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Strategic Asset Allocation and Consumption Decisions under Multivariate Regime Switching*

Massimo Guidolin
University of Virginia

Allan Timmermann
University of California San Diego

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Abstract

This paper studies optimal asset allocation to stocks, long-term bonds and T-bills and consumption choice in the presence of regime switching in asset returns. We find strong evidence that four separate regimes - characterized as crash, slow growth, bull and recovery states - are required to capture the joint distribution of stock and bond returns. Optimal asset allocations vary considerably across these states - both across bonds and stocks and among large and small stocks - and change significantly over time as investors revise their estimates of the current state probabilities. In the crash state investors always allocate more of their portfolio to stocks the longer their investment horizon, while the optimal allocation to stocks declines as a function of the investment horizon in bull markets. Consumption-to-wealth ratios are also found to depend on the underlying state. Welfare costs from ignoring regime switching are substantial, especially when frequent rebalancing is considered. Results are found to be robust to changes in risk aversion, the imposition of short sale restrictions, the inclusion of standard predictor variables such as the dividend yield and to parameter uncertainty.

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1. Introduction

For many investors the strategic asset allocation decision of how much to invest in major asset classes such as cash, stocks and bonds is the single most important investment decision. The strategic asset allocation decision can only be made in the context of and conditional on a model for the joint distribution of returns on such asset classes. Recent research suggest that stock and bond returns contain a predictable component that, though small particularly at short horizons, cannot be ignored in asset allocation decisions.[1]

So far, the workhorse in studies of optimal asset allocation decisions has been the standard linear forecasting model. Most studies assume that asset returns are generated by a stable process so the predictive power of state variables such as dividend yields, default and term spreads does not vary over time. However, there is mounting empirical evidence that asset returns follow a more complicated process with multiple "regimes", each of which is associated with a very different distribution of asset returns. Ang and Bekaert (2001, 2002), Gray (1996), Guidolin and Timmermann (2003), Perez-Quiros and Timmermann (2000), Turner, Startz and Nelson (1989) and Whitelaw (2001) all report evidence of regimes in stock returns or interest rates.

In this paper we characterize investors' strategic asset allocation and consumption decisions in the context of a regime-switching model with four states characterized as crash, slow growth, bull and recovery states. The presence of regimes imply that all conditional moments of the asset return distribution are time-varying, so we extend the previous literature on asset allocation under time-varying expected returns to cover the case in which all moments may need appropriate hedging. We find that the optimal asset allocation differs strongly across regimes. For instance, stocks are attractive to short-to-medium term investors in the bull state since the probability of staying in such a state is high. Stocks are far less attractive in the crash state even though this state is not very persistent. If the economy is in a non-persistent state or the investor is unsure of the identity of the current state, the importance attributed to the current regime is lowered and more weight is

[1] See, e.g., Ang and Bekaert (2001), Barberis (2000), Brandt (1999), Brennan et al. (1997), Brennan and Xia (2001), Campbell and Viceira (1999, 2001), Campbell, Chan and Viceira (2001), Chacko and Viceira (2000), Cocco et al (2001), Kandel and Stambaugh (1996), Tamayo (2001) and Xia (2001).

assigned to other states. Knowledge of the current regime thus affects optimal portfolio choices even if, as seems plausible, investors never know with certainty which regime the economy is currently in.

One of the key questions addressed in the literature on optimal asset allocation is how the investment horizon affects optimal portfolio weights. When investment opportunities remain constant over time, a power utility investor's horizon does not affect the optimal asset allocation, c.f. Samuelson (1969). In the absence of predictor variables such as the dividend yield, standard models therefore imply constant portfolio weights. In a thorough study of investment horizon effects that uses the dividend yield as a predictor, Barberis (2000) finds that the portfolio weight allocated to stocks should increase monotonically as a function of the investor's horizon. In the context of a larger set of assets and a joint consumption and savings problem, this finding is confirmed by Lynch (2001).

Even in the absence of predictor variables, regime switching models imply that investors' asset allocation varies over time as the different states offer different investment opportunities and investors revise their beliefs about the underlying state probabilities.[2] Interestingly, we find that horizon effects vary across states. Since stocks are not very attractive in the crash state, investors with a short horizon want to hold very little of their portfolio in stocks in this state. At longer investment horizons, there is a high chance that the economy will switch to a better state and so investors allocate more towards stocks. In the crash state we therefore get an upward sloping allocation to stocks as a function of the investment horizon. In the more persistent slow growth and bull states, investors with a short horizon hold large positions in stocks. At longer horizons the economy will almost certainly switch to the crash state so investors hold less in stocks, thereby creating a downward sloping relation between stocks holdings and the investment horizon.

In addition to these horizon effects we find interesting substitution effects across small and large firms. As the time horizon increases, the allocation to small caps as a proportion of the total equity portfolio typically declines, while the allocation to large caps increases. These findings are robust to changes in the coefficient of relative risk aversion, to allowing the investor to periodically rebalance portfolio holdings, and to parameter uncertainty.

[2] See Veronesi (1999) for a careful discussion of similar effects in a two-state asset pricing model.

Regime switching affects not only the optimal asset allocation but also the joint consumption-savings decision. For instance, a perception of being in a bull market induces investors to change current consumption since it changes both their perceived income and their current investment opportunities. In the crash state with poor investment opportunities, optimal consumption is relatively insensitive to the time horizon and uniformly below its steady-state value. Conversely, in the bull state, investment opportunities are very good and strong income effects induce investors to consume a higher percentage of wealth even at relatively long horizons.

The plan of the paper is as follows. Section 2 introduces the multi-state model used to capture predictability and regime switching for asset returns and reports empirical findings. Section 3 sets up the investor's asset allocation problem while Section 4 presents empirical results for a buy-and-hold investor. Section 5 introduces periodic rebalancing and Section 6 studies a joint consumption-asset allocation problem. Section 7 presents utility cost calculations and Section 8 concludes. Technical details are provided in appendices at the end of the paper.

2. Asset Returns under Regime Switching

A number of stylized features of asset returns have emerged from the empirical finance literature. Stock and bond returns are – to a limited extent – predictable, their volatility clusters over time and correlations are not the same in bull and bear markets. At most horizons (including monthly data), stock returns are also far from normally distributed and are affected by occasional outliers.

Regime switching models can capture such properties of the return distribution. These models typically identify bull and bear regimes with very different mean, volatility and correlations across assets. As the underlying state probabilities change over time this leads to time-varying expected returns, volatility persistence and correlations. Regime switching models are also capable of capturing even complicated forms of fat tails and skews in the underlying distribution of returns, c.f. Timmermann (2000).

Another attractive property of regime switching models comes from their interpretation as multivariate mixtures of normals. These have been widely used in the nonparametric literature to approximate densities of arbitrary form, see e.g. Marron and Wandt (1992). For example, in a multi-state model outliers are accommodated by having a separate crash state - capturing large negative returns -

and a bull burst state capturing large positive returns.

To capture the possibility of regimes in the joint distribution of asset returns, consider an $(n+m)\times 1$ vector of asset returns in excess of the T-bill rate, $\mathbf{r}_t = (r_{1t}, r_{2t}, ..., r_{nt})'$ extended by a set of m predictor variables, $\mathbf{z}_t = (z_{1t}, ..., z_{mt})'$. Suppose that the mean, covariance and possibly also (auto and cross-) serial correlations in returns are driven by a common state variable, S_t, that takes integer values between 1 and k:

$$\begin{pmatrix} \mathbf{r}_t \\ \mathbf{z}_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{s_t} \\ \boldsymbol{\mu}_{zs_t} \end{pmatrix} + \sum_{j=1}^{p} \mathbf{A}^*_{j,s_t} \begin{pmatrix} \mathbf{r}_{t-j} \\ \mathbf{z}_{t-j} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{\varepsilon}_{zt} \end{pmatrix}. \qquad (1)$$

where $\boldsymbol{\mu}_{s_t}$ and $\boldsymbol{\mu}_{zs_t}$ are intercept vectors for $\mathbf{r}_t$ and $\mathbf{z}_t$ in state s_t, $\{\mathbf{A}^*_{j,s_t}\}_{j=1}^{p}$ are $(n+m)\times(n+m)$ matrices of autoregressive coefficients in state s_t, and $(\boldsymbol{\varepsilon}_t' \ \boldsymbol{\varepsilon}_{zt}')' \sim N(0, \Omega^*_{s_t})$, where $\Omega^*_{s_t}$ is an $(n+m\times n+m)$ covariance matrix.

Regime switches in the state variable, S_t, are assumed to be governed by the transition probability matrix, $\mathbf{P}$, with elements

$$\Pr(s_t = i | s_{t-1} = j) = p_{ji}, \quad i, j = 1, .., k. \qquad (2)$$

Each regime is thus the realization of a first-order Markov chain with constant transition probabilities.

While simple, this model is quite general and allows asset returns to have different means, variances and correlations in different states. This means that the risk-return trade-off can vary over states and this is likely to have strong implications for asset allocation. For example, knowing that the current state is a persistent bull state will make most risky assets more attractive than in a bear state. Likewise, if stock market volatility is higher in recessions than in expansions, equity investments are less attractive in recessions unless their mean return rises commensurably.

Predictability in returns arises, most obviously, if the autoregressive terms are significant, i.e. if any coefficients in the $n\times m$ submatrices of $\mathbf{A}^*_{j,s_t}$ (explaining asset returns with lagged values of the predictors) are significant, as in Lynch (2001). However, even in the absence of autoregressive terms ($p=0$), predictability of mean returns arises as long as there are two states, s_t and s_t' for which $\boldsymbol{\mu}_{s_t} \neq \boldsymbol{\mu}_{s_t'}$. Variation in the state probabilities over time will then lead to time-variation in expected returns. Furthermore, if the covariance matrix differs across states, there will also be predictability in higher order moments such as volatility, correlations and skewness.

Notice that when $k = 1$, equation (1) simplifies to a standard vector autoregression. Our model thus nests as a special case the standard linear (single-state) model used in much of the asset allocation literature; see e.g. Barberis (2000) and Kandel and Stambaugh (1996). This model is selected if the data only supports a single regime.

Estimation of the model is relatively straightforward and proceeds by optimizing the likelihood function associated with our model. Since the underlying state variable, S_t, is unobserved we treat it as a latent variable and use the EM algorithm to update our parameter estimates, c.f. Hamilton (1989).

2.1. *Data*

Our analysis considers a US investor's asset allocation to three major asset classes, namely stocks, bonds and T-bills. We further divide the stock portfolio into large and small caps in light of the empirical evidence suggesting that these stocks have very different risk and return characteristics across different regimes, c.f. Perez-Quiros and Timmermann (2000).

Our analysis uses monthly returns on all common stocks listed on the NYSE. The first and second size-sorted CRSP decile portfolios are used to form a portfolio of small firm stocks, while deciles 9 and 10 are used to form a portfolio of large firm stocks. We also consider the return on a portfolio of 10-year T-bonds. Returns are calculated applying the standard continuous compounding formula, $\tilde{r}_{t+1} \equiv \ln S_{t+1} - \ln S_t$, where S_t is the asset price, inclusive of any cash distributions between time t and $t + 1$. To obtain excess returns r_t, we subtract the 30-day T-bill rate from these returns. Dividend yields are also used in the analysis and are computed as dividends on a value-weighted portfolio of stocks over the previous twelve month period divided by the current stock price. Our sample is January 1954 - December 1999, a total of 552 observations. Consistent with the literature (e.g. Barberis (2000, p. 233)) we only use data after the 1951 Treasury Accord. All data is obtained from the Center for Research in Security Prices.

2.2. *Model Specification*

Guidolin and Timmermann (2003) undertook a statistical specification analysis to determine the statistical evidence in support of the presence of regimes in the univariate and joint return distribution. Considering a range of values for the number of states, $k = 1, 2, 3, 4, 5, 6$ and the lag-order $p = 0, 1, 2, 3$, they used information criteria to select a four state model. Single-state models or models with a smaller number of states were strongly rejected.

Here we are less concerned with statistical evidence and more interested in ensuring that the return distribution is correctly specified. To determine the optimal asset allocation, an investor has to compute expected utility which requires integrating over the model for the return distribution. If this model is misspecified, suboptimal asset allocation decisions will almost certainly result, so it is important to make sure that the model is not misspecified. We therefore use recently developed predictive density tests proposed by, e.g. Diebold et al (1998) and Berkowitz (2001).

These tests are based on the so-called probability integral transform or $z-$score. This is the probability of observing a value smaller than or equal to the realization r_{t+1} of returns under the null that the model is correctly specified. Under the k-state mixture of normals, this is given by

$$\begin{aligned}\Pr(R_{t+1} &\leq r_{t+1}|\Im_t) = \sum_{i=1}^{k} \Pr(R_{t+1} \leq r_{t+1}|s_{t+1} = i, \Im_t) \Pr(s_{t+1} = i|\Im_t) \\ &= \sum_{i=1}^{k} \Phi\left(\sigma_i^{-1}(r_{t+1} - \mu_i - \sum_{j=1}^{p} a_{j,i} r_{t+1-j})\right) \Pr(s_{t+1} = i|\Im_t) \\ &\equiv z_{t+1}. \end{aligned} \quad (3)$$

Here R_t is the excess asset return and $\Phi(\cdot)$ is the cumulative density function of a standard normal variable. If the model is correctly specified, z_{t+1} should be independently and identically distributed (IID) on the interval $[0, 1]$, with a uniform distribution, c.f. Rosenblatt (1952). Berkowitz (2001) has proposed a likelihood-ratio test that inverts Φ to get a transformed $z-$score

$$z_{t+1}^{*} = \Phi^{-1}(z_{t+1}).$$

Under the null of a correctly specified model, z^* should be IID and normally distributed ($IIN(0, 1)$). This suggests conducting specification tests based on moment conditions such as

$$
\begin{aligned}
E[z_{t+1}^*] &= 0, \\
Var[z_{t+1}^*] &= 1, \\
Cov[z_{t+1}^*, z_t^*] &= 0, \\
Cov[(z_{t+1}^*)^2, (z_t^*)^2] &= 0, \\
Skewness[z_{t+1}^*] &= 0, \\
Kurtosis[z_{t+1}^*] &= 3.
\end{aligned}
\tag{4}
$$

We use a likelihood ratio test that focuses on a few salient moments of the return distribution. Suppose the log-likelihood function is evaluated under the null that $z_{t+1}^* \sim IIN(0,1)$:

$$L_{IIN(0,1)} \equiv -\frac{T}{2}\ln(2\pi) - \sum_{t=1}^{T}\frac{(z_t^*)^2}{2}, \tag{5}$$

where T is the sample size. Under the alternative of a misspecified model, the log-likelihood function incorporates deviations from the null, $z_{t+1}^* \sim IIN(0,1)$:

$$z_{t+1}^* = \mu + \sum_{j=1}^{p}\sum_{i=1}^{l}\rho_{ji}(z_{t+1-i}^*)^j + \sigma e_{t+1}, \tag{6}$$

where $e_{t+1} \sim IIN(0,1)$. The null of a correct return model implies $p \times l + 2$ restrictions — i.e., $\mu = \rho_{ji} = 0$ ($j = 1, ..., p$ and $i = 1, ..., l$) and $\sigma = 1$ — in equation (6). Let $L(\hat{\mu}, \{\hat{\rho}_{ji}\}_{j=1}^{p}{}_{i=1}^{l}, \hat{\sigma})$ be the maximized log-likelihood obtained from (6). To test that the forecasting model (1) is correctly specified, we use the following test statistic

$$LR = -2\left[L_{IIN(0,1)} - L(\hat{\mu}, \{\hat{\rho}_{ji}\}_{j=1}^{p}{}_{i=1}^{l}, \hat{\sigma})\right] \sim \chi^2_{p\times l+2}. \tag{7}$$

In addition to the standard Jarque-Bera (1980) test that considers skew and kurtosis in the z-scores, we present three likelihood ratio tests, namely a test of zero-mean and unit variance ($p = l = 0$), a test of lack of serial correlation in the $z-$scores and a test that further restricts their squared values to be serially uncorrelated.

Table I shows the results of these tests for the three asset classes under consideration. The smallest (and only) model to pass all predictive density tests is a model with four states that has regime-dependent expected returns and covariance matrix. Models with fewer states or whose volatility does not vary across states produce very large values of the Jarque-Bera test, particularly for the small stock portfolio and are hence clearly mis-specified.

2.3. *Model Estimates*

Since both statistical and economic criteria for model specification suggest a four state model with regime-dependent means and variances, Figure 2 plots the state probabilities while Table II shows the parameter estimates for this model.[3] It is easy to interpret the four regimes. Regime 1 is a 'crash' state characterized by large, negative mean excess returns and high volatility. It includes the two oil price shocks in the 1970s, the October 1987 crash, the early 1990s, and the 'Asian flu'. Regime 2 is a low growth regime characterized by low volatility and small positive mean excess returns on all assets. Regime 3 is a sustained bull state in which stock prices — especially those of the small stocks— grow rapidly on average. Interest rates frequently increase in this state and excess returns on long-term bonds are negative on average. The drawback to the high mean excess returns on small stocks is their rather high volatility, while large stocks and bonds have less volatile returns. Notice the big difference between mean returns on small and large stocks in regimes 2 and 3. In state 2 the mean return of large stocks exceeds that of small stocks by about 7% per annum, while this is reversed in state 3. Regime 4 is a recovery state with strong market rallies and high volatility for small stocks and bonds.

Correlations between returns also vary substantially across regimes. The correlation between large and small firms' returns varies from a high of 0.82 in the crash state to a low of 0.50 in the recovery state. The correlation between large cap and bond returns even changes signs across different regimes and varies from 0.37 in the recovery state to -0.40 in the crash state. Finally, the correlation between small stock and bond returns goes from -0.26 in the crash state to 0.12 in the slow growth state.

Mean returns and volatilities are greater in absolute terms in the crash and

[3] Attempts to simplify the number of parameters by imposing the restriction that mean returns are the same across the four states or that the covariance matrices are identical in the high volatility states (states 1 and 4) were clearly rejected at critical levels below 1 percent, c.f. Guidolin and Timmermann (2003).

recovery regimes, so it is perhaps unsurprising that persistence also varies considerably across states. The crash state has low persistence and on average only two months are spent in this regime. Interestingly, the transition probability matrix has a very particular form. Exits from the crash state are almost always to the recovery state and occur with close to 50 percent chance suggesting that, during volatile markets, months with large, negative mean returns cluster with months that have high positive returns. The slow growth state is far more persistent with an average duration of seven months. The bull state is the most persistent state with a 'stayer' probability of 0.88. On average the market spends eight successive months in this state. Finally, the recovery state is again not very persistent and the market is expected to stay just over three months in this state. The steady state probabilities, reflecting the average time spent in the various regimes are 9% (state 1), 40% (state 2), 28% (state 3) and 23% (state 4). Hence, although the crash state is clearly not visited as often as the other states, it is by no means an 'outlier' state that only picks up extremely rare events.

2.4. *Model with the Dividend Yield*

A large literature in finance reports evidence that variables related to the business cycle or to other slow-moving economic state variables have predictive power for stock and bond returns. While many predictor variables have been proposed, one of the key instruments is the dividend yield; see, e.g., Campbell and Shiller (1988), Fama and French (1988, 1993), Ferson and Harvey (1991), Goetzmann and Jorion (1993). Barberis (2000), Kandel and Stambaugh (1996), and Lynch (2001) investigate how predictability arising from the dividend yield affects optimal asset allocations.

To investigate the effect on our model of adding predictor variables such as the dividend yield, we extended the analysis to include this variable. This framework nests many of the models in the existing literature and therefore allows us to address how the presence of predictor variables such as the dividend yield affect the evidence of multiple states in asset returns.

Again we conducted a battery of tests to determine the best model specification. We found that a four-state VAR(1) model was strongly supported by the data. Unsurprisingly given the persistence in the dividend yield a single lag is required in the model extended to include the dividend yield. Table III shows the parameter estimates for the four-state regime switching model extended to include the dividend yield. Regime 1 continues to pick up market crashes, characterized by large, negative mean excess returns. The dividend yield is relatively high in this state (4%) and volatility is also above average. In steady state this regime occurs 15% of the time although it has an average duration of only two months. The autoregressive coefficients indicate substantial predictability of small and large firms' returns in this state. Regime 2 is a slow growth state with single-digit mean excess stock returns and moderate volatility. This state is highly persistent, lasting on average almost 16 months and occurring close to one-third of the time. Regime 3 continues to be a highly persistent bull state that lasts on average almost 15 months. Finally, regime 4 is again a recovery state with strong stock market rallies accompanied by substantial volatility. This state does not last long with an average duration of only 2 months. Nevertheless, at 18%, its steady-state probability is quite high.

3. The Investor's Asset Allocation Problem

Having determined the dynamics in the joint distribution of stock and bond returns we next study the asset allocation implications of the presence of persistent regimes in the investment opportunity set. First consider the 'pure' asset allocation problem for an investor with power utility defined over terminal wealth, W_{t+T}, coefficient of relative risk aversion $\gamma > 0$, and a time horizon T:

$$u(W_{t+T}) = \frac{W_{t+T}^{1-\gamma}}{1-\gamma}, \tag{8}$$

The investor is assumed to maximize expected utility by choosing at time t a portfolio allocation to large stocks, small stocks and bonds, $\boldsymbol{\omega}_t^T \equiv [\omega_t^l(T)\ \omega_t^s(T)\ \omega_t^b(T)]'$, while $1-(\boldsymbol{\omega}_t^T)'\boldsymbol{\iota}_3$ is invested in riskless, T-bills.[4] For simplicity we assume the investor has unit initial wealth. Portfolio weights are adjusted every $\varphi = \frac{T}{B}$ months at B equally spaced points t, $t+\frac{T}{B}$, $t+2\frac{T}{B}$, ..., $t+(B-1)\frac{T}{B}$. When $B=1$, $\varphi = T$ and the investor simply implements a buy-and-hold strategy.

Let $\boldsymbol{\omega}_b$ ($b = 0, 1, ..., B-1$) be the portfolio weights on the risky assets at these

[4] Following the literature we assume that the return on T-bills is known and constant each period. In the following, unless necessary, we will omit to formally indicate the investment horizon when referring to the vector of portfolio weights $\boldsymbol{\omega}_t^T$.

rebalancing times. Then $1-\boldsymbol{\omega}_b{}'\boldsymbol{\iota}_3$ is the weight on T-bills at time $t+b\frac{T}{B}$ and

$$u(W_B)=\frac{W_{t+T}^{1-\gamma}}{1-\gamma}=\frac{W_B^{1-\gamma}}{1-\gamma}.$$

With regular rebalancing the investor's optimization problem is therefore

$$\begin{aligned}
&\max_{\{\boldsymbol{\omega}_j\}_{j=0}^{B-1}} \quad E_t\left[\frac{W_B^{1-\gamma}}{1-\gamma}\right] \\
&s.t.\ W_{b+1}=W_b\left\{(1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right)+\boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{b+1}+\varphi r^f\boldsymbol{\iota}_3\right)\right\} \qquad (9)\\
&\mathbf{R}_{b+1}\equiv\mathbf{r}_{t_b+1}+\mathbf{r}_{t_b+2}+...+\mathbf{r}_{t_{b+1}},\quad b=0,1,...,B-1
\end{aligned}$$

Incorporating investors' use of predictor variables $\mathbf{z}_b$, at the decision times $b=0,1,...,B-1$, we get the following derived utility of wealth:

$$J(W_b,\mathbf{r}_b,\mathbf{z}_b,\boldsymbol{\theta}_b,\boldsymbol{\pi}_b,t_b)\equiv\max_{\{\boldsymbol{\omega}_j\}_{j=b}^{B-1}}E_{t_b}\left[\frac{W_B^{1-\gamma}}{1-\gamma}\right]. \qquad (10)$$

Here $\boldsymbol{\theta}_b=\left(\left\{\Omega_{i,b}^*,\{\mathbf{A}_{j,i,b}^*\}_{j=1}^{p}\right\}_{i=1}^{k},\mathbf{P}_b\right)$ is a vector that collects the parameters of the regime switching model and $\boldsymbol{\pi}_b$ is the vector of probabilities for each of the k possible states conditional on information at time t_b.

Consistent with common practice, we rule out short-selling. Letting $\mathbf{e}_j$ be a 3×1 vector of zeros with a 1 in the jth place and $\boldsymbol{\iota}_3$ be a 3×1 vector of ones, this means that $\mathbf{e}_j'\boldsymbol{\omega}_b\in[0,1]$ $(j=1,2,3)$ and $\boldsymbol{\omega}_b'\boldsymbol{\iota}_3\leq1$.[5]

Under power utility the Bellman equation conveniently simplifies to

$$J(W_b,\mathbf{r}_b,\mathbf{z}_b,\boldsymbol{\theta}_b,\boldsymbol{\pi}_b,t_b)=\frac{W_b^{1-\gamma}}{1-\gamma}Q(\mathbf{r}_b,\mathbf{z}_b,\boldsymbol{\theta}_b,\boldsymbol{\pi}_b,t_b)\quad(\gamma\neq1). \qquad (11)$$

Investors' learning is incorporated in this setup by letting them optimally revise their beliefs about the underlying state at each point in time using the updating equation:

$$\boldsymbol{\pi}_{b+1}(\hat{\boldsymbol{\theta}}_t)=\frac{\boldsymbol{\pi}_b(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi}\odot\boldsymbol{\eta}(\mathbf{y}_{b+1};\hat{\boldsymbol{\theta}}_t)}{(\boldsymbol{\pi}_b(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi}\odot\boldsymbol{\eta}(\mathbf{y}_{b+1};\hat{\boldsymbol{\theta}}_t))\boldsymbol{\iota}_k}. \qquad (12)$$

[5]Short-selling constraints were found not to be binding except for at the very short investment horizons and only have a marginal impact on our results.

Here $\odot$ denotes the element-by-element product, $\mathbf{y}_b \equiv (\mathbf{r}_b' \ \mathbf{z}_b')'$, and $\boldsymbol{\eta}(\mathbf{y}_{b+1})$ is the $k \times 1$ vector whose jth element gives the density of observation $\mathbf{r}_{b+1}$ in the jth state at time t_{b+1} conditional on $\hat{\boldsymbol{\theta}}_b$:

$$\boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_b) \equiv \begin{bmatrix} f(\mathbf{y}_{b+1}|s_{b+1} = 1, \{\mathbf{y}_{t_b-j}\}_{j=0}^{p-1}; \hat{\boldsymbol{\theta}}_b) \\ f(\mathbf{y}_{b+1}|s_{b+1} = 2, \{\mathbf{y}_{t_b-j}\}_{j=0}^{p-1}; \hat{\boldsymbol{\theta}}_b) \\ \vdots \\ f(\mathbf{y}_{b+1}|s_{b+1} = k, \{\mathbf{y}_{t_b-j}\}_{j=0}^{p-1}; \hat{\boldsymbol{\theta}}_b) \end{bmatrix}$$

$$= \begin{bmatrix} (2\pi)^{-\frac{N}{2}} |\hat{\Omega}_1^{-1}|^{\frac{1}{2}} \exp\left[-\frac{1}{2} \left(\mathbf{r}_b - \hat{\boldsymbol{\mu}}_1 - \sum_{j=0}^{p-1} \hat{\mathbf{A}}_{1j} \mathbf{y}_{t_b-j} \right)' \hat{\Omega}_1^{-1} \left(\mathbf{r}_b - \hat{\boldsymbol{\mu}}_1 - \sum_{j=0}^{p-1} \hat{\mathbf{A}}_{1j} \mathbf{y}_{t_b-j} \right) \right] \\ (2\pi)^{-\frac{N}{2}} |\hat{\Omega}_2^{-1}|^{\frac{1}{2}} \exp\left[-\frac{1}{2} \left(\mathbf{r}_b - \hat{\boldsymbol{\mu}}_2 - \sum_{j=0}^{p-1} \hat{\mathbf{A}}_{2j} \mathbf{y}_{t_b-j} \right)' \hat{\Omega}_2^{-1} \left(\mathbf{r}_b - \hat{\boldsymbol{\mu}}_2 - \sum_{j=0}^{p-1} \hat{\mathbf{A}}_{2j} \mathbf{y}_{t_b-j} \right) \right] \\ \vdots \\ (2\pi)^{-\frac{N}{2}} |\hat{\Omega}_k^{-1}|^{\frac{1}{2}} \exp\left[-\frac{1}{2} \left(\mathbf{r}_b - \hat{\boldsymbol{\mu}}_k - \sum_{j=0}^{p-1} \hat{\mathbf{A}}_{kj} \mathbf{y}_{t_b-j} \right)' \hat{\Omega}_k^{-1} \left(\mathbf{r}_b - \hat{\boldsymbol{\mu}}_k - \sum_{j=0}^{p-1} \hat{\mathbf{A}}_{kj} \mathbf{y}_{t_b-j} \right) \right] \end{bmatrix} \tag{13}$$

Such learning effects can be important since optimal portfolio choices depend not only on future values of asset returns and predictor variables ($\mathbf{r}_b$, $\mathbf{z}_b$), but also on future perceptions of the likelihood of being in each of the unobservable regimes ($\boldsymbol{\pi}_{t_b+j}$). In practice, the state probabilities are updated in calendar time and not at the frequency of the portfolio rebalancing. This approach is consistent with the notion that investors do not observe the true state. Appendix A derives equation (12) and provides further details on our implementation.

Solving (9) by standard backward induction techniques is, unfortunately, infeasible given the large number of parameters required to characterize the joint distribution of stock and bond returns so we assume that investors condition on their parameter estimates, $\hat{\boldsymbol{\theta}}_b$, although they recursively learn about the underlying state probabilities.

Since W_b is known at time t_b, $Q(\cdot)$ simplifies to

$$Q(\mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\pi}_b, t_b) = \max_{\boldsymbol{\omega}_b} \ E_{t_b} \left[\left(\frac{W_{b+1}}{W_b} \right)^{1-\gamma} Q\left(\mathbf{r}_{b+1}, \mathbf{z}_{b+1}, \boldsymbol{\pi}_{b+1}, t_{b+1}\right) \right]. \tag{14}$$

Optimal portfolio choices will reflect not only hedging demands for assets due to stochastic shifts in investment opportunities but also a hedging motive caused by changes in investors' beliefs about future state probabilities $\boldsymbol{\pi}_{t_b+j}$. In the absence of predictor variables, $\mathbf{z}_t$, the investor's perception of the regime probabilities, $\boldsymbol{\pi}_b$, is the only state variable and the basic recursions can be written

$$\begin{aligned} Q(\boldsymbol{\pi}_b, t_b) &= \max_{\boldsymbol{\omega}_b} E_{t_b}\left[\left(\frac{W_{b+1}}{W_b}\right)^{1-\gamma} Q\left(\boldsymbol{\pi}_{b+1}, t_{b+1}\right)\right], \\ \boldsymbol{\pi}_{b+1}(\hat{\boldsymbol{\theta}}_t) &= \frac{\boldsymbol{\pi}_b(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi} \odot \boldsymbol{\eta}(\mathbf{r}_{b+1}; \hat{\boldsymbol{\theta}}_t)}{\left(\boldsymbol{\pi}_b(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi} \odot \boldsymbol{\eta}(\mathbf{r}_{b+1}; \hat{\boldsymbol{\theta}}_t)\right)\boldsymbol{\iota}_k}. \end{aligned} \tag{15}$$

The paper whose modeling approach is most closely related to ours is Ang and Bekaert (2001). In a two-state model with observable states Ang and Bekaert evaluate the claim that international equity holdings might rationally have a home bias if return correlations increase in bull markets. Assuming observable states, they find that optimal portfolio weights depend both on the current regime and on the investment horizon. While our paper shares a similar regime switching setup, we address a very different question, namely a US investor's strategic asset allocation between bonds, stocks and cash and we find that a somewhat different model involving four states is required to capture their joint return distribution. Furthermore, we model regimes as unobservable and therefore explicitly address the effects on hedging demands arising from investors' recursive updating in their beliefs concerning the underlying state probabilities. We calculate asset allocations under optimal filtering, allowing for unobservable states. In our model investors therefore have to account for revisions in future beliefs $\boldsymbol{\pi}_{t_b+j}$ $(j \geq 1)$ when determining their current asset allocation.

4. Asset Allocation Results: Buy-and-Hold Investor

As a benchmark we first consider the asset allocation strategy of a buy-and-hold investor who only solves the asset allocation problem once, at time t. Following Barberis (2000), we approximate the integral in the expected utility functional through Monte Carlo methods:

$$\max_{\boldsymbol{\omega}_t} N^{-1} \sum_{n=1}^{N} \left\{ \frac{\left[(1-\boldsymbol{\omega}_t'\boldsymbol{\iota}_3)\exp\left(Tr^f\right) + \boldsymbol{\omega}_t' \exp\left(\sum_{i=1}^{T}(r^f\boldsymbol{\iota}_3 + \mathbf{r}_{t+i,n})\right)\right]^{1-\gamma}}{1-\gamma} \right\}, \tag{16}$$

where $\tilde{r}^p_{t+i,n} \equiv \boldsymbol{\omega}'_t \exp\left(\sum_{i=1}^{T}(r^f \boldsymbol{\iota}_3 + \mathbf{r}_{t+i,n})\right)$ is the portfolio return in the nth Monte Carlo simulation. Each simulated path of T excess returns is generated using draws from the estimated regime switching model conditional on $\widehat{\boldsymbol{\theta}}_t$. The simulations allow the regimes to switch stochastically as governed by the transition matrix, $\mathbf{P}$. We use $N = 30,000$ simulations and vary the investment horizon T between six and 120 in increments of six months.[6] The coefficient of relative risk aversion is initially set at $\gamma = 5$.

Figure 3 plots the optimal asset allocations at horizons $T = 6, 24$ and 120 months over the period 1980-1999. The portfolio weights are reasonably stable over time as investors acknowledge that the current state will not last indefinitely. The weight on small stocks fluctuates between zero and 50%, while the weight on large stocks varies between 0 and 70%. Bond holdings vary between 0 and 30%. T-bill holdings average around 20% for a long-term investor but fluctuate between 0 and 40% over the sample.

To put the effect of regime switching on optimal asset allocations into perspective, Figure 3 also shows the optimal asset holdings under independently and identically distributed returns where the optimal portfolio weights are constant across investment horizons. We refer to these as the 'myopic weights'. The optimal weights in the myopic portfolio are only non-zero for long-term bonds (70%) and large stocks (30%). Although T-bills were found to be important in some states in the regime switching model, surprisingly the myopic investor does not hold T-bills.

4.1. *Optimal Asset Allocation in the Four Regimes*

We found in Section 2 that the four regimes identified in the joint distribution of stock and bond returns had economic interpretations as crash, slow growth, bull and recovery states. To better understand the role of these economic states for asset allocation, Figure 4 shows optimal asset allocations starting from each of the current states, i.e. $\boldsymbol{\pi} = \mathbf{e}_j$ $(j = 1, 2, 3, 4)$.

State 1 is a low return state with little persistence. As the investment horizon grows, investors can be reasonably certain of leaving this state and move to better ones. The weight on stocks is therefore negligible for small T but increases rapidly

[6]A large number of simulations is needed to adequately account for the occurrence of regimes with low steady-state probabilities. Appendix C provides details on the accuracy of the numerical solutions when asset returns follow a regime switching model.

as T grows, producing an upward-sloping investment demand. Although it is sensible to avoid stocks almost completely at short horizons, the low persistence of regime 1 along with the high probability of switching to the recovery state with high mean returns leads to a rapid increase in the optimal allocation to stocks as the horizon expands. Even so, the optimal allocation to stocks never exceeds 35% when starting from the crash state. The allocation to bonds grows from zero to 30%, while the allocation to T-bills shows the opposite pattern, starting at 100% of the portfolio, declining to close to 40% at the longest 10-year horizon.

In the slow growth state (regime 2) the demand for small stocks is always zero while conversely that for large stocks is very high, starting at 100% at the shortest horizon before declining to a level near two-thirds of the portfolio at horizons longer than six months. The remainder of the portfolio is invested in bonds and T-bills with T-bills having approximately twice the weight of bonds.

In the bull state, small stocks take up 70% of the portfolio at short horizons before declining to 20% for horizons greater than six months. The reverse pattern is seen for large stocks that start at 30% for short horizons and grow to a level near 50% for horizons longer than six months. Bond and T-bill allocations starting from this state are close to zero at short horizons, rising to around 10% and 15%, respectively, at longer horizons.

Finally, in the recovery state, 100% of the portfolio is allocated to small stocks for an investor with a short horizon. This proportion declines to 40% for horizons longer than one year, while the allocation to large stocks and bonds rises from zero to 30% as the horizon is extended from 1 to 12 months. In this state practically nothing is invested in T-bills.

These results suggest that investors' perceptions of the current state probability is a key determinant of the relationship between the investment horizon and the optimal asset allocation, and therefore of the degree to which an investor can exploit predictability in asset returns.

At times, there may of course be considerable uncertainty surrounding the current regime and Figure 5 shows optimal asset allocations for two such scenarios. One assumes that the states have the same probability (25%) while the other scenario assumes that the states are represented by their steady-state probabilities (9%, 40%, 28% and 23% for states 1-4). The full extent to which the relationship between asset allocations and the investment horizon depends on the underlying state beliefs is again clear from this figure. For almost all assets the sign of the slope of the investment demand at short horizons is opposite in the two scenarios.

Overall, we find that the well-known investment advice of increased exposure to stocks the longer the horizon is not robust to how predictability in returns is modeled and may even be more of an exception than the rule.[7] In three of four states it makes sense for a buy-and-hold investor to be more cautious towards stocks as the investment horizon rises.

4.2. *Effects of Risk Aversion*

Up to this point we have assumed a coefficient of relative risk aversion, $\gamma = 5$, but investors may well have different degrees of risk aversion and it is of interest to see how strongly the results vary across different values of this parameter. Figure 6 therefore shows portfolio weights as a function of γ. In each graph we plot weights for each of the four regimes. To save space, we only focus on the combined allocation to stocks (small and large caps) and long-term bonds and present plots for three investment horizons, $T = 1, 24$, and 120 month. For comparison we also show results under the benchmark of no predictability (independently and identically distributed (IID) returns with the same mean and variance as in the data). Unsurprisingly, independent of the current state, the overall allocation to stocks declines as γ increases. As the investment horizon increases, the allocation to stocks becomes increasingly sensitive to γ at relatively low levels of risk aversion and the current state matters less. The difference between the allocation to stocks under the four state model and under IID returns decreases as the investment horizon grows. This is easy to explain since the current state probabilities matter less to a buy-and-hold investor, the longer the investment horizon. States clearly matter, however, at the short and medium horizons, irrespective of γ and their presence leads to a very different allocation from that under IID returns.

The bond allocation plots shown in the second column show that there is an inverse U-shaped relationship between γ and the allocation to bonds even under the IID model. The bond allocation rises sharply at low levels of risk aversion and

[7]For instance, Siegel (1994) argues that long-run investors should not try to time the stock market, but should buy and hold large equity portfolios because they involve moderate risks at long horizons. Our results show that both the proportion allocated to stocks and its decomposition into small and large stocks depends on the initial state.

peaks at a level close to 85% for γ just below 20. Far less is allocated to bonds under the regime switching model irrespective of the initial state although, on a much smaller scale, a similar non-monotonic pattern is observed. Irrespective of the assumed level of γ, the presence of regimes therefore lead to significantly different bond allocations than under no predictability.

4.3. *Asset Allocation and the Dividend Yield*

To preserve space we only report two exercises for the model that includes the dividend yield as a predictor variable. The first, presented in Figure 7, shows the optimal asset allocation as a function of the investment horizon in the extended model that includes the dividend yield set at its sample average. Asset allocations continue to vary significantly across the four states. Starting from state 1 the allocation to stocks (small stocks in particular) continues to rise as a function of the horizon and peaks at close to 40% of the portfolio at the 10-year horizon. The allocation to bonds is non-monotonic, starting from zero at the shortest horizon, rising to a level close to 90% at the six month horizon before declining to 60% at the longest horizons. T-bills form 100% of the portfolio at the shortest 1-month horizon but then see their allocation decline sharply to zero at horizons longer than six months.

The allocation to stocks continues to decline when the model starts from states two or four, although it now only declines to a level near 80-85% at the 10-year horizon. Consistent with this decline, the allocation to long-term bonds makes up for the remainder and there is no demand for T-bills in these two states. In the third (bull) state the allocation to stocks is now upwardsloping as a function of the horizon in contrast to what we found in the model without the dividend yield shown in Figure 4.

Figure 8 shows the effect of changing the dividend yield starting from steady state probabilities. We use a range of values of the dividend yield spanning its mean value plus or minus three standard deviations. As before, the equity investment schedules are either decreasing or essentially flat. The allocation to small stocks tends to be more sensitive to the yield than that of the large stocks. When the yield is very low, the allocation to stocks is very small and the allocation to T-bills very large, but declining in the horizon. As expected, the higher the dividend yield, the larger the allocation to stocks. This is consistent with the common finding of a positive correlation between the yield and expected returns. It shows that irrespective of the presence of regimes, we get very sensible results for the effect of changing the dividend yield on the optimal asset allocation.

We summarize these findings as follows. First, by comparing Figures 4 and 7, it is obvious that there are some differences between the optimal asset allocations depending on whether the dividend yield is part of the model. In the model extended to include the yield there is less of a role for T-bills in the optimal asset allocation, while conversely long-term bonds form a larger part of the portfolio. Furthermore, we also found that, irrespective of the presence of regimes, the larger the yield, the larger the typical allocation to stocks.

5. Rebalancing

The buy-and-hold results presented thus far ignore the possibility of rebalancing. However, in the presence of time-varying investment opportunities, investors should respond by changing their portfolio weights. In this section we therefore consider the effects of periodic rebalancing on optimal asset allocations. Once again we numerically solve the Bellman equation by discretizing the compact interval that defines the domain of *each* of the state variables on G equi-distant points[8] and using backward induction methods.[9] Suppose that $Q\left(\boldsymbol{\pi}_{b+1}, t_{b+1}\right)$ is known at all points $\boldsymbol{\pi}_{b+1} = \boldsymbol{\pi}^j_{b+1}$, $j = 1, 2, ..., G^{k-1}$. This will be true at time $t_B \equiv t + T$ as $Q(\boldsymbol{\pi}^j_B, t_B) = 1$ for all values of $\boldsymbol{\pi}^j_B$ on the grid. Then we can solve equation (9) to obtain $Q\left(\boldsymbol{\pi}_b, t_b\right)$ by choosing ω_b to maximize

$$E_{t_b}\left[\left\{(1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right)+\boldsymbol{\omega}_b'\exp(\mathbf{R}_{b+1,n}(s_b)+\varphi r^f\boldsymbol{\iota}_3)\right\}^{1-\gamma}Q(\boldsymbol{\pi}^j_{b+1}, t_{b+1})\right]. \tag{17}$$

The multiple integral defining the conditional expectation is again calculated by Monte Carlo methods. For each $\boldsymbol{\pi}_b = \boldsymbol{\pi}^j_b$, $j = 1, 2, ..., G^{k-1}$ on the grid we draw N samples of $\varphi-$period excess returns $\{\mathbf{R}_{b+1,n}(s_b)\}_{n=1}^N$ from the regime switching

[8]For example, this interval is $[0, 1]$ when the only state variable is future, updated state probabilities, while it is $[\overline{dy} - \alpha\sigma_{dy}, \overline{dy} + \alpha\sigma_{dy}]$ (where α is some constant determining the width of the interval) when the dividend yield enters the problem.

[9]For instance, when $G = 11$ and the dividend yield is not modeled, the points are defined as 0, 0.1, 0.2, ..., 1 and a $(k-1)$-dimensional grid on a maximum of G^{k-1} points is built. The grid has fewer than G^{k-1} points since each of the points must also satisfy the constraint $\boldsymbol{\pi}^j_b\boldsymbol{\iota}_k = 1$, $\forall j$.

model, where $\mathbf{R}_{b+1,n}(s_b) \equiv \sum_{i=1}^{\varphi} \mathbf{r}_{t_b+i,n}(s_b)$. The expectation (17) is then approximated as

$$N^{-1}\sum_{n=1}^{N}\left[\left\{(1-\boldsymbol{\omega}_b'\iota_3)\exp\left(\varphi r^f\right)+\boldsymbol{\omega}_b'\exp(\mathbf{R}_{b+1,n}(s_b)+\varphi r^f\boldsymbol{\iota}_3)\right\}^{1-\gamma}Q(\boldsymbol{\pi}_{b+1}^{(j,n)},t_{b+1})\right],$$

where $\boldsymbol{\pi}_{b+1}^{(j,n)}$ denotes the element $\boldsymbol{\pi}_{b+1}^{j}$ on the grid used to discretize the state space that is closest to

$$\boldsymbol{\pi}_{b+1,n}=\frac{\boldsymbol{\pi}_b\hat{P}_t^{\varphi}\odot\boldsymbol{\eta}(\mathbf{r}_{b+1,n};\hat{\boldsymbol{\theta}}_t)}{\left(\boldsymbol{\pi}_b\hat{P}_t^{\varphi}\odot\boldsymbol{\eta}(\mathbf{r}_{b+1,n};\hat{\boldsymbol{\theta}}_t)\right)\boldsymbol{\iota}_k},$$

using the distance measure $\sum_{i=1}^{k-1}|\boldsymbol{\pi}_{b+1}^{j}e_i-\boldsymbol{\pi}_{b+1,n}e_i|$. Starting from t_{B-1}, we work backwards through the B rebalancing points until $\hat{\boldsymbol{\omega}}_t \equiv \hat{\boldsymbol{\omega}}_0$ is obtained. Appendix B provides further details on the iterative backward solution to the asset allocation problem.

Table IV shows optimal portfolio weights for stocks and bonds for a range of investment horizons, T, under different values of the rebalancing frequency, $\varphi = 1, 3, 6, 12, 24$ months as well as the buy-and-hold scenario, $\varphi = T$ and the allocation under IID returns. For a given T, as φ declines, investors become more 'aggressive' in reacting to the current state probabilities. It is easy to see why: the smaller is φ, the shorter is the period over which the investor commits her wealth to a given portfolio. As a result, the investor puts less weight on the steady-state return distribution and increasing weight on the current state, s_t. Consequently, the weight on stocks in the crash state declines as φ decreases and rebalancing becomes more frequent. For instance, when $T = 120$ and $\varphi = 1$ (monthly rebalancing), investors hold no stocks in the bear state, preferring instead to wait for an improvement in the investment opportunity set. In contrast, when φ exceeds the average duration of this regime (e.g., $\varphi = 12$), it is optimal to invest some money in stocks (40%), although this weight is quite low. In states 2-4 investors increase their commitment to stocks as the time between rebalancing declines. In fact, when $\varphi = 1$, the optimal weight on stocks is close to 100% in these three regimes, irrespective of the investment horizon, T.

Keeping the rebalancing frequency, φ, constant, portfolio choices also depend on the investment horizon, T. In the presence of rebalancing opportunities, demand for stocks is no longer monotonically increasing or decreasing in regimes 2-4 and at first decreases before it increases again at longer horizons. In the crash state, demand for stocks is upward sloping although increasingly flat as φ declines. Once again, we find that, in general, it is not true that investors with longer horizons should allocate more to stocks. Finally, Table IV shows the IID weights as a benchmark. The resulting weight is independent of T and is strikingly different from most of the weights under the regime switching model.

As the investment horizon grows, non-monotonic patterns are observed in the allocation to bonds which in most cases first rises then declines. Starting from the crash state the allocation to bonds is generally lower, the more frequent the rebalancing (smaller φ) since the investor does not have to account for unexpected shifts to a better state but can afford to wait for such a shift to occur. In general, if rebalancing can occur very frequently, little is invested in bonds. This is because market timing opportunities are more significant for stocks (particularly small caps). The one exception is when the initial state reflects the steady-state probabilities where (excluding $\varphi = 1$) more frequent rebalancing may actually lead to a higher allocation to bonds.

6. **Optimal Savings and Portfolio Choices**

Up to this point we have ignored interim consumption. However, some recent papers (e.g., Brandt (1999), Campbell and Viceira (1999) and Lynch (2001)) have considered this issue and we follow this literature by letting the investor consume and rebalance the portfolio weights every φ months. Let $\boldsymbol{\omega}_b$ and C_b be the portfolio allocations and consumption flow at the rebalancing times $t+b\frac{T-t}{B}$, $b = 0, 1, ..., B-1$. Assuming the investor has additively separable power utility, expected lifetime utility is given by:

$$E_0\left[\sum_{b=0}^{B}\beta^b\frac{C_b^{1-\gamma}}{1-\gamma}\right], \tag{18}$$

where the expectation is taken conditional on the information available at time t ($b = 0$) and the subjective discount factor $\beta = (1+\rho)^{-\frac{\varphi}{12}}$, where ρ is the investor's annualized subjective rate of time preference. In the presence of consumption, the investor's wealth follows the process

$$\begin{aligned} W_{b+1} &= (W_b - C_b)\left[(1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{b+1} + \varphi r^f\boldsymbol{\iota}_3\right)\right] \\ &= W_b s_b\left[(1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{b+1} + \varphi r^f\boldsymbol{\iota}_3\right)\right], \end{aligned} \tag{19}$$

where $s_b \equiv (W_b - C_b)/W_b$ is the fraction of wealth saved at time b and $\mathbf{R}_{b+1} \equiv \mathbf{r}_{t_b+1} + \mathbf{r}_{t_b+2} + ... + \mathbf{r}_{t_{b+1}}$. The dynamic program implied by (18)-(19) can be solved by choosing a sequence of portfolio allocations and savings ratios $\{\boldsymbol{\omega}_b, s_b\}_{b=0}^{B-1}$ to get the following value function:[10]

$$J(W_b, \mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) \equiv \max_{\{\boldsymbol{\omega}_j, s_j\}_{j=b}^{B-1}} E_{t_b}\left[\sum_{j=0}^{B}\beta^j\frac{(1-s_{b+j})^{1-\gamma}W_{b+j}^{1-\gamma}}{1-\gamma}\right], \tag{20}$$

The Bellman equation is

$$J(W_b, \mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) \equiv \max_{\boldsymbol{\omega}_b, s_b}\left\{\frac{(1-s_b)^{1-\gamma}W_b^{1-\gamma}}{1-\gamma} + \beta E_{t_b}\left[J(W_{b+1}, \mathbf{r}_{b+1}, \mathbf{z}_{b+1}, \boldsymbol{\theta}_{b+1}, \boldsymbol{\pi}_{b+1}, t_{b+1})\right]\right\},$$

subject to the dynamic budget constraint (19). Under power utility this simplifies to

$$J(W_b, \mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) = \frac{W_b^{1-\gamma}}{1-\gamma}Q(\mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b).$$

It is well known that only in the special case where $\gamma = 1$ (logarithmic utility) can the optimal savings ratio be determined in closed-form (while $\hat{\boldsymbol{\omega}}_b$ cannot) independently of the investment opportunities. When $\gamma > 1$ both $\hat{s}_b$ and $\hat{\boldsymbol{\omega}}_b$ must be determined using numerical methods and, although the optimal portfolio choice does not depend on consumption choices, $\hat{s}_b$ always depends on the optimal portfolio structure.

Assuming once again that the investor uses the optimal filtering algorithm (12) to update state probabilities, $\hat{\boldsymbol{\pi}}_t$, the Bellman equation becomes

$$\begin{aligned} Q(\mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\pi}_b, t_b) = \max_{\boldsymbol{\omega}_b, s_b}\Big\{(1-s_b)^{1-\gamma} + \beta s_b^{1-\gamma}E_{t_b}\Big[&\left((1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{b+1} + \varphi r^f\boldsymbol{\iota}_3\right)\right)^{1-\gamma} \\ &\times Q(\mathbf{r}_{b+1}, \mathbf{z}_{b+1}, \boldsymbol{\pi}_{b+1}, t_{b+1})\Big]\Big\} \end{aligned} \tag{21}$$

[10] Obviously at time T, $s_B = 0$ holds while $\boldsymbol{\omega}_B$ is irrelevant.

In the absence of predictor variables, the problem simplifies to the basic recursions

$$\begin{aligned}
Q\left(\boldsymbol{\pi}_b, t_b\right) &= \max_{\boldsymbol{\omega}_b, s_b} \Big\{ (1-s_b)^{1-\gamma} + \beta s_b^{1-\gamma} \\
&\quad \times E_{t_b}\left[\left((1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3) \exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b' \exp\left(\mathbf{R}_{b+1} + \varphi r^f \boldsymbol{\iota}_3\right)\right)^{1-\gamma} Q\left(\boldsymbol{\pi}_{b+1}, t_{b+1}\right)\right]\Big\} \\
\boldsymbol{\pi}_{b+1}(\hat{\boldsymbol{\theta}}_t) &= \frac{\boldsymbol{\pi}_b(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi} \odot \boldsymbol{\eta}(\mathbf{r}_{b+1}; \hat{\boldsymbol{\theta}}_t)}{\left(\boldsymbol{\pi}_b(\hat{\boldsymbol{\theta}}_t)\hat{P}_t^{\varphi} \odot \boldsymbol{\eta}(\mathbf{r}_{b+1}; \hat{\boldsymbol{\theta}}_t)\right)\boldsymbol{\iota}_k},
\end{aligned}$$

where $\boldsymbol{\pi}_b$ is the only state variable. Again, we discretize the state space defined by $k-1$ regime probabilities in $\boldsymbol{\pi}_b$ and solve the Bellman equation by backward induction. Appendix B provides further details.

Figure 9 shows how the consumption-wealth ratio depends on the underlying state probabilities and the investment horizon by plotting $\hat{C}_t/W_t$ in each of the four regimes assuming consumption takes place at the beginning and end of the horizon. For comparison we also show the optimal value of C_b/W_b obtained when probabilities are set at their steady-state values. At short horizons all consumption schedules start at roughly 50%, i.e. half of the wealth is consumed and the other half is invested. However, the shape of the optimal consumption schedules depends on the initial state probabilities.

In the crash state, investment opportunities are poor, although (given the low persistence of this state) at long horizons an investor expects returns that exceed the subjective discount rate so the optimal consumption schedules are upward sloping, albeit rather flat and uniformly below those starting from the steady-state probabilities. Even at a 10-year horizon, starting from the crash state, no more than 55% of the wealth should be consumed. In regime 2, investment opportunities are moderately favorable so the consumption-wealth ratio is again upward sloping. However this state is also quite persistent, which accounts for the moderate steepness of the slopes starting from this state. Investment opportunities are both good and persistent in state 3, so strong income effects induce a rational investor to consume a higher percentage of wealth even at intermediate horizons (as high as 67% for a 10-year horizon). Finally, in state 4, investment opportunities are very good but transitory. While the excellent short-term investment prospects generate upward sloping consumption demand, they are also flat (intermediate between regime 1 and 2) reflecting the uncertainty about future states.

Overall, these results are consistent with those reported by Brandt (1999) and Campbell and Viceira (1999). In a two-period buy-and-hold exercise, Brandt finds that consumption choices are insensitive to the values assumed by a range of prediction variables (dividend yield, term and default spreads, past excess stock returns) and do not differ systematically from their unconditional estimates (that ignore predictability). Numerically, his estimates of the optimal consumption-wealth ratio are close to ours, the main difference being that we find a stronger sensitivity of $\hat{C}_t/W_t$ to the horizon in state three. The explanation for this is likely to be that the prediction variables used in Brandt's study mostly pick up low-frequency predictability patterns (e.g., the dividend yield typically changes very slowly) while our regime switching model captures predictability patterns at a higher frequency that further affect not just the conditional mean but the entire probability distribution of returns.

Table V presents results for $\varphi = 3, 12$, and T that allow for a more realistic setting with interim consumption over the interval $[t, T]$. Once again we show separate results starting from each of the four states. For comparison, we also report consumption-wealth ratios calculated under the assumption of logarithmic utility ($\gamma = 1$), when $\hat{C}_b/W_b$ can be found in closed-form and equals $(1-\beta_\varphi)/(1-\beta_\varphi^{B-b+1})$, where $B - b$ is the number of rebalancing points over the interval $[t, T]$ and $\beta_\varphi = (1+\rho)^{-\frac{\varphi}{12}}$. Substitution and income effects cancel exactly under this benchmark.

Under annual consumption and rebalancing ($\varphi = 12$) there is a clear pattern across regimes: In good states (2-4) the consumption-to-wealth ratio is systematically higher than under the logarithmic benchmark. This reflects consumption smoothing: in the good states, higher expected portfolio returns can support a higher C/W ratio than if low future returns are anticipated so C/W is higher than under logarithmic utility in states 2-4 and lower than this benchmark in state 1.

Finally, at long horizons the fact that investment opportunities (governed by the steady-state probabilities) are promising independently of the

current regime pushes $\hat{C}_t/W_t$ below the logarithmic benchmark in all states, e.g. to 1% vs. 3% for $T = 120$. Allowing for frequent rebalancing, and recalling that the expected long-run portfolio return exceeds the subjective rate of time preference, optimizing investors will reduce current consumption, invest as much as 99% of current wealth and collect the high payoffs/consumption streams later on.

7. Utility Cost and Parameter Estimation Uncertainty

It is natural to report a measure of the economic value of accounting for regimes in investors' asset allocation decisions. Following Lynch (2001), we obtain an estimate of this by comparing the investor's expected utility under the regime switching model to that assuming the investor is constrained to choose at time t an optimal saving ratio $\hat{s}_t^{IID}$ and portfolio weights $\hat{\boldsymbol{\omega}}_t^{IID}$ under the assumption that asset returns follow a multivariate Gaussian $N(\boldsymbol{\mu}, \Omega)$ process. In the latter case the portfolio choice and savings ratio are independent of the investment horizon and the value function for the constrained investor is

$$\begin{aligned} J_t^{IID} &\equiv \frac{1}{1-\gamma}(1-\hat{s}_t^{IID})^{1-\gamma}\sum_{b=0}^{B}\beta^b E_t\left[W_b^{1-\gamma}\right] \\ W_b &= W_{b-1}\hat{s}_t^{IID}\left[(1-(\boldsymbol{\omega}_t^{IID})'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right)+\boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{b+1}+\varphi r^f\boldsymbol{\iota}_3\right)\right]. \end{aligned}$$

The assumption of independent and identically distributed returns is a constrained special case of the model with regime switching and predictor variables, so

$$J_t^{IID} \leq J(W_t, \mathbf{r}_t, \mathbf{z}_t, \boldsymbol{\theta}_t, \boldsymbol{\pi}_t, t).$$

where $J(W_t, \mathbf{r}_t, \mathbf{z}_t, \boldsymbol{\theta}_t, \boldsymbol{\pi}_t, t)$ is the value function under the four state model. We compute the increase in initial wealth or compensatory premium, η_t^{IID}, an investor would require to derive the same level of expected utility from the constrained and unconstrained joint consumption and asset allocation problems:

$$\eta_t^{IID} = \left\{\frac{Q(\mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b)}{(1-\hat{s}_t^{IID})^{1-\gamma}\sum_{b=0}^{B}\beta^b E_t\left[(W_b)^{1-\gamma}\right]}\right\}^{\frac{1}{1-\gamma}} - 1. \tag{22}$$

For the buy-and-hold investor considered in Section 4, Figure 10 plots η_t^{IID}. The utility cost of ignoring regimes is modest at short horizons but is high when T is large. At short horizons the required compensation rate is close to zero, but it grows to a number close to 25% of wealth at the longest investment horizons although it should be recalled that this is over a 10-year horizon.

7.1. *Parameter Uncertainty*

So far our joint model for returns on small and large stocks and bonds does not involve any time-varying predictor variables, the presence of four regimes still complicates parameter estimation. For this reason we next consider the effect of parameter estimation uncertainty on our results. Since we use a different numerical method for deriving optimal portfolio weights, we cannot apply the approach used by Ang and Bekaert (2002). Instead we use the observation that, in large samples,

$$\sqrt{T}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)\overset{A}{\to}N(\mathbf{0},\mathbf{V}_{\theta}).$$

This allows us to set up the following bootstrap procedure. At step (iteration) q we draw (independently of previous iterations) a vector of parameters, $\widehat{\widehat{\boldsymbol{\theta}}}^{q}$, from the asymptotic distribution for $\widehat{\boldsymbol{\theta}}$, $N(\widehat{\boldsymbol{\theta}},T^{-1}\mathbf{V}_{\theta})$. Conditional on this draw, $\widehat{\widehat{\boldsymbol{\theta}}}^{q}$, we solve (9) obtaining a new vector of portfolio weights $\widehat{\hat{\boldsymbol{\omega}}}^{q}$. We repeat this process a large number of times Q. Confidence intervals for the optimal asset allocation $\hat{\boldsymbol{\omega}}_t$ can then be derived from the distribution for $\widehat{\hat{\boldsymbol{\omega}}}^{q}$, $l=1,2,...,Q$. Such an approach is computationally intensive, as (9) must be solved numerically thousands of times. For computational reasons we restrict the number of bootstrap trials to $Q=1,000$. Table VI shows the optimal asset allocation plus or minus one standard deviation of the bootstrapped distribution. Figures in bold indicate that this band does not include the IID asset allocation. These standard error bands are relatively wide, but never wide enough to overturn the validity of our conclusions concerning the optimal shape of equity investment schedules as a function of the investment horizon T. The allocation to stocks is upward sloping only in the crash state. In regimes 2-4, however, the equity investment schedules are downward sloping, as their bands 'decline' from a maximum of [0.7, 1] at $T=1$ to [0.4, 0.8] for long investment horizons.

These methods also allow us to reconsider the effect of parameter estimation uncertainty on the utility cost. We do so by calculating the compensatory variation $\eta_t^{IID,q}$ $q=1,2,...,1,000$ times using parameter estimates $\widehat{\boldsymbol{\theta}}$ drawn from their asymptotic distribution. Figure 10 shows 90% confidence intervals for the four regimes and two other configurations of the initial state probability vector $\hat{\boldsymbol{\pi}}_t$ (steady-state and equal probabilities) as a function of T. The null hypothesis of zero welfare loss implies that such intervals should systematically include 0 for all Ts; evidence that the lower bound of the interval is positive suggests that ignoring regime switching in solving asset allocation problems implies a statistically significant reduction in expected utility. Differences across regimes are modest and hardly visible in the plots.

The null of no significant welfare cost from ignoring regime switching is strongly rejected. Focussing on the lower bound of the confidence band, we notice that it is everywhere strictly positive and also economically significant (in excess of 3-4% of the initial wealth) for $T \geq 48$ months. At longer horizons, and independent of investors' initial state beliefs, such a lower bound always attains levels of 7-8%, which is a considerable fraction of wealth. Using a misspecified model in asset allocation decisions may thus be quite costly for long-term investors.

8. Conclusion

This paper proposed a model for the joint distribution of returns on small and large stocks and long-term bonds that provided evidence of the presence of four regimes, namely a crash, slow growth, bull and recovery state. Our model allows for rich patterns in the full return distribution and captures predictability not just in the conditional mean (which most of the existing literature has focused on) but also on predictability in the full (joint) return distribution, including the volatility, skew and degree of fat-tails. While two states were transitory (the crash and recovery state), the bear and bull state are more persistent with average durations of several months. This means that the regime switching model captures both short-term and long-term predictability.

We found that the presence of such regimes has a large impact on the optimal asset allocation. The optimal asset allocation varies significantly across regimes and the weights on the various asset classes strongly depend on which state the economy is perceived to be in. This means that asset allocations vary significantly over time even in the absence of 'outside' predictor variables such as the dividend yield.

We also found that the allocation to stocks is monotonically increasing as the investment horizon gets longer in only one of the four regimes. In the other regimes we observed a downward sloping allocation to stocks. The common investment advice of allocating more money to stocks the longer the investor's horizon should therefore be made conditional on the underlying market conditions.

Although the extension of the model to a joint consumption/asset allocation problem reveals that the effects of regime switching are only of second-order magnitude for optimal consumption/savings decisions, the qualitative differences across states remain noticeable. Moreover, we measure the welfare loss implied by the use of a simple model that disregards predictability and find that the compensatory variation measures are large not only in a statistical sense but also using an expected utility metric.

Importantly, these findings are robust to a number of issues, like the possibility for the investor to rebalance portfolio weights, the assumed coefficient of relative risk aversion, the presence of predictability from the dividend yield and the imposition of short sale constraints.

Appendix A - Derivation of Equation (12)

For simplicity, suppose that the calendar time governing the return process coincides with the rebalancing frequency φ. Let $\boldsymbol{\pi}_{b+1|b}$ be the vector of state probabilities for period t_{b+1} given the information available at time t_b, while $\mathbf{Y}_b \equiv (\mathbf{y}_0' \; \mathbf{y}_1' \; ... \; \mathbf{y}_b')'$ represents the history of the variables under consideration. Since $\boldsymbol{\pi}_{b+1|b}$ is a row vector that collects the state probabilities $\Pr\left(s_{b+1} = j|\mathbf{Y}_b; \hat{\boldsymbol{\theta}}_t\right)$ $(j = 1, 2, ..., k)$, the jth element of the $k \times 1$ vector $\boldsymbol{\pi}_{b+1|b} \odot \boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_t)$ can be interpreted as the conditional joint density of $\mathbf{y}_{b+1}$ and s_{b+1}:[11]

$$\Pr\left(\mathbf{y}_{b+1}, s_{b+1} = j|\mathbf{Y}_b; \hat{\boldsymbol{\theta}}_t\right) = \Pr\left(s_{b+1} = j|\mathbf{Y}_b; \hat{\boldsymbol{\theta}}_t\right) \times f(\mathbf{y}_{b+1}|s_{b+1} = j, \mathbf{Y}_b; \hat{\boldsymbol{\theta}}_t).$$

Furthermore, $f(\mathbf{y}_{b+1}|\mathbf{Y}_b; \hat{\boldsymbol{\theta}}_t)$, the density of $\mathbf{y}_{b+1}$ conditional on past information, is the sum of conditional densities $f(\mathbf{y}_{b+1}|s_{b+1} = j, \mathbf{Y}_b; \hat{\boldsymbol{\theta}}_t)$, $j = 1, 2, ..., k$, weighted by their respective regime probabilities. Hence it can be written as $(\boldsymbol{\pi}_{b+1|b} \odot \boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_t))\boldsymbol{\iota}_k$. It follows that the conditional distribution of s_{b+1} given all information available up to time t_{b+1} can be found as the ratio between $\Pr\left(\mathbf{y}_{b+1}, s_{b+1} = j|\mathbf{Y}_b; \hat{\boldsymbol{\theta}}_t\right)$ and $f(\mathbf{y}_{b+1}|\mathbf{Y}_b; \hat{\boldsymbol{\theta}}_t)$:

$$\Pr\left(s_{b+1} = j|\mathbf{Y}_b; \hat{\boldsymbol{\theta}}_t\right) = \frac{\Pr\left(\mathbf{y}_{b+1}, s_{b+1} = j|\mathbf{Y}_b; \hat{\boldsymbol{\theta}}_t\right)}{f(\mathbf{y}_{b+1}|\mathbf{Y}_b; \hat{\boldsymbol{\theta}}_t)} = \frac{\left(\boldsymbol{\pi}_{b+1|b} \odot \boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_t)\right)\mathbf{e}_j}{\left(\boldsymbol{\pi}_{b+1|b} \odot \boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_t)\right)\boldsymbol{\iota}_k},$$

so that the row vector $\boldsymbol{\pi}_{b+1|b+1}$ becomes

$$\boldsymbol{\pi}_{b+1|b+1} = \frac{\boldsymbol{\pi}_{b+1|b} \odot \boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_t)}{\left(\boldsymbol{\pi}_{b+1|b} \odot \boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_t)\right)\boldsymbol{\iota}_k}.$$

[11]Recall that $\odot$ is element-by-element multiplication.

Finally, from standard results on first-order Markov chains (e.g., Hamilton (1994, p.679)), it follows that $\boldsymbol{\pi}_{b+1|b} = \boldsymbol{\pi}_{b|b}\hat{\mathbf{P}}_t$ and hence

$$\boldsymbol{\pi}_{b+1|b+1} = \frac{\boldsymbol{\pi}_{b|b}\hat{\mathbf{P}}_t \odot \boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_t)}{\left(\boldsymbol{\pi}_{b|b}\hat{P}_t \odot \boldsymbol{\eta}(\mathbf{y}_{b+1}; \hat{\boldsymbol{\theta}}_t)\right) \boldsymbol{\iota}_k}.$$

Recalling that $\hat{\mathbf{P}}_t \in \hat{\boldsymbol{\theta}}_t$ and setting $\boldsymbol{\pi}_b \equiv \boldsymbol{\pi}_{b|b}$, we obtain equation (12).

Appendix B - Backward Solution of the Joint Consumption and Asset Allocation Problem under Regime Switching

Suppose the optimization problem has been solved backwards at the rebalancing points t_{B-1}, ..., t_{b+1} so that $Q(\boldsymbol{\pi}^j_{b+1}, t_{b+1})$ is known for all values $j = 1, 2, ..., G^{k-1}$ on the discretization grid. At each point $\boldsymbol{\pi}_b = \boldsymbol{\pi}^j_b$, it is possible to find $Q(\boldsymbol{\pi}^j_b, t_b)$ at time t_b. Monte Carlo approximation of the expectation in the objective function

$$(1-s_b)^{1-\gamma} + \beta s_b^{1-\gamma} E_{t_b}\left[\left((1-\boldsymbol{\omega}_b'\boldsymbol{\iota}_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b' \exp\left(\mathbf{R}_{b+1} + \varphi r^f \boldsymbol{\iota}_3\right)\right)^{1-\gamma} Q\left(\boldsymbol{\pi}_{b+1}, t_{b+1}\right)\right]$$

requires drawing N random samples of asset returns $\{\mathbf{R}_{b+1,n}(\boldsymbol{\pi}^j_b)\}_{n=1}^N$ from the $(b+1)\varphi$-step-ahead joint density conditional on period-t parameter estimates, $\widehat{\boldsymbol{\theta}}_t = \{\hat{\boldsymbol{\mu}}_{Mt}, \widehat{\Omega}_t, \hat{\mathbf{P}}_t\}$ assuming that, at each point, $\boldsymbol{\pi}^j_b$ is optimally updated to $\boldsymbol{\pi}_{b+1}(\boldsymbol{\pi}^j_b)$. The algorithm consists of the following steps:

1. For each possible value of the current regime, s_b, simulate N $\varphi-$period returns $\{\mathbf{R}_{b+1,n}(S_b)\}_{n=1}^N$ in calendar time from the regime switching model

$$\mathbf{r}_{t_b+j,n}(S_b) = \boldsymbol{\mu}_{s_{t_b+j}} + \boldsymbol{\varepsilon}_{t_b+j,n},$$

   where $\mathbf{R}_{b+1,n}(S_b) \equiv \sum_{j=1}^{\varphi} \mathbf{r}_{t_b+j,n}(s_b)$ and $\boldsymbol{\varepsilon}_{t_b+j,n} \sim N(0, \Omega_{s_{t_b+j}})$. At all rebalancing points this simulation allows for regime switching as governed by the transition matrix $\hat{\mathbf{P}}_t$. For example, if we start in regime 1, between t_b and $t_b + 1$ there is a chance $\hat{p}_{12} \equiv \mathbf{e}_1'\hat{\mathbf{P}}_t\mathbf{e}_2$ of switching to regime 2, and a chance $\hat{p}_{11} \equiv \mathbf{e}_1'\hat{\mathbf{P}}_t\mathbf{e}_1$ of staying in regime 1. At each point in time $\hat{\mathbf{P}}_t$ governs possible switches.

2. Combine the simulated $\varphi-$period returns $\{\mathbf{R}_{b+1,n}\}_{n=1}^{N}$ into a random sample of size N, using the probability weights contained in the row vector $\boldsymbol{\pi}_b^j$

$$\mathbf{R}_{b+1,n}(\boldsymbol{\pi}_b^i) = \sum_{j=1}^{k}(\boldsymbol{\pi}_b^i \mathbf{e}_j)\mathbf{R}_{b+1,n}(S_b = j).$$

3. Update the future regime probabilities perceived by the investor using the formula

$$\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}_b^i) = \frac{(\boldsymbol{\pi}_b^i)'\hat{\mathbf{P}}^{\varphi} \odot \boldsymbol{\eta}(\mathbf{R}_{b+1,n}(\boldsymbol{\pi}_b^i);\hat{\boldsymbol{\theta}}_t)}{\left((\boldsymbol{\pi}_b^i)'\hat{\mathbf{P}}^{\varphi} \odot \boldsymbol{\eta}(\mathbf{R}_{b+1,n}(\boldsymbol{\pi}_b^i);\hat{\boldsymbol{\theta}}_t)\right)\boldsymbol{\iota}_k}.$$

This gives an $N \times k$ matrix $\{\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}_b^i)\}_{n=1}^{N}$, whose rows correspond to simulated vectors of perceived regime probabilities at time t_{b+1}.

4. For all $n = 1, 2, ..., N$, calculate the value $\tilde{\boldsymbol{\pi}}_{b+1,n}^i$ on the discretization grid ($i = 1, 2, ..., G^{k-1}$) closest to $\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}_b^i)$ using the distance measure $\sum_{j=1}^{k-1}|\boldsymbol{\pi}_{b+1}^i\mathbf{e}_j - \boldsymbol{\pi}_{b+1,n}\mathbf{e}_j|$, i.e.

$$\tilde{\boldsymbol{\pi}}_{b+1,n}^i(\boldsymbol{\pi}_b^i) \equiv \underset{\mathbf{x}\in\times_{j=1}^{k-1}[0,1]}{\arg\min}\sum_{j=1}^{k-1}|\mathbf{x}\mathbf{e}_j - \boldsymbol{\pi}_{b+1,n}\mathbf{e}_j|.$$

Knowledge of the vector $\{\tilde{\boldsymbol{\pi}}_{b+1,n}^i(\boldsymbol{\pi}_b^i)\}_{n=1}^{N}$ allows us to build $\{Q(\boldsymbol{\pi}_{b+1}^{(i,n)}, t_{b+1})\}_{n=1}^{N}$, where $\boldsymbol{\pi}_{b+1}^{(i,n)} \equiv \tilde{\boldsymbol{\pi}}_{b+1,n}^i(\boldsymbol{\pi}_b^i)$ is a function of the assumed, initial vector of regime probabilities $\boldsymbol{\pi}_b^i$.[12]

5. Solve the program

$$\max_{\boldsymbol{\omega}_b(\boldsymbol{\pi}_b^j), s_b(\boldsymbol{\pi}_b^j)} (1-s_b)^{1-\gamma} \times +\beta s_b^{1-\gamma}$$
$$\times N^{-1}\sum_{n=1}^{N}\left[\left\{(1-\boldsymbol{\omega}_b'\iota_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{b+1,n}(\boldsymbol{\pi}_b^j) + \varphi r^f \boldsymbol{\iota}_3\right)\right\}^{1-\gamma} Q(\boldsymbol{\pi}_{b+1}^{(j,n)}, t_{b+1})\right],$$

[12] This step may be avoided when $Q(\boldsymbol{\pi}_{b+1}^i, t_{b+1})$ is constant for all values on the discretization grid. This happens when $t_{b+1} = T$ and implies that the portfolio weights determined at step $b+1$ $\{\hat{\boldsymbol{\omega}}_{b+1}(\boldsymbol{\pi}_{b+1}^i)\}$ are invariant to changes in $\boldsymbol{\pi}_{b+1}^i$. In this case the step simplifies to

$$\max_{\boldsymbol{\omega}_b(\boldsymbol{\pi}_b^i)} N^{-1}\sum_{n=1}^{N}\left[\left\{(1-\boldsymbol{\omega}_b'\iota_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b'\exp\left(\mathbf{R}_{b+1,n}(\boldsymbol{\pi}_b^i) + \varphi r^f \mathbf{e}_1\right)\right\}^{1-\gamma}\right].$$

which for large values of N provides an arbitrarily precise Monte-Carlo approximation to expected utility. The value function evaluated at the optimal portfolio weights $\hat{\boldsymbol{\omega}}_b(\boldsymbol{\pi}_b^j)$ and savings ratio $\hat{s}_b(\boldsymbol{\pi}_b^j)$ defines $Q(\boldsymbol{\pi}_b^j, t_b)$ for the jth point on the initial grid.

The algorithm is applied to all possible values $\boldsymbol{\pi}_b^j$ on the discretization grid until all values of $Q(\boldsymbol{\pi}_b^j, t_b)$ are obtained for $j = 1, 2, ..., G^{k-1}$. It is then iterated backwards until $t_{b+1} = t + \varphi$. At that stage the algorithm is applied one last time, taking $Q(\boldsymbol{\pi}_{t+\varphi}^j, t+\varphi)$ as given and using the actual row vector of smoothed regime probabilities $\boldsymbol{\pi}_t$. The resulting vector $\hat{\boldsymbol{\omega}}_t$ and value for $\hat{s}_t$ are the desired optimal portfolio allocation at time t and the optimal savings rate, respectively, while $Q(\boldsymbol{\pi}_t, t)$ is the optimal value function.

In the simpler buy-and-hold case ($\varphi = T - t$) step 2 is replaced with a simulation routine that for each possible future regime, s_b, simulates N asset returns of length T, $\{\mathbf{R}_{T,s}(s_b)\}_{n=1}^N$ from the Markov switching model

$$\mathbf{r}_{t+i,s}(s_b) = \boldsymbol{\mu}_{s_{t+i}} + \boldsymbol{\varepsilon}_{t+i,n},$$

where $\mathbf{R}_{T,s}(s_b) \equiv \sum_{i=1}^T \mathbf{r}_{t+i,n}(s_b)$ and $\boldsymbol{\varepsilon}_{t+i,n} \sim \Omega_{s_{t+i}}$. In other words, a matrix of monthly returns $\left\{\{\mathbf{r}_{t+i,s}(s_b)\}_{n=1}^S\right\}_{i=1}^T$ is first drawn and then summed into N long-term asset returns $\{\mathbf{R}_{T,s}(s_b)\}_{n=1}^N$. Steps 1 and 4-6 are irrelevant in the buy-and-hold case since the objective simplifies to:

$$\max_{\boldsymbol{\omega}_t, s_t} \frac{(1-s_t)^{1-\gamma}}{1-\gamma} + \beta s_t^{1-\gamma}$$
$$\times N^{-1} \sum_{n=1}^N \left\{ \frac{\left[(1-\boldsymbol{\omega}_t'\boldsymbol{\iota}_3)\exp\left(Tr^f\right) + \boldsymbol{\omega}_b' \exp\left(\mathbf{R}_{T,n}(\boldsymbol{\pi}_b^j) + \varphi r^f \boldsymbol{\iota}_3\right)\right]^{1-\gamma}}{1-\gamma} \right\},$$

where $\mathbf{R}_{T,n} = \sum_{i=1}^4 (\boldsymbol{\pi}_t \mathbf{e}_i)\mathbf{R}_{T,n}(s_b = i)$. The absence of several steps under rebalancing makes computations much faster in the buy-and-hold case.

When solving pure asset allocation problems (i.e. $s_b = 1$ at all rebalancing points), step 5 simplifies to solving the program

$$\max_{\boldsymbol{\omega}_b(\boldsymbol{\pi}_b^j)} N^{-1} \sum_{n=1}^N \left\{ \left[(1-\boldsymbol{\omega}_b'\iota_3)\exp\left(\varphi r^f\right) + \boldsymbol{\omega}_b' \exp\left(\mathbf{R}_{b+1,n}(\boldsymbol{\pi}_b^j) + \varphi r^f \boldsymbol{\iota}_3\right)\right]^{1-\gamma} Q(\boldsymbol{\pi}_{b+1}^{(j,n)}, t_{b+1}) \right\}.$$

The value function corresponding to the portfolio weights $\hat{\boldsymbol{\omega}}_b(\boldsymbol{\pi}_b^j)$ defines $Q(\boldsymbol{\pi}_b^j, t_b)$ at the jth point on the initial discretization grid. The algorithm is again applied to all possible values $\boldsymbol{\pi}_b^j$ on the grid. The other calculations are similar to the joint consumption-asset allocation case. This gives the vector of optimal portfolio weights $\hat{\boldsymbol{\omega}}_t$, and the expected lifetime utility from optimal consumption and portfolio choices, $Q(\boldsymbol{\pi}_t, t)$.

Finally, extending these methods to the case in which further predictors (e.g. the dividend yield) are useful to forecast excess asset returns is straightforward and only implies generalizing step 1 to simulate for each possible value of the regime s_b *and* each possible configuration of the state vector $[\mathbf{r}_b' \ \mathbf{z}_b']'$ on a suitable grid, N $\varphi-$period returns $\{\mathbf{R}_{b+1,n}(S_b)\}_{n=1}^N$ in calendar time from the regime switching model

$$\begin{pmatrix} \mathbf{r}_{t_b+j,n}(S_b) \\ \mathbf{z}_{t_b+j,n}(S_b) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{s_{t_b+j}} \\ \boldsymbol{\mu}_{zs_{t_b+j}} \end{pmatrix} + \mathbf{A}_{s_t}^* \begin{pmatrix} \mathbf{r}_{t_b+j-1,n}(S_b) \\ \mathbf{z}_{t_b+j-1,n}(S_b) \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_{t_b+j,n} \\ \boldsymbol{\varepsilon}_{zt_b+j,n} \end{pmatrix}$$

where $\mathbf{R}_{b+1,n}(S_b) \equiv \sum_{j=1}^{\varphi} \mathbf{r}_{t_b+j,n}(s_b)$ and $(\boldsymbol{\varepsilon}_{t_b+j,n}' \ \boldsymbol{\varepsilon}_{zt_b+j,n}')' \sim N(0, \Omega_{s_{t_b+j,n}}^*)$. Step 3 is virtually identical, the only difference being represented by the fact that the updating of $\boldsymbol{\pi}_b^i$ must now take into account a likelihood function $\boldsymbol{\eta}(\mathbf{y}_{b+1,n}(\boldsymbol{\pi}_b^i); \hat{\boldsymbol{\theta}}_t)$ defined over $\mathbf{y}_{b+1,n}(\boldsymbol{\pi}_b^i) \equiv [\mathbf{r}_{b+1,n}'(\boldsymbol{\pi}_b^i) \ \mathbf{z}_{b+1,n}'(\boldsymbol{\pi}_b^i)]'$. Step 4 must be adjusted to define distances on the discretization grid to also account for values of $\mathbf{y}_{b+1,n}(\boldsymbol{\pi}_b^i)$ generated on each of the simulated paths.

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Table I

Specification Tests for Regime Switching Models

The table reports tests for the transformed z-scores generated by the extended multivariate regime-switching model:

$$y_t = \mu_{s_t} + \sum_{j=1}^{p} A_{js_t} y_{t-j} + \varepsilon_t$$

where $\mathbf{y}_t$ is a $(n+m \times 1)$ random vector collecting excess asset returns in the first n positions followed by m predictor variables, $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t, A_{js_t} is the matrix of autoregressive coefficients associated with lag $j \geq 1$ in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}\ \varepsilon_{4t}]' \sim N(\mathbf{0}, \Omega_{s_t})$ .The unobserved state variable s_t is governed by a first-order Markov chain that can assume k distinct values. p autoregressive terms are considered. The three monthly return series comprise a portfolio of large stocks (ninth and tenth CRSP size decile portfolios), a portfolio of small stocks (first and second CRSP deciles), and 10-year bonds all in excess of the return on 30-day T-bills. The predictor is the dividend yield. The data was obtained from the CRSP tapes. The sample period is 1954:01 – 1999:12. The tests are based on the principle that under the null of correct specification of the model, the probability integral transform of the one-step-ahead standardized forecast errors should follow an IID uniform distribution over the interval (0,1). A further Gaussian transform described in Berkowitz (2001) is applied to perform LR tests of the null that (under correct specification) the transformed z-scores are IIN(0,1) distributed.

Model	Skew	Kurt.	Jarque -Bera	LR_2	LR_3	LR_6	Skew	Kurt.	Jarque -Bera	LR_2	LR_3	LR_6	Skew	Kurt.	Jarque -Bera	LR_2	LR_3	LR_6
	Panel A – Large Caps Portfolio						**Panel B –Small Caps Portfolio**						**Panel C –Long-Term Bonds**					
MSIA(1,0)	0.051	7.593	**473.65 (0.000)**	1.846 (0.397)	**24.22 (0.000)**	**30.02 (0.000)**	-0.385	4.978	**101.72 (0.000)**	1.928 (0.381)	4.076 (0.253)	10.044 (0.123)	0.386	4.671	**76.57 (0.000)**	2.278 (0.320)	**7.824 (0.050)**	**17.004 (0.009)**
MSIA(1,1)	-0.287	4.864	**87.32 (0.000)**	**31.56 (0.000)**	**32.63 (0.000)**	**34.05 (0.190)**	0.305	8.294	**630.60 (0.000)**	**7.848 (0.020)**	**8.742 (0.033)**	9.136 (0.166)	0.324	4.395	**52.60 (0.000)**	**6.345 (0.042)**	7.727 (0.052)	9.715 (0.137)
MSIA (2,0)	-0.385	4.976	**99.50 (0.000)**	0.002 (0.999)	2.142 (0.543)	8.116 (0.230)	0.043	7.548	**457.80 (0.000)**	0.003 (0.987)	**20.91 (0.000)**	**26.814 (0.000)**	0.389	4.674	**75.40 (0.000)**	0.000 (1.000)	5.568 (0.135)	**14.830 (0.022)**
MSIA (2,1)	-0.229	5.247	**109.09 (0.000)**	0.022 (0.989)	2.274 (0.518)	6.862 (0.334)	0.196	8.152	**553.99 (0.000)**	**8.434 (0.015)**	**11.598 (0.009)**	**17.760 (0.007)**	0.259	4.283	**39.70 (0.000)**	**16.34 (0.000)**	**17.454 (0.000)**	**23.344 (0.001)**
MSIH (2,0)	-0.497	5.084	**115.76 (0.000)**	0.058 (0.971)	2.254 (0.521)	8.382 (0.211)	-0.124	7.065	**360.01 (0.000)**	0.062 (0.969)	**23.76 (0.000)**	**30.330 (0.000)**	0.245	3.890	**22.42 (0.000)**	0.042 (0.979)	5.540 (0.136)	10.338 (0.111)
MSIAH (2,1)	-0.098	4.797	**66.49 (0.000)**	0.090 (0.956)	2.280 (0.516)	6.684 (0.351)	0.092	8.512	**618.50 (0.000)**	0.054 (0.973)	1.908 (0.592)	6.988 (0.322)	0.189	4.190	**31.73 (0.000)**	0.052 (0.974)	1.900 (0.593)	8.258 (0.220)
MSIAH (3,1)	-0.083	4.029	**20.57 (0.000)**	0.048 (0.976)	2.452 (0.484)	5.556 (0.475)	-0.022	5.855	**154.32 (0.000)**	0.020 (0.990)	2.030 (0.566)	6.580 (0.361)	0.026	3.630	**7.56 (0.023)**	0.068 (0.967)	2.002 (0.575)	7.970 (0.240)
MSIAH (3,2)	-0.115	3.802	**11.77 (0.003)**	0.072 (0.965)	3.006 (0.391)	5.362 (0.498)	0.034	5.101	**74.580 (0.000)**	0.060 (0.970)	2.206 (0.531)	7.241 (0.302)	-0.008	3.559	5.27 (0.072)	0.082 (0.960)	2.234 (0.525)	7.126 (0.309)
MSIA (4,1)	0.172	3.445	5.897 (0.052)	0.030 (0.985)	2.158 (0.540)	5.738 (0.453)	0.557	6.912	**308.96 (0.000)**	0.058 (0.971)	2.566 (0.463)	5.804 (0.446)	0.177	4.085	**24.35 (0.000)**	**16.73 (0.000)**	**17.222 (0.001)**	**23.306 (0.001)**
MSIAH (4,1)	-0.106	3.423	3.908 (0.142)	0.168 (0.919)	2.212 (0.530)	5.266 (0.510)	0.116	3.468	4.763 (0.092)	0.014 (0.993)	2.192 (0.534)	5.898 (0.435)	0.045	3.505	4.59 (0.101)	0.008 (0.996)	1.850 (0.604)	6.816 (0.338)
MSIAH (4,2)	0.010	3.194	0.558 (0.757)	0.052 (0.974)	3.550 (0.314)	6.046 (0.418)	0.161	3.748	**9.770 (0.008)**	0.312 (0.856)	2.792 (0.425)	5.592 (0.470)	0.127	3.194	1.50 (0.472)	0.818 (0.664)	3.620 (0.306)	7.882 (0.251)

Table II

Estimates of Regime Switching Model for Stock and Bond Returns

The table reports the estimation output for the regime switching model:

$$r_t = \mu_{s_t} + \varepsilon_t$$

where $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}]' \sim N(\mathbf{0}, \Omega_{s_t})$ is the vector of unpredictable return innovations. The unobserved state variable s_t is governed by a first-order Markov chain that can assume *k* values. The three monthly return series comprise a portfolio of large stocks (ninth and tenth CRSP size decile portfolios), a portfolio of small stocks (first and second CRSP deciles), and 10-year bonds all in excess of the return on 30-day T-bills. The data was obtained from the CRSP tapes. The sample is 1954:01 – 1999:12. The first panel refers to the case (*k* = 1) and represents a single-state benchmark. The data reported on the diagonals of the correlation matrices are annualized volatilities. Asterisks attached to correlation coefficients refer to covariance estimates. For mean coefficients and transition probabilities, standard errors are reported in parenthesis.

	Panel A – Single State Model		
	Large caps	Small caps	Long-term bonds
1. Mean excess return	0.0066 (0.0018)	0.0082 (0.0026)	0.0008 (0.0009)
2. Correlations/Volatilities			
Large caps	0.1428***		
Small caps	0.7215**	0.1481***	
Long-term bonds	0.2516	0.1196	0.0748***

	Panel B – Four State Model			
	Large caps	Small caps	Long-term bonds	
1. Mean excess return				
Regime 1 (crash)	-0.0510 (0.0146)	-0.0810 (0.0219)	-0.0131 (0.0047)	
Regime 2 (slow growth)	0.0069 (0.0027)	0.0008 (0.0033)	0.0009 (0.0016)	
Regime 3 (bull)	0.0116 (0.0032)	0.0167 (0.0048)	-0.0023 (0.0007)	
Regime 4 (recovery)	0.0226 (0.0055)	0.0458 (0.0098)	0.0098 (0.0033)	
2. Correlations/Volatilities				
Regime 1 (crash):				
Large caps	0.1625***			
Small caps	0.8233***	0.2479***		
Long-term bonds	-0.4060*	-0.2590	0.0902***	
Regime 2 (slow growth):				
Large caps	0.1118***			
Small caps	0.7655***	0.1099***		
Long-term bonds	0.2043***	0.1223	0.0688***	
Regime 3 (bull):				
Large caps	0.1133***			
Small caps	0.6707***	0.1730***		
Long-term bonds	0.1521	-0.0976	0.0261***	
Regime 4 (recovery):				
Large caps	0.1479***			
Small caps	0.5013***	0.2429***		
Long-term bonds	0.3692***	-0.0011	0.1000***	
3. Transition probabilities	Regime 1	Regime 2	Regime 3	Regime 4
Regime 1 (crash)	0.4940 (0.1078)	0.0001 (0.0001)	0.02409 (0.0417)	0.4818
Regime 2 (slow growth)	0.0483 (0.0233)	0.8529 (0.0403)	0.0307 (0.0110)	0.0682
Regime 3 (bull)	0.0439 (0.0252)	0.0701 (0.0296)	0.8822 (0.0403)	0.0038
Regime 4 (recovery)	0.0616 (0.0501)	0.1722 (0.0718)	0.0827 (0.0498)	0.6836

* denotes 10% significance, ** significance at 5%, *** significance at 1%.

Table III

Estimates of Regime Switching Model for Stock and Bond Returns and the Dividend Yield

The table shows estimation results for the regime switching model:

$$y_t = \mu_{s_t} + A_{s_t} y_{t-1} + \varepsilon_t$$

where $\mathbf{y}_t$ is a 4×1 vector collecting excess asset returns in the first three positions plus an additional prediction variable (the dividend yield), $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t, A_{js_t} is the matrix of autoregressive coefficients associated to lag 1 in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}\ \varepsilon_{4t}]' \sim N(\mathbf{0}, \Omega_{s_t})$. The unobservable state s_t is governed by a first-order Markov chain that can assume *k* distinct values. The three monthly return series comprise a portfolio of large stocks (ninth and tenth CRSP size decile portfolios), a portfolio of small stocks (first and second CRSP deciles), and 10-year bonds all in excess of the return on 30-day T-bills. The predictor is the dividend yield. The first panel refers to the single-state case *k* = 1. Asterisks attached to correlation coefficients refer to covariance estimates. For mean coefficients and transition probabilities, standard errors are reported in parenthesis.

	Panel A – VAR(1) (single state) Model			
	Large caps	Small caps	Long-term bonds	Dividend Yield
1. Intercept term	0.0021 (0.0070)	-0.0160 (0.0102)	-0.0032 (0.0036)	0.0004(0.0003)
2. VAR(1) Matrix				
Large caps	-0.0466 (0.0635)	0.0370 (0.0925)	0.2299(0.0330)	0.1261(0.0024)
Small caps	0.1236 (0.0412)	0.1244 (0.0600)	0.2624 (0.0214)	0.6641 (0.0016)
Long-term bonds	-0.0442 (0.0839)	-0.0261 (0.1223)	0.1070 (0.0436)	0.1322 (0.0032)
Dividend Yield	-0.0005 (0.2028)	-0.0005 (0.2953)	-0.0098 (0.1054)	0.9856 (0.0077)
3. Correlations/Volatilities				
Large caps	0.1417***			
Small caps	0.7285***	0.2063***		
Long-term bonds	0.2466*	0.1353	0.0736***	
Dividend Yield	-0.9243***	-0.7695***	-0.2413	0.0056***
	Panel B – Four State Model			
	Large caps	Small caps	Long-term bonds	Dividend Yield
1. Intercept term				
Regime 1 (crash)	-0.0848 (0.1065)	-0.1152 (0.1528)	-0.0150 (0.0396)	0.0014 (0.0514)
Regime 2 (slow growth)	-0.0232 (0.0338)	-0.0188 (0.0516)	-0.0016 (0.0115)	0.0011 (0.0010)
Regime 3 (bull)	0.0122 (0.0539)	-0.0323 (0.0471)	0.0048 (0.0278)	0.0002 (0.0021)
Regime 4 (recovery)	0.0370 (0.0490)	0.0179 (0.0940)	-0.0038 (0.0324)	0.0007 (0.0019)
2. VAR(1) Matrix				
Regime 1 (crash):				
Large caps	-0.0494 (0.5360)	0.2391 (0.3875)	0.3092 (0.7164)	1.2089 (2.8282)
Small caps	-0.0357 (0.9401)	0.2424 (0.6332)	0.7277 (1.0894)	1.5972 (4.0047)
Long-term bonds	0.0136 (0.4381)	-0.0059 (0.2641)	-0.0215 (0.4246)	0.1838 (1.0283)
Dividend Yield	0.0002 (0.0262)	-0.0076 (0.0192)	-0.0170 (0.0301)	1.0074 (0.1302)
Regime 2 (slow growth):				
Large caps	-0.0563 (0.3064)	-0.0311 (0.1609)	0.0526 (0.4049)	1.1417 (1.1539)
Small caps	-0.0029 (0.5142)	0.2710 (0.2795)	-0.0077 (0.7227)	0.8963 (1.7180)
Long-term bonds	-0.0430 (0.1539)	-0.0056 (0.0896)	0.4234 (0.1888)	0.0813 (0.3948)
Dividend Yield	0.0010 (0.0096)	0.0007 (0.0051)	-0.0013 (0.0132)	0.9552 (0.0340)
Regime 3 (bull):				
Large caps	-0.0535 (0.3682)	-0.0789 (0.3452)	-0.0800 (0.4560)	-0.0810 (1.4631)
Small caps	0.0200 (0.3399)	0.1878 (0.3256)	-0.1707 (0.4503)	1.0675 (1.2817)
Long-term bonds	-0.0272 (0.1568)	-0.0550 (0.1518)	-0.0057 (0.1809)	-0.1571 (0.7925)
Dividend Yield	-0.0022 (0.0124)	0.0032 (0.0113)	0.0055 (0.0162)	0.9924 (0.0566)

* denotes 10% significance, ** significance at 5%, *** significance at 1%.

Table III (continued)

Estimates of Regime Switching Model for Stock and Bond Returns and the Dividend Yield

	Panel B (cont'd) – MMSIAH(4,1) Model			
	Large caps	Small caps	Long-term bonds	Dividend Yield
2. VAR(1) Matrix (cont'd)				
Regime 4 (recovery):				
Large caps	-0.1994 (0.4243)	-0.0419 (0.2394)	0.2603 (0.4992)	-0.0123 (1.3605)
Small caps	0.3832 (0.7902)	-0.1739 (0.4847)	0.0481 (1.0007)	1.1191 (2.6891)
Long-term bonds	-0.1465 (0.3439)	-0.0113 (0.1973)	0.0606 (0.3846)	0.4777 (0.8776)
Dividend Yield	0.0047 (0.0154)	0.0024 (0.0086)	-0.0105 (0.0180)	0.9428 (0.0504)
3. Correlations/Volatilities				
Regime 1 (crash):				
Large caps	0.1206*			
Small caps	0.7530	0.2044*		
Long-term bonds	-0.2128	-0.1487	0.0906*	
Dividend Yield	-0.9289	-0.7885	0.1688	0.0056
Regime 2 (slow growth):				
Large caps	0.0896***			
Small caps	0.7496***	0.1513***		
Long-term bonds	0.2344	0.0006	0.0431***	
Dividend Yield	-0.9322***	-0.7939***	-0.1808	0.0027***
Regime 3 (bull):				
Large caps	0.1224***			
Small caps	0.7524***	0.1239***		
Long-term bonds	0.1083**	0.1450	0.0577***	
Dividend Yield	-0.9099***	-0.7261***	-0.1174	0.0043***
Regime 4 (recovery):				
Large caps	0.1191*			
Small caps	0.3668	0.2189***		
Long-term bonds	0.2600	-0.1320	0.0949**	
Dividend Yield	-0.9312*	-0.5573	-0.1909	0.0041*
3. Transition probabilities	Regime 1	Regime 2	Regime 3	Regime 4
Regime 1 (crash)	0.4606 (0.1868)	0.0623 (0.1117)	4.51e-19 (0.0733)	0.4771
Regime 2 (slow growth)	2.29e-05 (0.0541)	0.9151 (0.0670)	9.07e-15 (0.0440)	0.0848
Regime 3 (bull)	0.0598 (0.0727)	5.71e-22 (0.0106)	0.9329 (0.0696)	0.0074
Regime 4 (recovery)	0.3223 (0.1939)	0.0809 (0.0935)	0.1160 (0.1063)	0.4808

* denotes 10% significance, ** significance at 5%, *** significance at 1%.

Table IV

Effects of Rebalancing

This table reports the optimal weight to be invested in stocks (small and large caps) as a function of the rebalancing frequency φ for an investor with power utility and a constant relative risk aversion coefficient of 5. Excess returns are assumed to be generated by the model the regime switching model:

$$r_t = \mu_{s_t} + \varepsilon_t$$

where $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}]' \sim N(\mathbf{0}, \Omega_{s_t})$ is the vector of return innovations.

Rebalancing Frequency φ	**Investment Horizon T (in months)**					
Panel A - Optimal Allocation to Stocks						
	T=1	**T=6**	**T=12**	**T=24**	**T=60**	**T=120**
IID (no predictability)	0.33	0.33	0.33	0.33	0.33	0.33
Crash regime 1						
φ = T (buy-and-hold)	0.00	0.24	0.34	0.48	0.58	0.60
φ = 24 months	0.00	0.24	0.34	0.48	0.5	0.5
φ = 12 months	0.00	0.24	0.34	0.37	0.39	0.40
φ = 6 months	0.00	0.24	0.28	0.31	0.33	0.34
φ = 3 months	0.00	0.00	0.00	0.00	0.00	0.00
φ = 1 month	0.00	0.00	0.00	0.00	0.00	0.00
Slow growth regime 2						
φ = T (buy-and-hold)	1.00	0.68	0.65	0.65	0.65	0.64
φ = 24 months	1.00	0.68	0.65	0.65	0.70	0.80
φ = 12 months	1.00	0.68	0.65	0.72	0.82	0.93
φ = 6 months	1.00	0.68	0.71	0.77	0.88	0.96
φ = 3 months	1.00	0.92	0.85	0.89	0.95	0.99
φ = 1 month	1.00	1.00	1.00	1.00	1.00	1.00
Bull regime 3						
φ = T (buy-and-hold)	1.00	0.67	0.66	0.65	0.65	0.65
φ = 24 months	1.00	0.67	0.66	0.65	0.72	0.83
φ = 12 months	1.00	0.67	0.66	0.74	0.85	0.88
φ = 6 months	1.00	0.67	0.74	0.80	0.90	0.95
φ = 3 months	1.00	0.94	0.96	0.98	1.00	1.00
φ = 1 month	1.00	1.00	1.00	1.00	1.00	1.00
Recovery regime 4						
φ = T (buy-and-hold)	1.00	0.82	0.71	0.69	0.68	0.66
φ = 24 months	1.00	0.82	0.71	0.69	0.71	0.74
φ = 12 months	1.00	0.82	0.71	0.72	0.74	0.77
φ = 6 months	1.00	0.82	0.75	0.79	0.82	0.85
φ = 3 months	1.00	0.98	1.00	1.00	1.00	1.00
φ = 1 month	1.00	1.00	1.00	1.00	1.00	1.00
Steady-state probabilities						
φ = T (buy-and-hold)	1.00	0.73	0.68	0.67	0.65	0.64
φ = 24 months	1.00	0.73	0.68	0.67	0.71	0.77
φ = 12 months	1.00	0.73	0.68	0.73	0.78	0.81
φ = 6 months	1.00	0.73	0.78	0.81	0.84	0.83
φ = 3 months	1.00	0.88	0.85	0.84	0.84	0.84
φ = 1 month	1.00	0.98	0.98	0.98	0.98	0.98

Table IV (continued)

Effects of Rebalancing

Rebalancing Frequency φ	Investment Horizon T (in months)					
Panel B - Optimal Allocation to Long-Term Bonds						
	T=1	**T=6**	**T=12**	**T=24**	**T=60**	**T=120**
IID (no predictability)	0.67	0.67	0.67	0.67	0.67	0.67
Crash regime 1						
φ = T (buy-and-hold)	0.00	0.34	0.29	0.19	0.12	0.08
φ = 24 months	0.00	0.34	0.29	0.19	0.16	0.10
φ = 12 months	0.00	0.34	0.29	0.21	0.17	0.11
φ = 6 months	0.00	0.34	0.28	0.18	0.15	0.10
φ = 3 months	0.00	0.18	0.16	0.13	0.11	0.05
φ = 1 month	0.00	0.00	0.00	0.00	0.00	0.00
Slow growth regime 2						
φ = T (buy-and-hold)	0.00	0.32	0.34	0.19	0.14	0.08
φ = 24 months	0.00	0.32	0.34	0.19	0.17	0.13
φ = 12 months	0.00	0.32	0.34	0.20	0.14	0.01
φ = 6 months	0.00	0.32	0.21	0.13	0.04	0.00
φ = 3 months	0.00	0.05	0.13	0.04	0.00	0.00
φ = 1 month	0.00	0.00	0.00	0.00	0.00	0.00
Bull regime 3						
φ = T (buy-and-hold)	0.00	0.17	0.12	0.10	0.08	0.08
φ = 24 months	0.00	0.17	0.12	0.10	0.05	0.00
φ = 12 months	0.00	0.17	0.12	0.06	0.03	0.00
φ = 6 months	0.00	0.17	0.07	0.02	0.00	0.00
φ = 3 months	0.00	0.02	0.00	0.00	0.00	0.00
φ = 1 month	0.00	0.00	0.00	0.00	0.00	0.00
Recovery regime 4						
φ = T (buy-and-hold)	0.00	0.00	0.00	0.00	0.01	0.02
φ = 24 months	0.00	0.00	0.00	0.00	0.01	0.01
φ = 12 months	0.00	0.00	0.00	0.00	0.00	0.00
φ = 6 months	0.00	0.00	0.00	0.00	0.00	0.00
φ = 3 months	0.00	0.00	0.00	0.00	0.00	0.00
φ = 1 month	0.00	0.00	0.00	0.00	0.00	0.00
Steady-state probabilities						
φ = T (buy-and-hold)	0.00	0.03	0.04	0.05	0.07	0.06
φ = 24 months	0.00	0.03	0.04	0.05	0.10	0.12
φ = 12 months	0.00	0.03	0.04	0.08	0.12	0.14
φ = 6 months	0.00	0.03	0.07	0.10	0.16	0.17
φ = 3 months	0.00	0.06	0.15	0.12	0.11	0.10
φ = 1 month	0.00	0.02	0.02	0.02	0.02	0.02

Table V

Optimal Consumption-Wealth Ratio – Effects of Rebalancing

This table reports the optimal consumption-wealth ratio as a function of the rebalancing frequency φ and the investment horizon T for an investor with power utility, a constant relative risk aversion coefficient of 5, and (annualized) subjective rate of time preference of 5%. Excess returns are assumed to be generated by the regime switching model:

$$r_t = \mu_{s_t} + \varepsilon_t$$

where $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}]' \sim N(\mathbf{0}, \Omega_{s_t})$ is the vector of return innovations.

Rebalancing Frequency φ	Investment Horizon T (in months)					
	T=1	T=6	T=12	T=24	T=60	T=120
Crash regime 1						
φ = T (buy-and-hold)	0.50	0.51	0.52	0.54	0.60	0.69
φ = T, γ = 1 (myopic) case	0.51	0.51	0.51	0.52	0.56	0.62
φ = 12 months	0.50	0.51	0.52	0.34	0.18	0.10
φ = 12, γ = 1 (myopic) case	0.51	0.51	0.51	0.35	0.19	0.11
φ = 3 months	0.50	0.32	0.19	0.11	0.02	0.01
φ = 3, γ = 1 (myopic) case	0.50	0.34	0.20	0.12	0.05	0.03
Slow growth regime 2						
φ = T (buy-and-hold)	0.51	0.51	0.52	0.54	0.60	0.69
φ = T, γ = 1 (myopic) case	0.51	0.51	0.51	0.52	0.56	0.62
φ = 12 months	0.51	0.51	0.52	0.46	0.28	0.21
φ = 12, γ = 1 (myopic) case	0.51	0.51	0.51	0.35	0.19	0.11
φ = 3 months	0.51	0.35	0.21	0.13	0.03	0.01
φ = 3, γ = 1 (myopic) case	0.50	0.34	0.20	0.12	0.05	0.03
Bull regime 3						
φ = T (buy-and-hold)	0.50	0.51	0.54	0.56	0.61	0.70
φ = T, γ = 1 (myopic) case	0.51	0.51	0.51	0.52	0.56	0.62
φ = 12 months	0.50	0.51	0.54	0.36	0.20	0.14
φ = 12, γ = 1 (myopic) case	0.51	0.51	0.51	0.35	0.19	0.11
φ = 3 months	0.50	0.33	0.20	0.11	0.02	0.01
φ = 3, γ = 1 (myopic) case	0.50	0.34	0.20	0.12	0.05	0.03
Recovery regime 4						
φ = T (buy-and-hold)	0.50	0.51	0.52	0.54	0.60	0.67
φ = T, γ = 1 (myopic) case	0.51	0.51	0.51	0.52	0.56	0.62
φ = 12 months	0.50	0.51	0.52	0.38	0.22	0.14
φ = 12, γ = 1 (myopic) case	0.51	0.51	0.51	0.35	0.19	0.11
φ = 3 month	0.50	0.33	0.20	0.12	0.02	0.01
φ = 3, γ = 1 (myopic) case	0.50	0.34	0.20	0.12	0.05	0.03

Table VI

Effect of Parameter Estimation Uncertainty

This table reports 90% confidence bands for a buy-and-hold investor's optimal portfolio weights at different investment horizons, T, assuming a constant relative risk aversion coefficient of 5. Under regime switching, portfolio returns are assumed to be generated by the model

$$r_t = \mu_{s_t} + \varepsilon_t$$

where $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t}\ \varepsilon_{2t}\ \varepsilon_{3t}]' \sim N(\mathbf{0}, \Omega_{s_t})$ is the vector of return innovations. In the IID case, k = 1. Highlighted blocks of cells signal a portfolio weights confidence interval that fails to include the IID weight.

		Investment Horizon T						
		T=1	T=6	T=24	T=48	T=72	T=96	T=120
Panel A: Allocation to Small Caps								
Crash regime 1	Mean + 1*SD	0.000	**0.319**	**0.393**	**0.392**	**0.395**	**0.390**	**0.394**
	Mean	0.000	**0.173**	**0.230**	**0.228**	**0.228**	**0.225**	**0.226**
	Mean - 1*SD	0.000	**0.028**	**0.067**	**0.063**	**0.061**	**0.060**	**0.058**
Slow growth regime 2	Mean + 1*SD	0.211	0.277	**0.357**	**0.375**	**0.385**	**0.383**	**0.383**
	Mean	0.061	0.127	**0.197**	**0.212**	**0.217**	**0.218**	**0.217**
	Mean - 1*SD	0.000	0.000	**0.037**	**0.049**	**0.050**	**0.053**	**0.052**
Bull regime 3	Mean + 1*SD	**0.915**	**0.530**	**0.432**	**0.410**	**0.404**	**0.403**	**0.401**
	Mean	**0.632**	**0.313**	**0.258**	**0.242**	**0.235**	**0.233**	**0.231**
	Mean - 1*SD	**0.349**	**0.096**	**0.083**	**0.073**	**0.067**	**0.064**	**0.060**
Recovery regime 4	Mean + 1*SD	**1.000**	**0.607**	**0.457**	**0.424**	**0.417**	**0.410**	**0.411**
	Mean	**0.890**	**0.406**	**0.279**	**0.252**	**0.245**	**0.238**	**0.236**
	Mean - 1*SD	**0.706**	**0.205**	**0.101**	**0.080**	**0.073**	**0.066**	**0.061**
Steady-state	Mean + 1*SD	**1.000**	**0.573**	**0.447**	**0.418**	**0.407**	**0.405**	**0.401**
	Mean	**0.827**	**0.361**	**0.270**	**0.247**	**0.238**	**0.235**	**0.231**
	Mean - 1*SD	**0.634**	**0.149**	**0.092**	**0.076**	**0.069**	**0.065**	**0.061**
Panel B: Allocation to Large Caps								
Crash regime 1	Mean + 1*SD	**0.050**	**0.290**	0.497	0.553	0.573	0.579	0.590
	Mean	**0.005**	**0.114**	0.275	0.323	0.341	0.347	0.355
	Mean - 1*SD	**0.000**	**0.000**	0.053	0.093	0.109	0.116	0.119
Slow growth regime 2	Mean + 1*SD	**1.000**	0.709	0.629	0.616	0.613	0.611	0.613
	Mean	**0.834**	0.470	0.395	0.384	0.380	0.379	0.380
	Mean - 1*SD	**0.621**	0.232	0.161	0.151	0.148	0.147	0.147
Bull regime 3	Mean + 1*SD	0.630	0.703	0.632	0.620	0.616	0.619	0.616
	Mean	0.351	0.441	0.393	0.384	0.382	0.384	0.381
	Mean - 1*SD	0.073	0.179	0.154	0.148	0.147	0.148	0.146
Recovery regime 4	Mean + 1*SD	**0.275**	0.500	0.570	0.591	0.592	0.603	0.604
	Mean	**0.101**	0.268	0.336	0.356	0.360	0.368	0.369
	Mean - 1*SD	**0.000**	0.039	0.102	0.122	0.128	0.132	0.135
Steady-state	Mean + 1*SD	0.724	0.648	0.611	0.608	0.609	0.610	0.608
	Mean	0.174	0.406	0.386	0.381	0.380	0.378	0.380
	Mean - 1*SD	0.195	0.145	0.135	0.137	0.139	0.139	0.140

Table VIII - continued

		Investment Horizon T						
		T=1	T=6	T=24	T=48	T=72	T=96	T=120
Panel C: Allocation to Long-term Bonds								
Crash regime 1	Mean + 1*SD	**0.033**	**0.481**	**0.406**	**0.375**	**0.363**	**0.360**	**0.356**
	Mean	**0.000**	**0.264**	**0.221**	**0.200**	**0.190**	**0.190**	**0.186**
	Mean - 1*SD	**0.000**	**0.047**	**0.036**	**0.024**	**0.018**	**0.019**	**0.015**
Slow growth regime 2	Mean + 1*SD	**0.229**	**0.383**	**0.359**	**0.348**	**0.345**	**0.343**	**0.343**
	Mean	**0.084**	**0.206**	**0.191**	**0.183**	**0.180**	**0.179**	**0.178**
	Mean - 1*SD	**0.000**	**0.028**	**0.025**	**0.019**	**0.015**	**0.014**	**0.012**
Bull regime 3	Mean + 1*SD	**0.000**	**0.043**	**0.221**	**0.276**	**0.296**	**0.307**	**0.313**
	Mean	**0.000**	**0.010**	**0.095**	**0.130**	**0.143**	**0.151**	**0.156**
	Mean - 1*SD	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
Recovery regime 4	Mean + 1*SD	**0.037**	**0.401**	**0.371**	**0.357**	**0.350**	**0.346**	**0.347**
	Mean	**0.006**	**0.230**	**0.203**	**0.191**	**0.185**	**0.180**	**0.182**
	Mean - 1*SD	**0.000**	**0.059**	**0.036**	**0.024**	**0.021**	**0.014**	**0.017**
Steady-state	Mean + 1*SD	**0.000**	**0.125**	**0.255**	**0.295**	**0.309**	**0.318**	**0.321**
	Mean	**0.000**	**0.043**	**0.117**	**0.143**	**0.152**	**0.158**	**0.161**
	Mean - 1*SD	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.001**
Panel D: Allocation to 1-month T-bills								
Crash regime 1	Mean + 1*SD	**1.000**	**0.607**	**0.489**	**0.453**	**0.442**	**0.438**	**0.433**
	Mean	**0.996**	**0.349**	**0.275**	**0.250**	**0.240**	**0.238**	**0.233**
	Mean - 1*SD	**0.966**	**0.091**	**0.060**	**0.046**	**0.039**	**0.038**	**0.034**
Slow growth regime 2	Mean + 1*SD	0.083	**0.391**	**0.408**	**0.413**	**0.415**	**0.416**	**0.418**
	Mean	0.024	**0.202**	**0.217**	**0.221**	**0.223**	**0.224**	**0.225**
	Mean - 1*SD	0.000	**0.012**	**0.027**	**0.030**	**0.031**	**0.032**	**0.032**
Bull regime 3	Mean + 1*SD	0.000	**0.392**	**0.435**	**0.431**	**0.430**	**0.424**	**0.423**
	Mean	0.000	**0.225**	**0.249**	**0.240**	**0.237**	**0.229**	**0.229**
	Mean - 1*SD	0.000	**0.059**	**0.064**	**0.049**	**0.044**	**0.035**	**0.035**
Recovery regime 4	Mean + 1*SD	0.000	0.222	0.356	**0.385**	**0.396**	**0.401**	**0.402**
	Mean	0.000	0.090	0.178	**0.198**	**0.207**	**0.211**	**0.211**
	Mean - 1*SD	0.000	0.000	0.000	**0.012**	**0.019**	**0.022**	**0.019**
Steady-state	Mean + 1*SD	0.000	**0.347**	**0.410**	**0.418**	**0.421**	**0.420**	**0.419**
	Mean	0.000	**0.188**	**0.226**	**0.228**	**0.228**	**0.227**	**0.226**
	Mean - 1*SD	0.000	**0.030**	**0.043**	**0.038**	**0.036**	**0.033**	**0.033**
Panel E: Overall Allocation to Stocks (Small and Large Caps)								
Crash regime 1	Mean + 1*SD	**0.000**	0.478	0.701	**0.745**	**0.766**	**0.769**	**0.779**
	Mean	**0.000**	0.284	0.500	**0.545**	**0.565**	**0.569**	**0.576**
	Mean - 1*SD	**0.000**	0.091	0.299	**0.346**	**0.363**	**0.369**	**0.374**
Slow growth regime 2	Mean + 1*SD	**1.000**	**0.794**	**0.781**	**0.786**	**0.790**	**0.789**	**0.792**
	Mean	**0.893**	**0.590**	**0.586**	**0.591**	**0.593**	**0.592**	**0.593**
	Mean - 1*SD	**0.736**	**0.387**	**0.392**	**0.396**	**0.396**	**0.395**	**0.394**
Bull regime 3	Mean + 1*SD	**1.000**	**0.925**	**0.836**	**0.816**	**0.810**	**0.814**	**0.809**
	Mean	**1.000**	**0.760**	**0.651**	**0.625**	**0.616**	**0.617**	**0.611**
	Mean - 1*SD	**1.000**	**0.595**	**0.468**	**0.434**	**0.423**	**0.418**	**0.412**
Recovery regime 4	Mean + 1*SD	**1.000**	**0.872**	**0.808**	**0.802**	**0.799**	**0.804**	**0.805**
	Mean	**0.994**	**0.676**	**0.614**	**0.607**	**0.603**	**0.604**	**0.604**
	Mean - 1*SD	**0.962**	**0.481**	**0.421**	**0.411**	**0.407**	**0.404**	**0.403**
Steady-state	Mean + 1*SD	**1.000**	**0.926**	**0.839**	**0.817**	**0.809**	**0.808**	**0.807**
	Mean	**1.000**	**0.764**	**0.652**	**0.625**	**0.615**	**0.611**	**0.610**
	Mean - 1*SD	**1.000**	**0.602**	**0.466**	**0.433**	**0.420**	**0.414**	**0.411**

Figure 1

Smoothed State Probabilities: Four-state model for Stock and Bond Returns

The graphs plot the smoothed probabilities of regimes 1-4 for the multivariate Markov Switching model comprising returns on large and small firms and 10-year bonds all in excess of the return on 30-day T-bills.

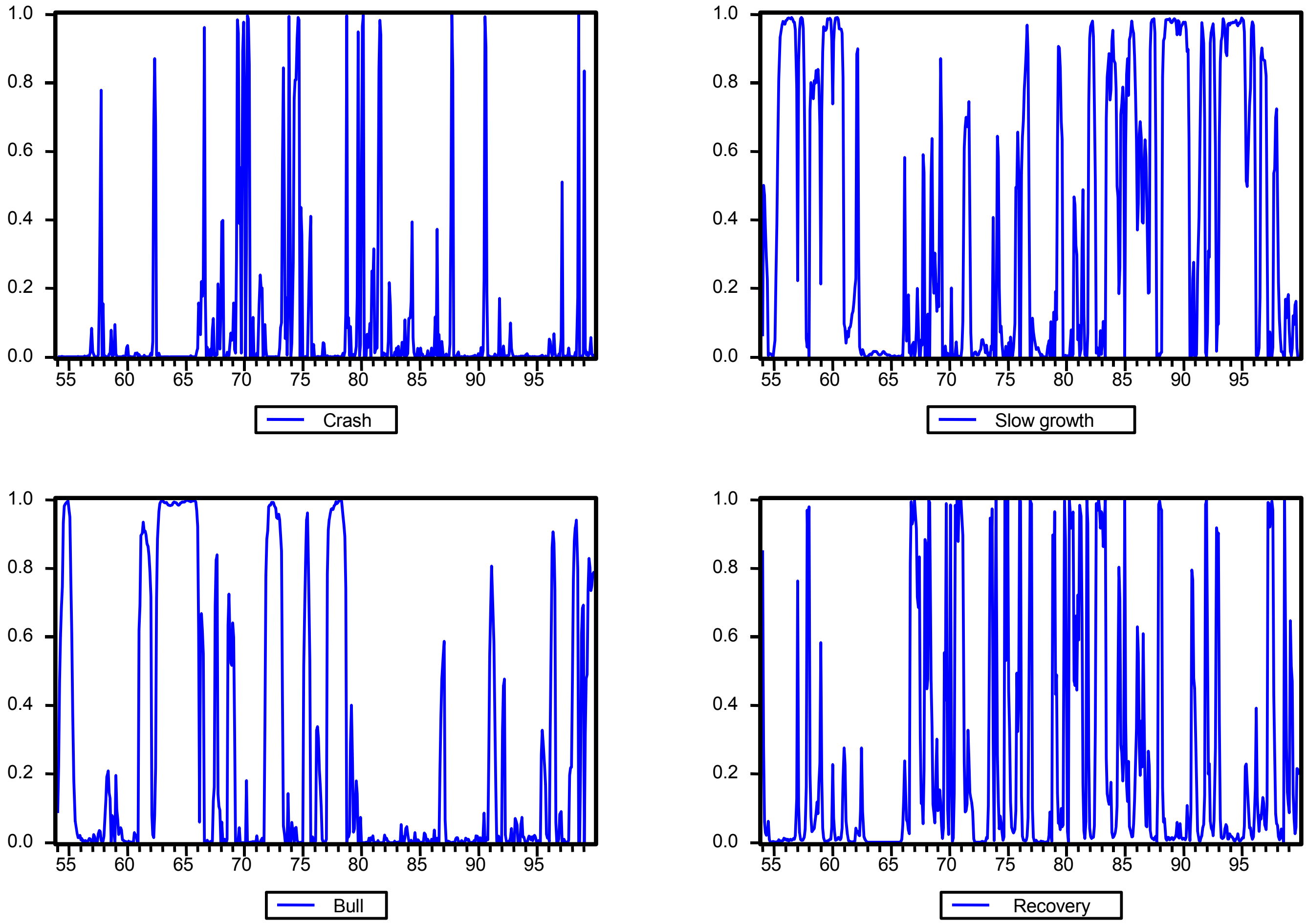

Figure 2

Scatter Plot of Empirical Quintiles of Z-Scores vs. Standard Normal Quintiles

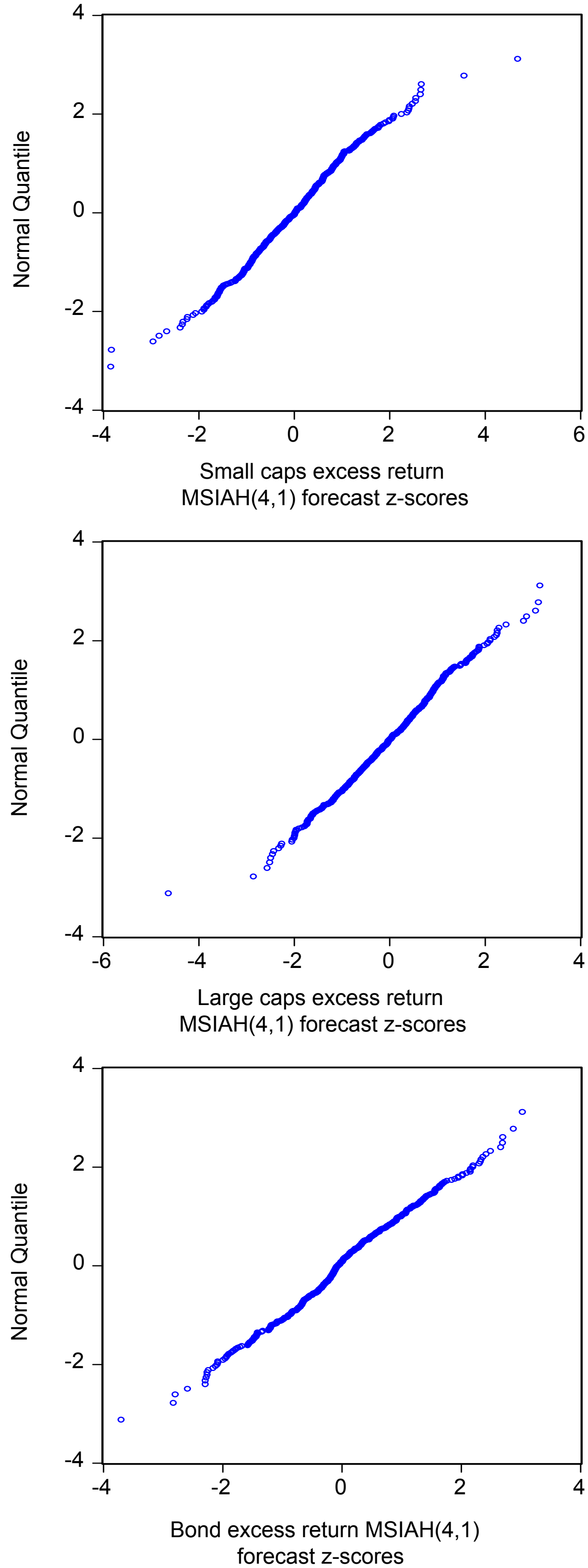

Figure 3

Optimal Portfolio Allocation for a Buy-and-Hold Investor

Plots of the optimal allocation to stocks —both small and large caps —bonds, and cash at various investment horizons. The plots assume the investor has power utility and coefficient of relative risk aversion $\gamma = 5$. Allocations are shown both for the four-state regime switching model and for a myopic investor who ignores the presence of regimes.

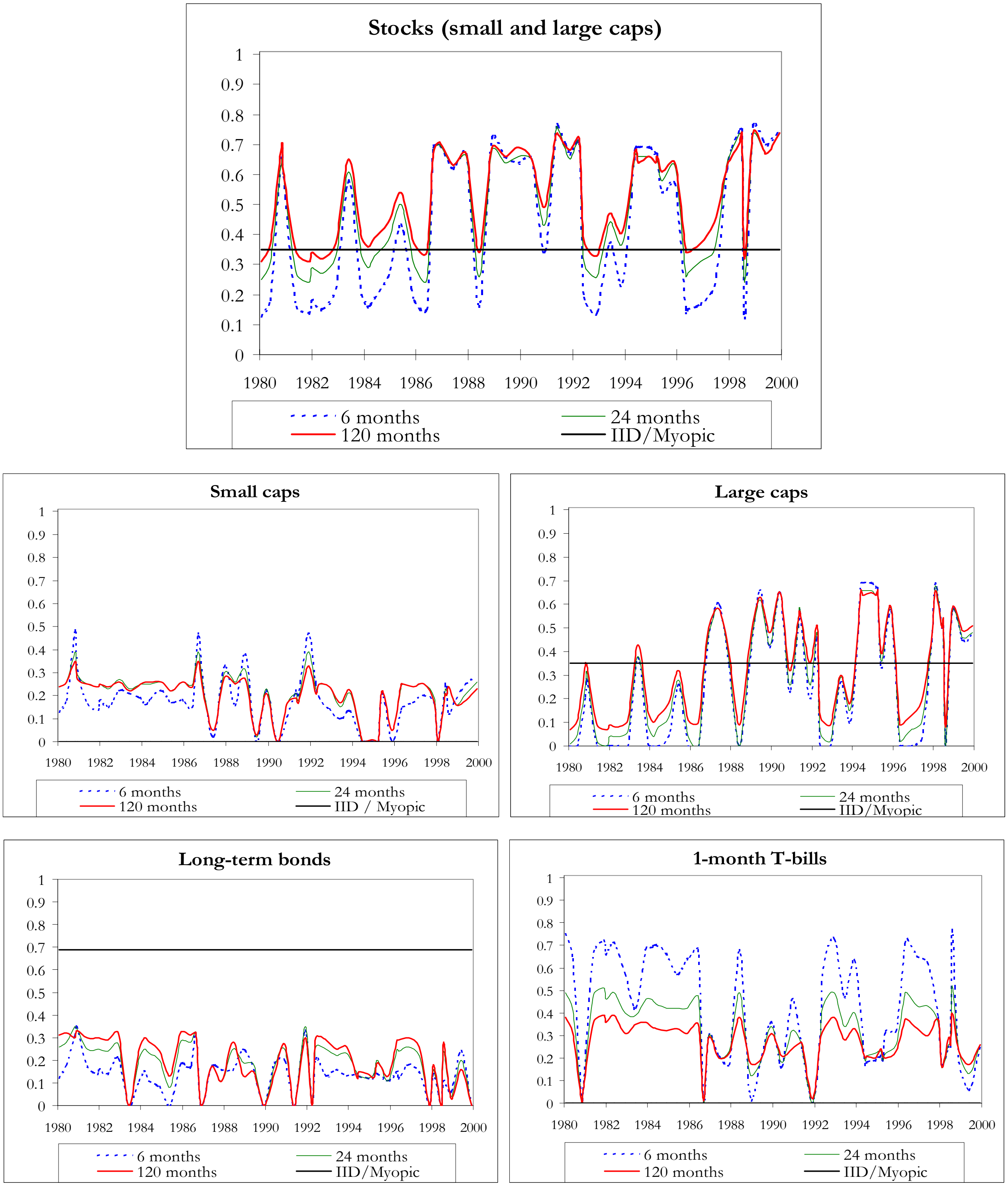

Figure 4

Optimal Portfolio Allocation as a Function of the Investment Horizon: Known States

This graph varies the state probabilities perceived by the investor and traces out the resulting asset allocation. The graphs show the optimal allocation to four asset classes —small and large caps, long-term bonds, and 1-month T-bills— as a function of the investment horizon for an investor with constant relative risk aversion $\gamma = 5$.

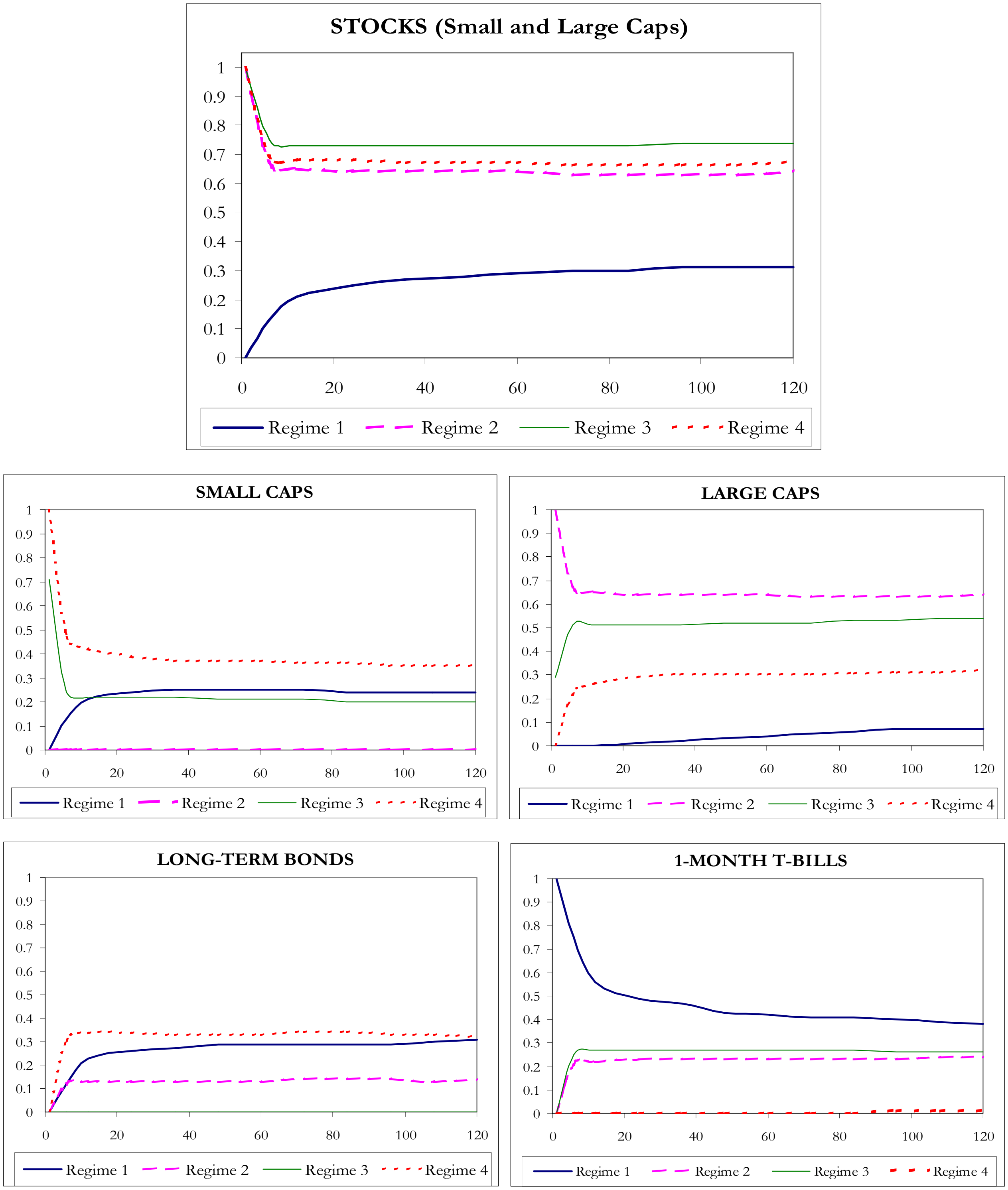

Figure 5

Optimal Portfolio Allocation as a Function of the Investment Horizon: Uncertain States

This Figure considers the case with great uncertainty about the current regime. The graphs show the optimal allocation to four asset classes —small and large caps, long-term bonds, and 1-month T-bills —as a function of the investment horizon for an investor with constant relative risk aversion $\gamma = 5$.

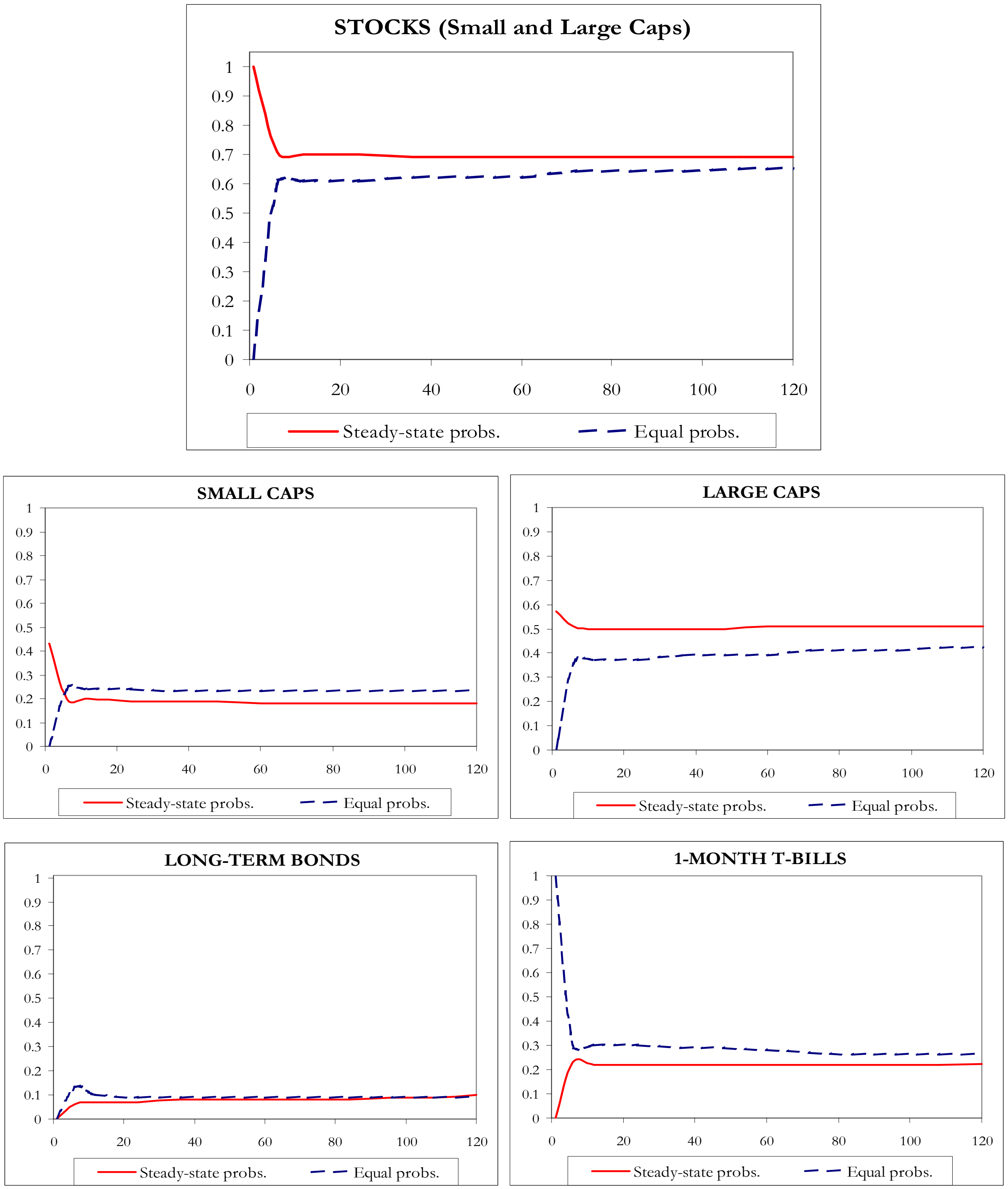

Figure 6

Optimal Portfolio Allocation for a Buy-and-Hold Investor – Effects of Risk Aversion

Plots of the optimal allocation to stocks —both small and large caps —and long-term bonds at three investment horizons. The plots assume the investor has power utility and coefficient of relative risk aversion γ in the interval [1, 50]. Allocations are shown both for the four-state regime switching model and for a myopic investor who ignores the presence of regimes.

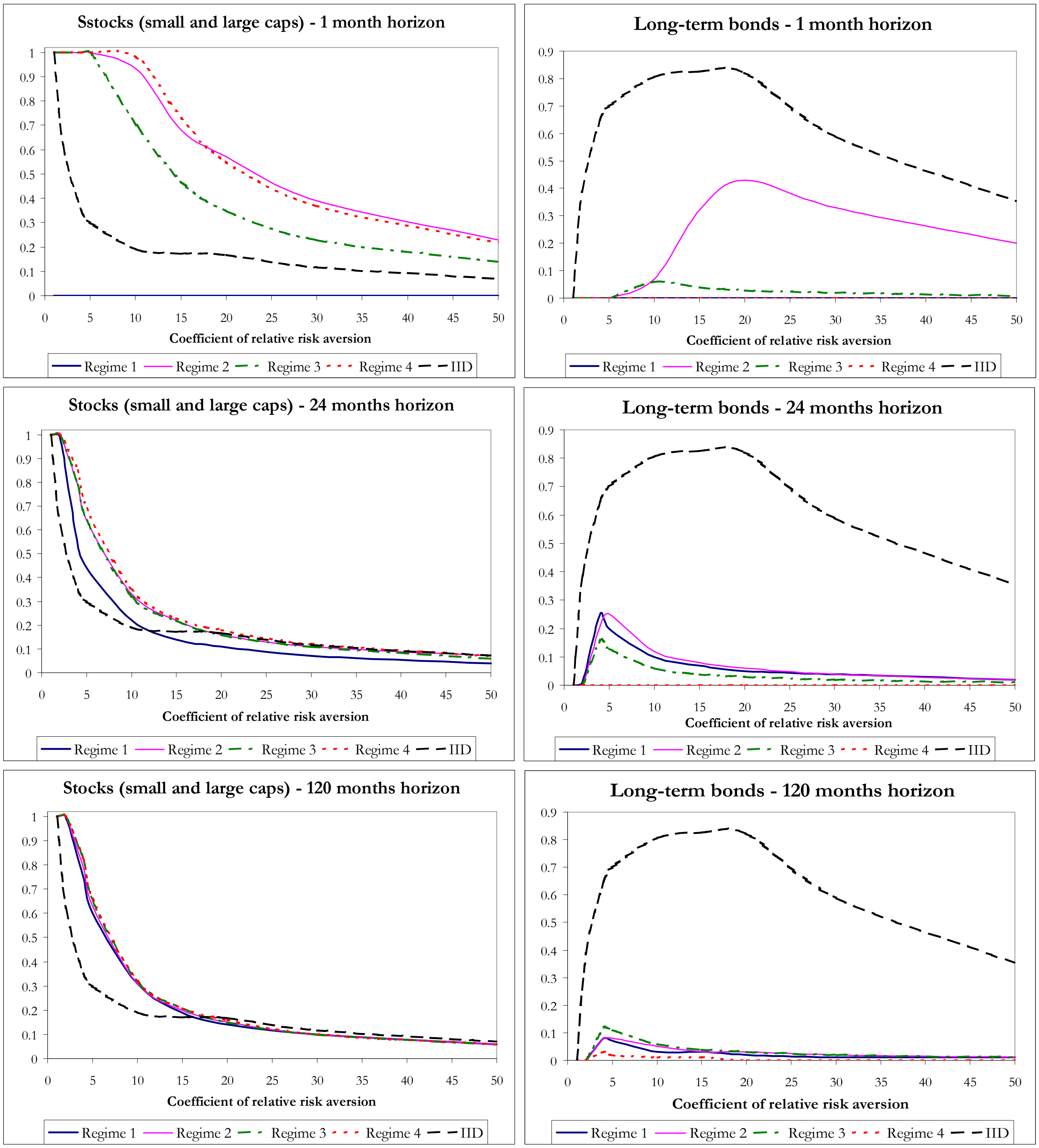

Figure 7

Optimal Portfolio Allocation as a Function of T – Predictability from the Dividend Yield

For each asset class, the five graphs plot the optimal allocation as a function of the investment horizon for an investor with constant relative risk aversion γ = 5 for six alternative configuration of state probabilities: certainty of being in regimes 1-4, equal probability assigned to each regime (i.e. maximum uncertainty), and ergodic probabilities. In each graph, the state variables (y_{t-1}) are held at their unconditional sample means, which implies that excess asset returns are essentially zero, while the dividend yield is 3.3%.

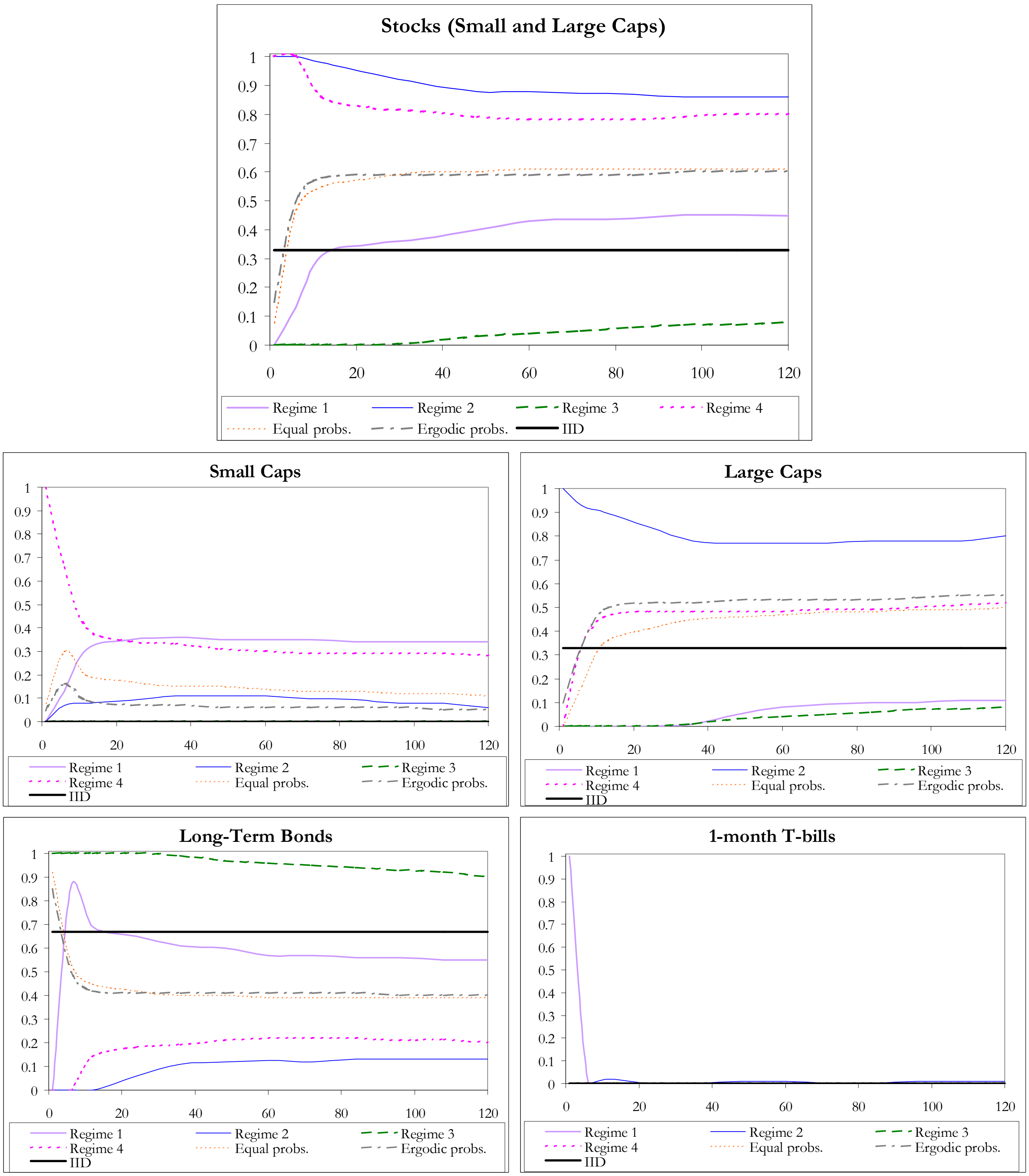

Figure 8

Optimal Portfolio Allocation to Stocks as a Function of T and the Dividend Yield

For each asset class, the five three-dimensional graphs plot the optimal allocation as a function of the investment horizon and of the current dividend yield level for an investor with constant relative risk aversion $\gamma = 5$. In each graph, the perceived regime probabilities are fixed at their ergodic, full-sample values while the lagged value of asset returns is fixed at the unconditional sample mean (essentially zero).

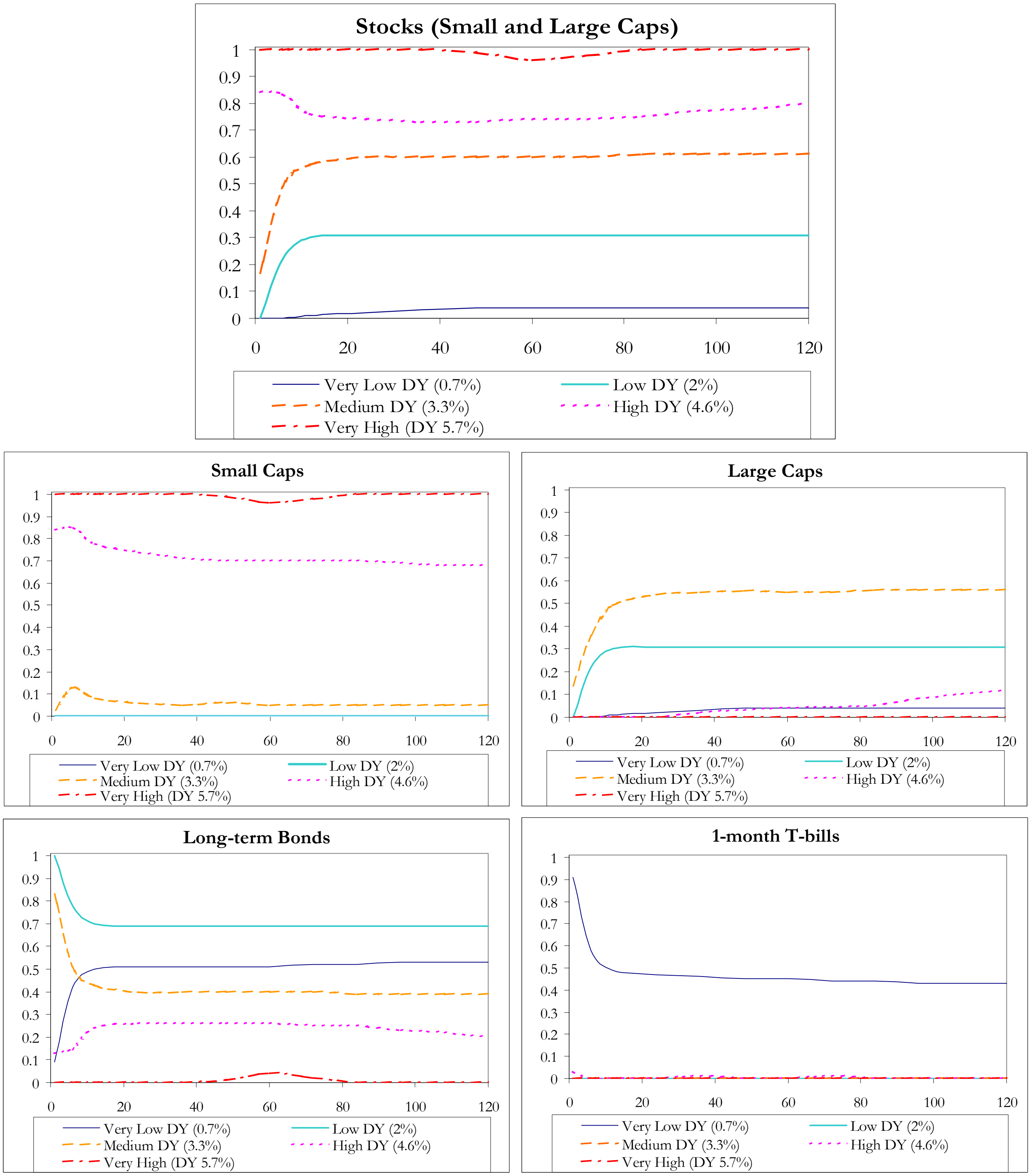

Figure 9

Optimal Consumption-Wealth Ratio – Buy-and-Hold Case

The graph plot the optimal consumption-wealth ratio as a function of the investment horizon for the four the possible regimes under a a MMSIH(4,0) model. The investor has a power utility function with relative risk aversion coefficient γ = 5 and (annualized) rate of subjective time preference of 5%. He implements a buy-and-hold strategy that implies that consumption occurs at time t (today) and at time t+T. As a benchmark, the solid bold line in each graph reports the consumption-wealth ratio obtained when the regime probabilities are set at their steady-state values.

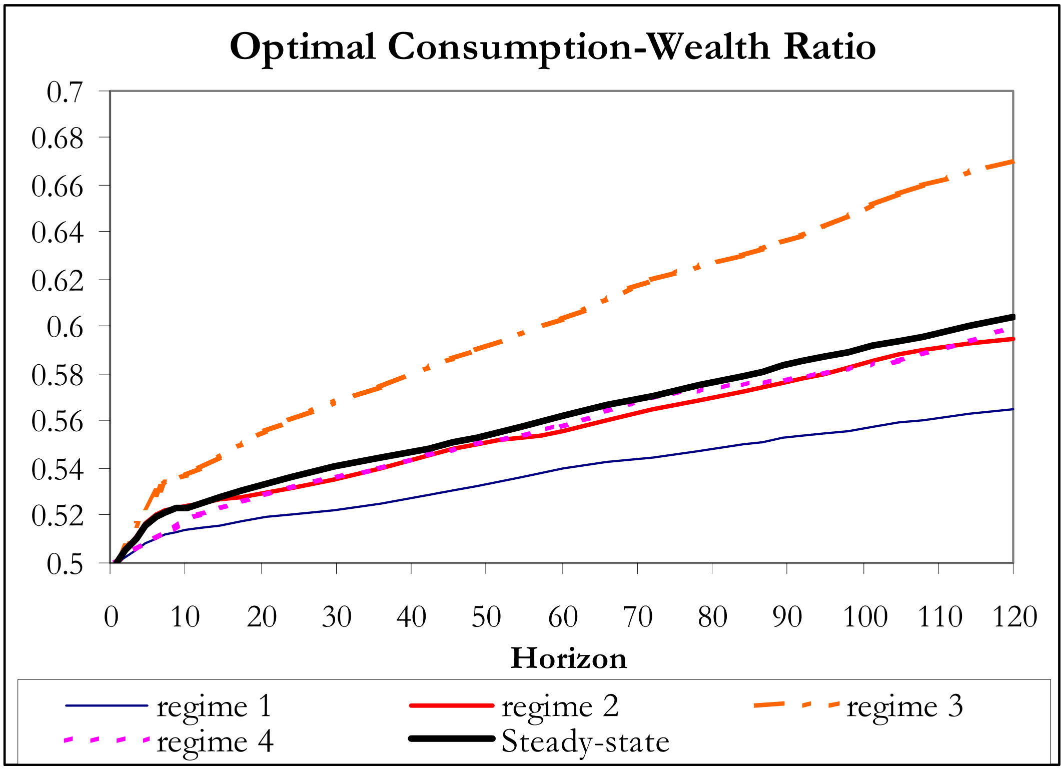

Figure 10

Bootstrap 90% Confidence Bands for Utility Costs from Ignoring Regimes

The graphs plot means, medians, and bootstrap confidence intervals for the compensatory variation required to persuade a buy-and-hold investor with power utility (and $\gamma = 5$) to be willing to ignore regimes in asset returns.