm and Shapiro (2008). They estimate the mean risk tolerance to be 0.21 and the standard deviation to be 0.17. The distribution they estimate has skew of 1.84. We match these three moments by giving 92% of agents a low risk tolerance $\tau_L$ of 0.156 and 8% of agents a high risk tolerance $\tau_H$ of 0.797. We refer to the low-risk-tolerance agents as *cautious agents* and the high-risk-tolerance agents as *daring agents*.[4]

The dividend growth shock process is assumed to be i.i.d. with two states of equal probability. In the bad state, the detrended gross growth rate is $G_1 = 0.67 \approx (\frac{1.005}{1.025})^{20}$ and in the good state, it is $G_2 = 1.5 \approx (\frac{1.045}{1.025})^{20}$. With our 20-year time horizon, these parameters imply that the dividend growth rate is 2.5% per year in expected terms, 0.5% per year in the bad state and 4.5% per year in the good state. The dividend growth rate shocks capture generational risk. The mean scenario mimics the experience of the United States, the good scenario mimics that of South Korea, and the bad scenario mimics that of the Japan in the last 2 decades. Since all key variables below are expressed on a per dividend basis, and the

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[4]Note that this parameterization differs from the illustration in Section 2.3.

utility function is isoelastic with an elasticity of intertemporal substitution equal to 1, the trend growth rate of 2.5% per year does not affect any of our analysis.

Table 2: Benchmark Parameter Values

| | | |
|---|---|---|
| Capital's Share | $\alpha$ | 0.33 |
| Discount Factor | $\beta$ | 0.67 |
| Middle-aged Average Propensity to Consume | $\rho_M$ | 0.75 |
| Young Average Propensity to Consume | $\rho_Y$ | 0.69 |
| | | |
| Low Risk Tolerance | $\tau_L$ | 0.156 |
| High Risk Tolerance | $\tau_H$ | 0.797 |
| Fraction of Low Risk Tolerance Agents | $\psi_L$ | 0.92 |
| Fraction of High Risk Tolerance Agents | $\psi_H$ | 0.08 |
| | | |
| i.i.d. Dividend Growth Shock Process | | |
| Detrended Growth Rate | $(G_1, G_2)$ | $(0.67, 1.50)$ |
| Probability | $(\pi_1, \pi_2)$ | $(0.50, 0.50)$ |

Our model does not have an analytical solution, and so we solve the model numerically. To make the model stationary, we detrend relevant variables with same-dated dividends. Given six types of agents and two types of assets in our model, it might seem that the dimension of the state space would be large. As shown in section 3.4, however, we are able to reduce the state space to two endogenous variables: the wealth levels of the middle-aged cautious and middle-aged daring agents $(w_{Mt}^H, w_{Mt}^L)$. We are able to omit the dividend growth rate from the state space due to the i.i.d. assumption. We discretize the state space in our numerical solution, but we allow the decision variables to be continuous. We use the recursive dynamic technique to solve the model. For the detailed computational algorithm see the Appendix.

## 4.2   Price and Return Dynamics

We now present the quantitative behavior of our model economy, beginning with price and return dynamics. Table 3 presents the summary statistics based on a model simulation of 10,000 periods, which is a sufficiently long sample to eliminate the simulation uncertainty. Returns are expressed as annual rates. The log risk free rate has a mean of 1.92% and

a standard deviation of 0.97%. The log realized tree return is on average 3.62% with a standard deviation of 8.95%. Thus, the model economy generates a 1.70% risk premium with a standard deviation of 8.94%. Although our model is not specifically designed to deliver a large risk premium, it goes a substantial fraction of the way toward explaining the historical premium and its volatility.

Table 3: Summary Statistics on Prices, Annualized

| Variable | Mean | Standard Deviation | Correlation with $G$ | Auto Correlation |
|---|---|---|---|---|
| log(Risk Free Rate) | 1.92% | 0.97% | 0.80 | −0.05 |
| log(Realized Tree Return) | 3.62% | 8.95% | 1.00 | 0.02 |
| log(Realized Excess Return) | 1.70% | 8.94% | 1.00 | −0.07 |
| log(Expected Tree Return) | 3.99% | 0.15% | 0.56 | −0.19 |
| log(Expected Excess Return) | 2.07% | 0.48% | −0.87 | 0.25 |
| Tree Price-Dividend Ratio | 18.94 | 0.54 | −0.47 | −0.25 |

The model's expected returns and price-dividend ratio, on the other hand, are much less volatile than in historical data. For instance, the price-dividend ratio is on average 19, with a standard deviation of 0.54. Because shocks to the fundamentals are i.i.d. in the growth rate, movements in the realized return on the risky asset move essentially one-for-one with the dividend shock. Moreover, the i.i.d. growth shock means that any forecastability of returns comes from general equilibrium effects, not from the fundamentals *per se*. It turns out that these general equilibrium have a contribution to returns that is relatively small compared to sheer good and bad luck, so there is relatively little serial correlation in the realized risky return and the risk-free rate.

The expected excess return is critical for portfolio choice. It is strongly negatively correlated with the dividend growth rate shock and is somewhat mean reverting. Mechanically, these results are coming from movements in the risk-free rate, which is highly correlated with the shock to the dividend growth rate. Intuitively, the risk-free rate is moving because the shock to dividends changes the wealth of daring agents and therefore shifts the aggregate risk tolerance.

The endogenous wealth dynamics across agents are central to understanding the price dynamics. The history of shocks matters because it affects the distribution of wealth and

therefore the aggregate risk tolerance. To see how the history of shocks matters, Figure 1 shows the equilibrium relationship between price-dividend ratios and the aggregate risk tolerance in period $t$ for different combinations of the shocks in period $t-1$ and $t$ for a representative simulation. For example, "Bad Good" denotes a history where the dividend growth rate shock is bad in period $t-1$ and good in period $t$. There is a distribution of points on the graph for each of the four possibilities for the two-period histories because earlier periods affect the equilibrium. But as can be seen in the figure, the two most recent shocks do a good job of characterizing the current state of the economy.

Figure 1: Equilibrium Price-Dividend Ratio and Aggregate Risk Tolerance
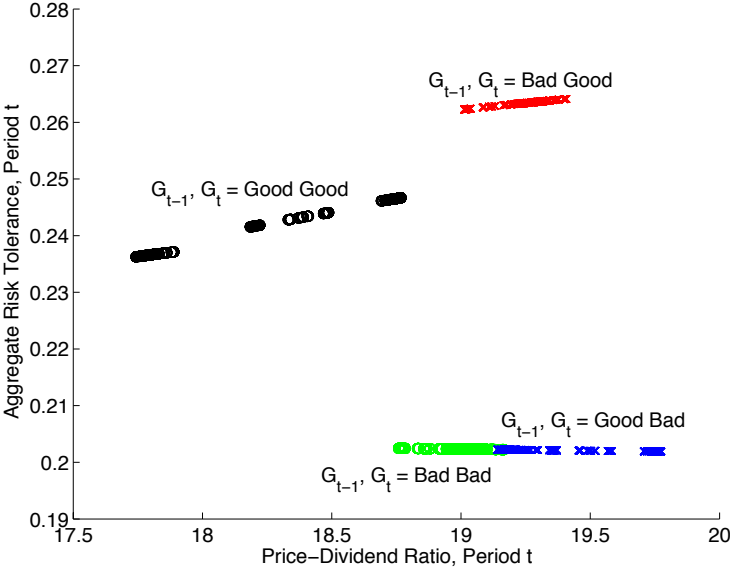


To clarify the workings of the model, let us pursue a detailed analysis of these regularities, beginning with an explanation of the behavior of aggregate risk tolerance. Figure 1 shows that aggregate risk tolerance in period $t$ depends mainly on the dividend growth rate shock in period $t$. The average aggregate risk tolerance is higher after a good shock than after a bad shock in period $t$: 0.25 versus 0.2. The $t-1$ shock hardly matters for aggregate risk tolerance when $G_t$ is bad, and matters only somewhat when it is good. To explain these patterns, the interlocking structure of general equilibrium requires us to start in the middle of things, eventually circling around to explain how everything fits together. Anticipating

our more detailed discussion of portfolio choice below, note that the daring agents make levered investments in the risky asset when young, and thus have relatively high shares of invested wealth when middle-aged after a good shock and relatively low shares of invested wealth after a bad shock. As a result, aggregate risk tolerance comoves with the most recent dividend-growth-rate shock. Moreover, due to the positive risk premium, daring agents' share of invested wealth rises more (relative to what would result from safe investment) after a good shock than it falls after a bad shock. Consequently, the economy-wide aggregate risk tolerance falls slightly to 0.202 after a bad shock, and rises significantly to a value about 0.25, while each cohort enters economic life with a cohort average risk tolerance of 0.207.

What explains the dependence of aggregate risk tolerance on the earlier shock $G_{t-1}$? A good shock at time $t-1$ puts a large share of invested wealth in the hands of daring middle-aged agents, who then hold a large fraction of all available shares of the risky asset, leaving less for their more cautious peers and for the young cohort as a whole. Thus, the overall holdings of the young cohort are less aggressive when they enter economic life on the heels of a good dividend-growth-rate shock than when they enter economic life on the heels of a bad dividend-growth-rate shock. This in turn means that their share of invested wealth in middle age after a good shock $G_t$ will be higher if they took more risks under a bad shock in their youth than if they took fewer risks under a good shock in their youth. Thus, the aggregate risk tolerance is higher than a history of bad and good shocks than a history of two goods shocks.

The dependence of the risky asset share of the young cohort as a whole on the shock right before they begin investing also makes it possible to explain the behavior of the price-dividend ratio. Recall that the price-dividend ratio is determined exactly by the total middle-aged wealth in each period. See equation 23. Because of the Cobb-Douglas production function and the constant average propensity to consume, the invested wealth of the young cohort is a fixed fraction of output, or equivalently, in a fixed ratio to the dividend. Their later wealth in middle age in comparison with the dividend at that time depends on their overall portfolio return compared to the growth rate of the dividend. Let us explain the general level of the

price-dividend ratio after each of the four possible histories of the last two shocks.

A good shock leading into the youth of the current middle-aged cohort depressed their risky asset share when young, and they were significantly underlevered relative to the economy as a whole. Then a second good shock will actually cause the growth rate of their wealth to fall behind the growth rate of the dividend (which equals the growth rate of economic production as a whole). Middle-aged wealth that is smaller relative to the expanded size of the economy will then lead to a low price-dividend ratio (in part because dividends are large). In contrast, a bad shock as they go into middle age causes the economy to grow more slowly than their relatively safer portfolio, leading to a high middle-aged wealth to dividend ratio, which in turn implies a high price-dividend ratio.

The bad shock leading into the youth of the current middle-aged cohort led that cohort to have slightly elevated risky asset holdings and thus to be slightly overlevered relative to the economy as a whole. A good shock at middle age then leads them to have a high level of wealth relative to the economy as a whole, leading in turn to a high price-dividend ratio. In contrast, a bad shock at middle age causes them to have a low level of wealth relative to the economy as a whole, leading in turn to a low price-dividend ratio.

Note that the last shock makes a bigger difference for the price-dividend ratio between the (Good, Good) and (Good, Bad) histories than it does between the (Bad, Good) and (Bad, Bad) histories. The reason is that the daring middle-aged agents in the previous period pulled more dramatically ahead of the economy after a good shock leading into their middle age than they fell behind the economy after a bad shock leading into their middle age. This led the current middle-aged cohort to have a significantly underlevered portfolio after a good shock leading into their youth, but only slightly overlevered portfolio after a bad shock leading into their youth.

To summarize the history dependence of prices, we present the key price statistics conditional on the current shock, the past shock, and the two past shocks in Table 4. The kind of reasoning above can be extended to interpret the consequences of a longer history of shocks, but the most recent shocks have the most powerful effect on the model economy. One reason

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we include the effects of a three-shock history is the identity relating the realized tree return to the history of the price-dividend ratio:

$$R_t^{\text{Tree}} = \frac{P_t + D_t}{P_{t-1}} = \frac{D_t}{D_{t-1}} \left( \frac{1 + (P_t/D_t)}{P_{t-1}/D_{t-1}} \right) = G_t \left( \frac{1 + p_t}{p_{t-1}} \right).$$

Thus, the realized tree return depends on the values of the price-dividend ratio in two different periods. Among three-shock histories, the realized tree return is highest after (Good, Good, Good) since the two good shocks at $t-2$ and $t-1$ depress $p_{t-1}$, while the good shock at $t$ gives a high $G_t$.[5] The realized tree return is lowest after the three-shock history (Good, Bad, Bad) because the combination of a good shock at $t-2$ followed by a bad shock at $t-1$ elevates $p_{t-1}$, while the bad shock at $t$ gives a low $G_t$.

Table 4: History Dependence of Key Variables

| $G_t$ | $P_t/D_t$ | | $\log R_t^{\text{Tree}}$ | | $\log R_t$ | | $E(\log R_t^{\text{Tree}})$ | | $E(\log Z_t)$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Bad | Good | Bad | Good | Bad | Good | Bad | Good | Bad | Good |
| Mean | 19.20 | 18.69 | 1.61 | 5.60 | 1.75 | 2.10 | 3.91 | 4.08 | 2.16 | 1.98 |
| $G_{t-1}$ | | | | | | | | | | |
|   Bad | 18.99 | 19.23 | 1.51 | 5.60 | 1.81 | 1.96 | 3.96 | 3.95 | 2.15 | 1.98 |
|   Good | 19.41 | 19.23 | 1.71 | 5.60 | 1.68 | 2.23 | 3.86 | 4.20 | 2.18 | 1.97 |
| $G_{t-2}, G_{t-1}$ | | | | | | | | | | |
|   Bad, Bad | 19.07 | 19.32 | 1.58 | 5.67 | 1.79 | 1.94 | 3.94 | 3.93 | 2.15 | 1.99 |
|   Good, Bad | 18.90 | 19.15 | 1.45 | 5.54 | 1.84 | 1.99 | 3.98 | 3.97 | 2.14 | 1.98 |
|   Bad, Good | 19.21 | 17.81 | 1.53 | 5.41 | 1.74 | 2.33 | 3.91 | 4.29 | 2.17 | 1.95 |
|   Good, Good | 19.59 | 18.52 | 1.87 | 5.79 | 1.62 | 2.12 | 3.81 | 4.11 | 2.19 | 1.99 |

Note: $P_t/D_t$ denotes the price-dividend ratio, $\log R_t$ denotes the log risk free interest rate, $\log R_t^{\text{Tree}}$ denotes the log realized tree return, given by $\log(D_t + P_t) - \log P_{t-1}$, $E(\log R_t^{\text{Tree}})$ denotes the expected log tree return, and $E(\log Z_t)$ denotes the expected log excess return. All prices are annualized. All prices are in percentage, except the price-dividend ratio. $G_t$, $G_{t-1}$ and $G_{t-2}$ denote the dividend growth rate shock in period $t$, $t-1$ and $t-2$, respectively.

Table 4 also shows that the risk free rate is higher after a good shock than after a bad shock in period $t$. This pattern becomes particularly pronounced after a good shock in period $t-1$. The expected tree return is higher after a good shock in period $t$. However, the pattern does not hold when we consider a two-period history of shocks. After a good shock in period $t-1$, the expected tree return is higher after a good shock. In contrast, after a bad shock in period $t-1$, the expected tree return is higher after a bad shock. The expected excess

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[5]The realized tree return is second-highest after the three-shock history (Bad, Bad, Good) not only because two bad shocks at $t-2$ and $t-1$ depress $p_{t-1}$ somewhat, but also because the tail end of a bad shock at $t-1$ followed by a good shock at $t$ puts $p_t$ at a reasonably high level as well as giving a high $G_t$.

return is lower after a good shock in period $t$, given the high aggregate risk tolerance after a good shock in period $t$. The expected excess return is especially low after a history of the bad and good shocks.

## 4.3    Portfolio Choice and Rebalancing

Most modern discussions of asset pricing in general equilibrium focus on the properties of price and returns as in Table 3 and 4. Asset holdings are left implicit. Since we are particularly concerned with implications for individual asset demands, we look directly at portfolio choice and rebalancing in response to shocks in this subsection. The first three columns of Table 5 report the portfolio share of each type of agents. We find the three important effects: leverage, aggregate risk tolerance and intertemporal hedging.

First, leverage arises endogenously, with the daring agents being highly levered and with the cautious agents holding a risky, but unlevered portfolio. As shown in the first column of Table 5, the cautious agents hold only about 48% of their invested wealth in the risky asset and the rest in the risk-free bonds. By contrast, the daring agents invest on average 568% of their invested wealth in the risky asset by selling the risk-free bonds to the cautious agents. This difference in portfolio shares across types enables shocks to change the distribution of wealth across types.

Second, as the second and third columns of Table 5 show, one of the most striking implications of the model is that all types have a higher mean portfolio tree share after a bad shock than after a good shock due to the aggregate risk tolerance effect. Our market crash example in Section 2.3 also has this feature. A bad shock pushes down the aggregate risk tolerance and raises the expected excess return. As a result, all types of agents choose a larger portfolio share. After a good shock, the aggregate risk tolerance rises, the expected excess return declines, and all agents decrease their portfolio shares. It may seem surprising that all agents are able to move their portfolio shares in the same direction in a general equilibrium, but Section 4.3.1 will show that changes in the wealth distribution—in particular, changes in the share of all invested wealth in the hands of middle-aged daring agents—allow all portfolio

Table 5: Portfolio Choice, Savings Weight and Asset Demand by Type (Percent)

| $G_t$ | Portfolio Share $\theta$ | | | Savings Weight $\frac{\mathcal{W}_i}{\mathcal{W}}$ | | Tree Amount $\frac{\theta \mathcal{W}_i}{\mathcal{W}}$ | |
| --- | --- | --- | --- | --- | --- | --- | --- |
| | Average | Bad | Good | Bad | Good | Bad | Good |
| Cautious | 48 | 53 | 43 | 92.8 | 85.0 | 49 | 36 |
| Young | 47 | 53 | 42 | 60.1 | 61.7 | 32 | 26 |
| Middle | 49 | 53 | 44 | 32.7 | 23.3 | 17 | 10 |
| | | | | | | | |
| Daring | 568 | 704 | 432 | 7.2 | 15.0 | 51 | 65 |
| Young | 570 | 706 | 433 | 5.2 | 5.4 | 37 | 23 |
| Middle | 566 | 701 | 432 | 2.0 | 9.6 | 14 | 42 |
| | | | | | | | |
| Total | 90 | 105 | 74 | 100.0 | 100.0 | 100 | 100 |
| Young | 89 | 105 | 73 | 65.3 | 67.1 | 69 | 48 |
| Middle | 90 | 105 | 75 | 34.7 | 32.9 | 31 | 52 |

Note: The portfolio share is the ratio of risky assets and savings of each type, the savings weight is the fraction of each type's savings in total savings, and the tree amount is the amount of tree held by each type. In particular, the tree amount equals the savings weight times the portfolio share for each type. Both the tree amount and the savings weight should sum up to 100 percent across types. Savings is after-consumption wealth. $G_t$ denotes the dividend growth rate shock in period $t$.

shares to move in the same direction.

Third, Table 5 indicates that for each level of risk tolerance the portfolio share of the young is similar to that of the middle-aged. Recall that we abstract from the dynamic interaction between human capital and portfolio choice by having labor income only at the beginning of the life. The comparison between young and middle-aged portfolios for a given level of risk tolerance thus isolates the pure intertemporal hedging effect. Table 5 indicates that the hedging effect is small quantitatively. Qualitatively, the cautious young have a lower portfolio share than the cautious middle-aged, while the daring young have a higher portfolio share than the daring middle-aged. Section 4.3.3 discusses these implications in detail.

The average young portfolio share has important implications for the dynamics of the total middle-aged wealth and the price-dividend ratio next period. Recall that the cautious agents account for 92% of the population and the daring agents account for the other 8%. The average young portfolio share is 105% ($= 0.92 \times 53 + 0.08 \times 704$) after a bad shock and 74% ($= 0.92 \times 43 + 0.08 \times 433$) after a good shock. Thus, the young hold a slightly more daring portfolio (105%) than the market portfolio (100%) after a good shock, but a much more conservative portfolio (74%) after a bad shock. These patterns for the portfolio share

of the young generate the history dependence of the price-dividend ratio that we discussed in the previous subsection.

There are two natural ways to track portfolio changes: (a) by comparing those in similar roles such as "middle-aged daring" over time from one generation to the next (*types*), and (b) by following individuals across time (*cohorts*). Subsection 4.3.1 tracks portfolio holdings by type while Subsection 4.3.2 tracks portfolio holdings by cohort.

### 4.3.1 Rebalancing by Type

It may seem surprising that all types of agents move their portfolios in the same direction in a general equilibrium. The key forces behind this finding come from the changes in the wealth and savings levels of the middle-aged agents that generate changes in the tree holdings $\frac{\theta \mathcal{W}_i}{\mathcal{W}}$ of each type to satisfy the adding-up constraint

$$1 = \frac{\mathcal{W}_{Yt}^L}{\mathcal{W}_t}\theta_{Yt}^L + \frac{\mathcal{W}_{Yt}^H}{\mathcal{W}_t}\theta_{Yt}^H + \frac{\mathcal{W}_{Mt}^L}{\mathcal{W}_t}\theta_{Mt}^L + \frac{\mathcal{W}_{Mt}^H}{\mathcal{W}_t}\theta_{Mt}^H. \tag{25}$$

To understand the implications of this constraint, look first at the "saving weights"—that is the weights for invested wealth or after-consumption wealth $\frac{\mathcal{W}_i}{\mathcal{W}}$—reported in the middle columns of Table 5. The cautious agents have a larger savings weight than the daring because the cautious have a larger mass. The savings weight of the cautious is, however, lower after a good shock (85%) than after a bad shock (92.8%), so the aggregate risk tolerance increases after a good shock. Both the daring and cautious young agents have a slightly larger savings weight after a good shock than after a bad shock. The cautious middle-aged have a much smaller savings weight after a good shock (23.3%) than after a bad shock (32.7%). In contrast, the daring middle-aged have a much larger savings weight after a good shock (9.6%) than after a bad shock (2%).

Now, examine the amount of trees held by each type of agents in the last two columns of Table 5. The quantity of trees held by each type equals that type's portfolio share multiplied by the savings weight of each type. Even though the economy has relatively fewer daring agents, the tree holdings of the daring agents are comparable in total size to those of the

cautious agents due to their heavily levered position. After a good shock, the young agents have slightly higher savings weights but much lower portfolio shares, leading to much lower total tree holdings. After a good shock, the middle-aged cautious agents have a much lower savings weight and thus an even lower quantity of tree holdings given their decreased portfolio share. On the other hand, the middle-aged daring agents have a much higher savings weight after a good shock, almost four times higher. The increase in their savings weight dominates the decline in their portfolio share, implying a larger total quantity of tree holdings by the middle-aged daring agents.

In sum, all types of agents in the economy adjust their portfolio shares in the same direction in response to the dividend growth shocks. In particular, all increase their portfolio shares after a bad shock and decrease them after a good shock. The changes in the underlying wealth distribution make such portfolio behavior consistent with general equilibrium.

### 4.3.2 Rebalancing by Cohort

So far we have been focusing on the portfolio dynamics of each type of agent at a point in time. Now we track the portfolio rebalancing of each cohort over time. When young, the agents choose an optimal portfolio share after the dividend shock. At the beginning of their middle age, before any rebalancing, the value of their tree holdings as a share of their middle-aged wealth changes passively in response to the realized dividend shock and the realized tree return. We refer to this portfolio share as the realized portfolio share. The agents then choose the optimal portfolio share for their middle age (governing wealth accumulation between then and old age). We present the sequence of the three portfolio shares in Table 6. We group equilibrium observations by the history of shocks. $G_Y$ and $G_M$ denote the shocks that the agents experience when young and when middle-aged, respectively.

Let's first examine the passive movements of the portfolio share of each cohort. Since the cautious young agents are not levered, the realized middle-age portfolio share rises substantially after a good shock at their middle age, and declines slightly after a bad shock. By contrast, since the daring young agents are highly levered, their realized middle-age portfolio

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Table 6: Rebalancing: Portfolio Share, by Cohort (Percent)

| $G_Y, G_M$ | Cautious Agents | | | Daring Agents | | |
|---|---|---|---|---|---|---|
| | Young | Realized | Middle-Aged | Young | Realized | Middle-Aged |
| Bad, Bad | 53.1 | 52.0 | 53.4 | 706 | 988 | 700 |
| Good, Bad | 41.8 | 39.9 | 53.4 | 433 | 594 | 702 |
| Bad, Good | 53.1 | 71.0 | 42.7 | 706 | 166 | 395 |
| Good, Good | 41.8 | 59.0 | 45.9 | 433 | 161 | 467 |

Note: $G_Y$ and $G_M$ denote the shocks that agents experience when young and when middle-aged, respectively.

share rises after a bad shock, but declines after a good shock. In addition, the magnitude of the changes in their portfolio share is quite large. For instance, the realized portfolio share rises from 706% to 988% after a history of two bad shocks, and declines from 706% to 166% after a bad shock when young followed by a good shock at middle age.

If the conventional portfolio rebalancing advice were more or less correct, the agents would undo the movements in the realized middle-aged portfolio shares and return to the portfolio share when young. In terms of the direction of the adjustment at middle age, the conventional advice is consistent with the behavior of the cautious agents. However, the cautious agents do not return to the same portfolio share when young. The new optimal level is typically quite different from the optimal share when young. The difference in the optimal shares between the two periods is particularly large when the shock they experience when young is different from the shock at middle age.

In addition to the quantitative differences, there are two types of qualitative differences between the optimal rebalancing shown in Table 6 and standard rebalancing advice. First, even returning to the same tree share requires opposite adjustments for people who seek to be more levered than the economy as a whole as compared to agents who seek to be less levered than the economy as a whole. Second, for the daring agents, the direction of their adjustments at the middle age is sometimes opposite to the direction that would restore their tree share to what they had earlier. In particular, after a bad shock when young followed by a good shock at middle age, the daring middle-aged agents further increase their portfolio share above the realized portfolio share.

### 4.3.3 Intertemporal Hedging

Intertemporal hedging in our model arises when the portfolio shares shift consistently by age to take into account any predictability in returns. Referring back to Table 5, there is little difference in the portfolio choice between the young and the middle-aged. Given that our generational model produces only limited predictability in returns, it is not surprising that we see little intertemporal hedging. On the other hand, the qualitative implications of intertemporal hedging are still interesting and deserve some discussions.

To see the intertemporal hedging effect clearly, we plot the ratio of the young and middle-aged portfolio shares (in the lower panel) together with the dividend growth rate shock (in the upper panel) for a specific simulation in Figure 2. For the cautious agents, the young have a smaller portfolio share than the middle-aged, especially after a good dividend growth shock. For the daring agents, the young have a larger portfolio share than the middle-aged, especially after a bad dividend growth shock. The portfolio share difference between the middle-aged and the young is larger in proportional terms for the cautious agents than for the daring agents.

Figure 2: Intertemporal Hedging
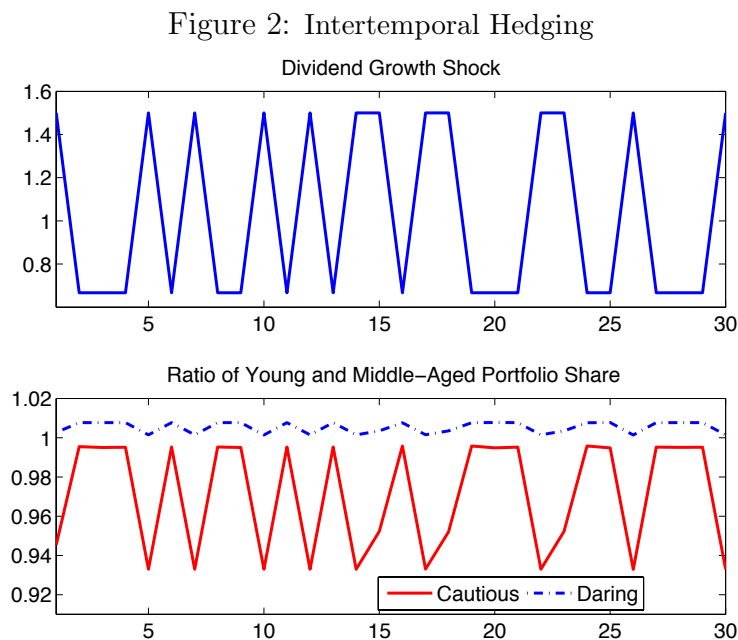


To understand these patterns of intertemporal hedging, recall the portfolio choice problem

of the young in equation (14) and of the middle-aged in equation (8). Unlike the middle-aged, who care about simple real returns, the young care about returns relative to the returns of the target variable annuity, as indicated by (17). Table 7 presents the annuity price for the middle-aged agents conditional the current and the past shock.

Table 7: Period-$t$ Target Annuity Price for Middle-Aged

|  | Cautious | | Daring | |
| --- | --- | --- | --- | --- |
| $G_t$ | Bad | Good | Bad | Good |
| Mean | 0.894 | 0.882 | 0.785 | 0.790 |
| $G_{t-1}$ |  |  |  |  |
| Bad | 0.891 | 0.889 | 0.762 | 0.802 |
| Good | 0.897 | 0.876 | 0.767 | 0.779 |

For the cautious agents, the target annuity price is higher after a bad shock than after a good shock. The combination of a low risk-free rate and lower expected tree returns late in life after a bad shock makes it harder to finance a comfortable retirement. See Table 3. This unfavorable movement in the target annuity price after a bad shock increases the effective risk of holding tree shares when young. By contrast, this target annuity price is irrelevant for the middle-aged cautious agents; they care only about one-period returns. Thus, the young cautious agents invest less in the tree than the middle-aged cautious agents. The annuity price variation in period $t$ is larger when the dividend shock is good in period $t-1$. This finding is reinforced when the cautious agents experienced a good shock when young. Note in particular that the future risk-free rate covaries much more positively with the later shock after a good shock when young. By contrast, after a bad shock when young, not only does the risk-free rate covary less with the later shock, but the set of two possible tree returns is now negatively correlated with the later shock.

For the daring agents, however, the target annuity price is lower after a bad shock than after a good shock, since they tend to be so highly-levered that the increase in the equity premium after a bad shock makes it easier for them to finance a comfortable retirement. This target annuity price pattern helps the daring agents smooth consumption at their middle age, and thus the young daring agents invest more in the risky asset than the middle-aged daring agents. The variation in the period-$t$ annuity price is larger when the dividend shock is bad

(which yields a greater covariance of the equity premium with the later shock) than when the dividend growth shock is good. Thus, quantitatively, the young daring agents invest more in risky assets than the middle-aged daring agents especially when they experience a bad shock in their youth.

# 5   Relation to Literature

There is a great deal of research about asset allocation and portfolio choice in finance and economics. These topics have become particularly interesting recently as economists have begun to study household finance in detail, following Campbell (2006). Relatively little of this research considers the general equilibrium constraints that we apply in our model. Two recent papers that feature general equilibrium are Cochrane, Longstaff and Santa-Clara (2008) and Garleanu and Panageas (2008). Garleanu and Panageas (2008), like our paper, models investor heterogeneity in risk preference and age in an overlapping generations model. Both of these papers focus on the behavior of asset prices, showing that general equilibrium can generate return predictability that is consistent with empirical evidence. Neither explores the implications of market returns for either trading or the portfolio choice of different types of individuals that is the focus of our paper.

The focus of our paper is much closer to the focus of Campbell and Viceira (2002). They find an important intertemporal hedging component of portfolio choice. In our model the leverage effect and the time-varying aggregate risk tolerance effect are much more significant. We have experimented with several different sets of parameter values to see if we can make the hedging effect larger, but we have not been able to find a scenario in which it matters very much. Moreover, we do not allow for the possibility of large variations in expected returns because of mispricing. In general, our framework presumes perfect markets and optimizing agents. Approaches such as Campbell and Viceira that use actual asset returns, which might reflect mispricing, have greater scope for finding intertemporal hedging.

There is a large and important literature about labor income risk and portfolio choice, including Bodie, Merton and Samuelson (1992), Jaganathan and Kocherlakota (1996), Cocco,

Gomes and Maenhout (2005), and Gomes and Michaelides (2005). In a recent paper that is related to our work, Heathcote, Krueger and Rios-Rull (2010) find that young people can benefit from severe recessions if old people are forced to sell assets to finance their consumption, causing market prices to fall more than wages. We purposely abstract from these considerations by assuming that agents only earn labor income before they have an opportunity to rebalance. Our aim is to focus on the general equilibrium effects while suppressing the important effects of the human capital.

The leverage effect that we find is of course related to the effect discussed by Black (1976). Geanakoplos (2010) has recently developed a model emphasizing how shocks to valuation shift aggregate risk tolerance. Daring investors, who take highly levered risky positions, may go bankrupt after bad shocks thereby decreasing aggregate risk tolerance and further depressing prices. This analysis hence features both the leverage and aggregate risk tolerance effects. The details of his model are quite different from ours. Moreover, the focus of his analysis is defaults, which will not occur in our setting.

Two recent empirical papers, Calvet, Campbell and Sodini (2009) and Brunnermeier and Nagel (2008) examine portfolio rebalancing with holdings and survey data, respectively. Both papers find that people do not completely rebalance, or that investors do not completely reverse the passive changes in their asset allocations. They both suggest that some type of inertia drives the lack of complete rebalancing. Our work provides a new benchmark for these and future empirical studies. An important implication of our model is that it is neither feasible nor optimal for investors to completely rebalance in aggregate, so the evidence from these papers is not inconsistent with optimization. Moreover, the leverage effect implies that investors with high initial shares will change their allocations in the opposite way than that predicted by full rebalancing. In fact, regressions in the online appendix of Calvet, Campbell and Sodini (2009) indicate that the coefficients on passive changes in allocation are increasing in initial stock share. This is exactly what we would expect to find.[6]

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[6]Since Calvet, Campbell and Sodini (2009) have detailed data on investors' actual holdings and returns, some portion of the returns their investors experience is idiosyncratic. This rationalizes the fact that the coefficients in the online appendix of Calvet Campbell and Sodini (2009) decline in initial share but they do not become positive for investors with high initial shares.

When academic researchers think about asset allocation, they generally include all risky assets (including all types of corporate bonds) into their definition of the risky mutual fund that all investors should hold. In our model, regardless of how one might divide assets into safe or risky categories, we define the "risky asset" or the "tree" to be the set of all assets in positive net supply and we define the risk-free asset to be in zero net supply. The levered position in trees in our model encompasses both stock holdings and the risky part of the corporate bonds.

When practitioners think about portfolio choice they generally treat stocks, bonds, and cash-like securities as separate asset categories in their allocation and rebalancing advice, as Canner, Mankiw and Weil (1997) point out. To understand the size of the leverage effect, it is important to remember that stock is a levered investment in underlying assets. Thus, for example, if the corporate sector is financed 60% by stock and 40% by debt, an investor with more than 60% in stock is actually levered relative to the economy as a whole. If some corporate debt can be considered essentially risk-free then it should be possible to express such an investor's position as a portfolio of a levered position in the underlying assets of the economy and some borrowing at the risk-free rate. The leverage effect will be important for all investors that hold a levered position in the underlying assets of the economy.

# 6 Conclusion

Standard portfolio advice is that investors should hold a constant share of risky assets. This advice cannot hold in equilibrium because the demand for risky assets must equal its supply. This paper develops an equilibrium model to illustrate how optimizing agents rebalance in equilibrium. The model includes three periods of life, so there is meaningful rebalancing over time, and agents with different risk tolerance, so there is meaningful trade. The model has a partial analytic solution. In particular, the state of the economy can be summarized by the distribution of wealth of middle-aged investors. This analytic solution both helps make the findings of the paper transparent and simplifies the solution of the full dynamic equilibrium model.

The paper finds that there are several effects that are key to understanding rebalancing in equilibrium. First, there is a leverage effect. In the model, leverage arises endogenously because of the heterogeneity in risk preferences. Shocks to the valuation of risky assets cause levered agents to rebalance in the opposite direction as unlevered agents. Second, there is an aggregate risk tolerance effect. Shocks to valuation of risky assets affect agents' wealth differently depending on their initial position in the risky asset. Such shifts in the distribution of wealth change aggregate risk tolerance and therefore the demand for assets going forward. Note that the leverage and aggregate risk tolerance effects interact because leverage magnifies how valuation shocks change the distribution of wealth.

Third, there is an intertemporal hedging effect whereby equilibrium time variation in expected returns affects the portfolio choices of young investors as they exploit mean reversion in asset returns. This intertemporal hedging effect arising from mean reversion is what is often highlighted in analyses emphasizing time- or age-variation in portfolio shares. While it is present in our model, it is weak quantitatively. The model is solved under a specification with i.i.d. growth rates, so mean reversion in returns is not built into the specification. Moreover, other analyses of intertemporal hedging are based on empirical asset returns, which might include mean reversion arising from mispricing. Our model has all agents frictionlessly optimizing, so many sources of mean reversion that may be present in the data are not present in the model.

Hence, our analysis highlights two important and interacting effects for understanding equilibrium demand for risky assets—the leverage effect and the aggregate risk tolerance effect. The distribution of wealth across investors is key to understanding how returns and portfolios evolve in response to shocks to the fundamentals.

Our framework is useful for understanding how portfolios would respond to a stock market crash in an economy where all investors were behaving optimally. Contrary to the standard advice that all investors should return their portfolio shares to their pre-crash levels, our investors after a crash increase their share in risky assets relative to the pre-crash levels—though by different amounts depending on their risk tolerance.

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# Appendix: Solution Technique

Let's consider discrete grid points for the state variable. However, we allow the choice variables to be continuous. We assume that the world ends at the end of period $T$, and solve the model backward from the last period. Behavior at time $T$ is very simple: all remaining agents consume their wealth and the price-dividend ratio is zero. Thus, we can derive the value functions $v_{YT}^j(w_{MT}^H, w_{MT}^L)$ and $v_{MT}^j(w_{MT}^H, w_{MT}^L)$.

Consider now time $T-1$ for a state $(w_{MT-1}^H, w_{MT-1}^L)$. The price-dividend ratio is given by equation (22).[7] The optimal consumption allocation can be easily derived. To solve for the portfolio allocation, we need to know $R_{T-1}(w_{MT-1}^H, w_{MT-1}^L)$. For any guess of $R_{T-1}$, the optimal portfolio share $\theta_{YT-1}^j = \theta_{MT-1}^j$ can be computed as follows:

$$
\theta_{MT-1}^j(w_{MT-1}^H, w_{MT-1}^L) = \arg\max_\theta \sum_{n'=1}^{N} \pi_{n'} \left[ (1-\theta)R_{T-1} + \theta \frac{G^{n'}}{p_{T-1}} \right]^{1-\gamma_j}.
$$

We check the bond market clearing condition and update $R_{T-1}$ until the market clears. Given the optimal portfolio shares, we compute the certainty equivalent total return functions for the young $\phi_{YT-1}^j(w_{MT-1}^H, w_{MT-1}^L)$ and for the middle-aged $\phi_{MT-1}^j(w_{MT-1}^H, w_{MT-1}^L)$. Thus, we can derive the value function of the young and the middle-aged.

We now study period $T-2$ for any state $(w_{MT-2}^H, w_{MT-2}^L)$, which has the same fundamental structure as a generic period. The price-dividend ratio is a function of the sum of the state variables, and the optimal consumption allocation can be easily derived using the first-order conditions. To solve for the model, we need to know the risk-free rate and the portfolio choice of the young. Given any guess of $(R_{T-2}, \theta_{YT-2}^H, \theta_{YT-2}^L)$, the portfolio choice of the young implies the next-period state variable $(w_{MT-1}^{Hn'}, w_{MT-1}^{Ln'})$ and the price-dividend ratio $p_{T-1}^{n'}$, for any dividend growth realization $G^{n'}$. Thus, we can compute the middle-aged optimal portfolio and certainty equivalent total return $\hat{\phi}_{MT-1}^j(w_{MT-1}^{Hn'}, w_{MT-1}^{Ln'})$. We then can compute the optimal portfolio choice of the young $\theta_{YT-2}^j$, which is used to update our initial guess. We also update the interest rate using the bond market clearing condition. We repeat the above procedures until our guesses converge. We then compute the certainty equivalent total return function and the value functions of the middle-aged $v_{MT-2}^j(w_{MT-2}^H, w_{MT-2}^L)$ and the young $v_{YT-2}^j(w_{MT-2}^H, w_{MT-2}^L)$.

We repeat the above procedure for period $T-3$, $T-4$, etc, until the value functions of the young and the middle-aged across the two periods converge.

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[7]The asset-market and good-market equilibrium conditions are modified: the marginal propensity to consume and the optimal risky asset share for the young become the same as for the middle aged.