

Parameter Learning in General Equilibrium: The Asset Pricing Implications

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Abstract

Parameter learning strongly amplifies the impact of macro shocks on marginal utility when the representative agent has a preference for early resolution of uncertainty. This occurs as rational belief updating generates subjective long-run consumption risks. We consider general equilibrium models with unknown parameters governing either long-run economic growth, the variance of shocks, rare events, or model selection. Overall, parameter learning generates long-lasting, quantitatively significant additional macro risks that help explain standard asset pricing puzzles.

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1 Introduction

Asset pricing theories commonly assume a particularly strong form of knowledge by economic agents: they know the true model and true parameter values. This is, in part, motivated by conventional wisdom suggesting that parameter learning has negligible asset pricing implications. To see why, assume normally distributed consumption growth, $\Delta \ln(C_t) = y_t \stackrel{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma^2)$, μ is unknown, and that $\mu \sim \mathcal{N}(\mu_0, A_0\sigma^2)$ a priori. Bayesian updating implies the posterior is $\mathcal{N}(\mu_t, A_t\sigma^2)$, where μ_t and A_t are recursively defined and y^t is data up to time t . Under power utility preferences, the ‘equity’ premium on a single-period consumption claim is $\gamma\sigma^2(1 + A_t)$. Since A_t decreases rapidly over time, parameter uncertainty has a small initial and quickly dissipating impact on asset prices.

We show that this conventional wisdom does not hold generally because rational parameter learning generates subjective long run consumption risks that have important asset pricing implications when the representative agent has Epstein-Zin recursive utility (see Bansal and Yaron (2004)). Long run risks arise because posterior distributions, $\mathbb{P}(\theta \in A|y^t)$, as well as posterior moments, are martingales (e.g., Doob (1949)), which implies that shocks to rational beliefs are permanent and impact consumption growth in all future periods. For agents who prefer early resolution of uncertainty, assets whose payoffs are affected by unknown parameters may therefore be particularly risky. The same logic holds when there is uncertainty over the model specification and agents learn about the ‘true’ model over time.

Parameter uncertainty is intuitively important, especially in asset pricing models with numerous parameters and increasingly complex dynamics. In many common specifications, the main asset pricing implications arise from long-run properties of consumption dynamics and/or rare events, both of which are extremely hard-to-estimate with observed histories of macroeconomic aggregates. This fact suggests the importance of accounting for parameter uncertainty when analyzing the relation between macroeconomic risks and asset prices, as emphasized in Hansen (2007).

This paper studies qualitatively and quantitatively how parameter uncertainty impacts asset prices when the temporal resolution of uncertainty matters. The overall finding is that reasonable calibrations of parameter and model uncertainty can tremendously amplify the perceived quantity of aggregate risk, with corresponding strong implications for asset prices. Parameter learning affects asset prices principally because belief *updating* leads to large shocks to continuation utility. This is distinct from other conduits (e.g., Weitzmann (2007)), where parameter uncertainty leads to fat-tailed predictive distributions, increasing

the probability of very high marginal utility states. Since the shocks to beliefs are permanent, even a small amount of parameter uncertainty can have a large effect on continuation utility, which is a function of the conditional distribution of consumption in all future periods. As continuation utility shocks arise from *subjective* beliefs updates, learning can drive a large wedge between the known-parameters consumption dynamics and the dynamic behavior of the pricing kernel. For instance, large time series variation in the price of risk and the equity risk premium can arise from homoskedastic macro fundamentals, even though the preferences are time- and state-invariant.¹

Since there are many different types of parameters (means, variances, transition probabilities, etc.), we are particularly interested in understanding when parameter learning may generate large and long-lasting effects on central asset pricing quantities like the risk premium and return volatility of the aggregate consumption and dividend claims, as well as real yields on short- and long-term default-free bonds. To this end, we consider a range of models with uncertainty about parameters governing different aspects of aggregate consumption dynamics.

We first consider the simple consumption growth model described above and document, analytically and numerically, the asset pricing implications of an unknown mean growth rate. While overly simple to be realistic, this model highlights the underlying economics. In addition to this case, we consider learning about the variance of shocks (see Weitzmann (2007) and Bakshi and Skoulakis (2010)), as well as parameters governing the persistence and severity of rare events (see, Rietz (1988), Barro (2006, 2009), and Gourio (2012) for models on disaster risk). Finally, we consider parameter uncertainty in the form of model uncertainty where the agent is learning whether consumption growth is iid or contains a small, persistent component. This range of model and types of parameter uncertainty provide a taxonomy of how Bayesian parameter learning impacts asset prices.

In terms of realistic calibrations, learning about the persistence of rare events or otherwise hard to measure bad states of the economy has dramatic and quite realistic asset pricing effects. For example, consider a Markov-switching model where the bad state occurs once every 100 years on average, with a mean and persistence calibrated to U.S. consumption data during the Great Depression. With uncertainty over the persistence of the bad state, assuming a 100 year training sample, the model delivers an equity risk premium over the last

¹Related, long-run consumption risk due to parameter learning does not imply predictability of consumption growth moments—a controversial feature of standard long-run risk models (see Beeler and Campbell (2012)).

100 years of about 6% when the agent has relative risk aversion (RRA) coefficient of 3.9 and an elasticity of intertemporal substitution (EIS) of 2. The pricing kernel volatility is high, a prerequisite for explaining asset price moments (see Hansen and Jagannathan (1991)). With a 200 year training sample, the equity risk premium falls to 5%. The corresponding ‘full-information’ model with known parameters generates a risk premium of 1%. Thus, realistic parameter uncertainty provides a dramatic magnification of macroeconomic risks on asset prices. Further, the model can match the very high equity return volatility at the onset of the Great Depression relative to normal times, as well as the drop in aggregate price-dividend ratios, as the risk premium increases strongly in bad times.

Learning about rare event parameters, though realistic and important, has only a minor impact on dynamics during ‘normal’ times, since learning about bad states is rare. We therefore also consider learning about parameters governing business cycle fluctuations in consumption via a case of model uncertainty. Here, the representative agent is uncertain whether consumption growth is i.i.d. or it contains a small, persistent component a la Bansal and Yaron (2004).² In this case, the impact of learning on asset prices is quantitatively large and endogenously long lasting. In fact, learning generates a high price of risk even if agents assign a very small probability to the more risky economy with persistent shocks. The price of risk and the risk premium vary substantially even though both are constant in each individual model and can, in certain states, be more than double their size in either of the individual models. When feeding this model the actual consumption realizations over the post-WW2 U.S. sample, the price of risk and the equity market risk premium are high in recessions relative to expansions, as in the data.

Alvarez and Jermann (2004) argue that asset prices imply small welfare costs from business cycle fluctuations if business cycles are comprised of only transitory fluctuations, consistent with the analysis in Lucas (1987). In the model uncertainty case we analyze, a non-trivial fraction of the variation in beliefs stems from business-cycle fluctuations during the post-WW2 U.S. sample. Thus, business cycle fluctuations are associated with permanent shocks to subjective consumption dynamics, and carry therefore ‘long-run’ risk. Alvarez and Jermann (2004, 2005) document empirically that permanent components in the pricing kernel are the main source of large welfare costs. Our learning channel provides an endogenous mechanism for how such large permanent macro shocks arise.

²A similar problem is analyzed in Hansen and Sargent (2010), but they focus on learning under a preference for robustness.

2 Parameter learning as a source of long-run risks

Parameter uncertainty and rational updating generates truly ‘long-run’ risks. Intuitively, this occurs because the forecast errors of optimal beliefs are unpredictable, which implies that shocks to beliefs are permanent. Mathematically, long run risks arise due to various martingale properties associated with conditional probabilities.

Denote the time- t posterior density of a vector of parameters θ as $p(\theta|y^t)$. By the law of iterated expectations, $\mathbb{P}[\theta \in A|y^t]$, expectations of functions of the parameters ($\mathbb{E}[h(\theta)|y^t]$), and likelihood ratio statistics are all martingales. This is easy to see: defining $\mu_t \equiv \mathbb{E}[\theta|y^t]$,

$$\mathbb{E}[\mu_{t+1}|y^t] = \mathbb{E}[\mathbb{E}[\theta|y^{t+1}]|y^t] = \mathbb{E}[\theta|y^t] = \mu_t. \quad (1)$$

Thus, the belief process, μ_t , is a martingale, and evolves via $\mu_{t+1} = \mu_t + \eta_{t+1}$, where $\mathbb{E}[\eta_{t+1}|y^t] = 0$. From this, it is clear that the shocks to rational beliefs, η_{t+1} , are not just persistent, but *permanent* as they have a unit root. This property of Bayesian parameter learning has been noted before, see, e.g., Hansen (2007).

This intuition also holds for learning about competing model specifications. Consider two models, denoted model 0 and model 1, and define an indicator variable, M , such that $M = 1$ (0) indicates that model 1 (0) is true. The data is then generated from $p(y_{t+1}|M = 0, y^t)$ or $p(y_{t+1}|M = 1, y^t)$, where the dependence on y^t as a conditioning variable reflects the fact there could be learning about other parameters or state variables within the model. From the agent’s perspective, M is a random variable whose value can be learned. Given initial probabilities, $p_0 = \mathbb{P}[M = 0]$, rational learning generates the posterior $p_t = \mathbb{P}[M = 0|y^t]$, which is defined recursively by Bayes rule as

$$p_{t+1} = \frac{p(y_{t+1}|M = 0, y^t)p_t}{p(y_{t+1}|M = 0, y^t)p_t + p(y_{t+1}|M = 1, y^t)(1 - p_t)}. \quad (2)$$

As in the case of fixed parameter uncertainty, shocks to beliefs regarding the true model specification (the random variable M) are martingales and have permanent effects.

We consider an Epstein-Zin (1989) agent with utility, V , over consumption, C :

$$V_t = \left\{ (1 - \beta) C_t^{1-1/\psi} + \beta (E_t[V_{t+1}^{1-\gamma}])^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}, \quad (3)$$

where γ is RRA, ψ is the EIS, and β is the time discount factor. The stochastic discount

factor (SDF) in this economy is

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\beta \frac{PC_{t+1} + 1}{PC_t} \right)^{\theta-1}, \quad (4)$$

where PC_t is the wealth-consumption ratio at time t and where $\theta = (1 - \gamma) / (1 - \psi^{-1})$. The first component, $\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$, is the usual power utility component. When there is a preference for the timing of the resolution of uncertainty, (i.e., if $\theta \neq 1$), the SDF has a second term, $\left(\beta \frac{PC_{t+1} + 1}{PC_t} \right)^{\theta-1}$, through which long-run risks impact asset prices.

When the underlying structural parameters governing consumption dynamics are unknown, parameter learning impacts marginal intertemporal rates of substitution. In particular, revisions in beliefs generate permanent shocks to the conditional distribution of future consumption, impacting the price-consumption ratio via changes in growth expectations and/or discount rates. From Equation (4) it is immediate that these shocks are priced risk factors in this economy.

The rest of the paper quantifies the impact of various types of parameter uncertainty in a range of models. We first consider the simplest model, where consumption growth is i.i.d. lognormal, but the mean growth rate is unknown. This case transparently provides intuition and a sense of magnitudes. We then move on to more interesting consumption dynamics, including learning about rare events and model uncertainty.

2.1 Relation to existing literature

Our focus on parameter learning connects to a long-standing debate in macroeconomics. One common critique of rational expectations models assuming perfect knowledge is precisely the assumption that agents know ‘fixed but unknown’ parameters, e.g., Modigliani (1977). Of course there is nothing about parameter or model learning inconsistent with rational expectations, as noted by Lucas and Sargent (1978, p. 68): “... *it has been only a matter of analytical convenience and not of necessity that equilibrium models have used the assumption of stochastically stationary "shocks" and the assumption that agents have already learned the probability distributions that they face. Both of these assumptions can be abandoned, albeit at a cost in terms of the simplicity of the model... While models incorporating Bayesian learning and stochastic nonstationarity are both technically feasible and consistent with the equilibrium modeling strategy, almost no successful applied work along these lines has come to light. One reason is probably that nonstationary time series models are cumbersome and*

come in so many varieties.” As discussed below, numerical solutions are generally required and can be quite complicated.

Hansen (2007) stresses the importance of studying how parameter and model uncertainty impacts asset valuation, forcing economic agents to face the inference problems as econometricians. Hansen (2007), and also Hansen and Sargent (2010), take a robustness approach, with agents making decisions that are robust to model uncertainty and consider the case of an EIS of one. In contrast, we focus on Bayesian learning with Epstein-Zin preferences and consider EIS values different from unity and also consider the pricing of long-horizon risky claims—notably claims to the infinite streams of consumption and dividends, as well as long-term bonds.

Other related papers considering general equilibrium implications of parameter learning include Veronesi (2000), Cogley and Sargent (2008), Jobert, Platania, and Rogers (2006), Benzoni, Collin-Dufresne and Goldstein (2011), Johannes, Lochstoer, and Mou (2010), and Kumar and Gvozdeva (2012). Relative to these our paper focuses on the impact of priced parameter uncertainty on asset price moments in a general equilibrium model with Epstein-Zin preferences.³

Pastor and Veronesi (2009, 2012) consider parameter learning applications with power utility preferences over final wealth. As shown in Timmermann (1993) and Lewellen and Shanken (2002), parameter learning about dividend dynamics induces excess return predictability in in-sample forecasting regressions as typically undertaken in the literature. The hallmark of the learning channel, however, is poor out-of-sample performance of such regressions, consistent with the data (see Goyal and Welch (2008) and Johannes, Korteweg and Polson (2013)).

A number of papers consider state uncertainty, where the state evolves discretely via a Markov chain or smoothly via a Gaussian process.⁴ Veronesi (2000) considers learning about mean-dividend growth rates in a power utility setting and focuses on the role of information quality. Learning about a fixed parameter is a special case, and with common preference parameters, Veronesi shows that in this case the equity premium falls and could even be negative when parameters are uncertain.

³Benzoni, Collin-Dufresne and Goldstein (2011) also solve for a continuous time general equilibrium model where there is parameter uncertainty associated with a rare crash-event probability. Their focus is on explaining the implied option skew however. Kumar and Gvozdeva (2012) also consider a general equilibrium model with Epstein-Zin preferences and parameter uncertainty, but their numerical solution is only an approximation to the true problem.

⁴Earlier contributions include Detemple (1986), Dothan and Feldman (1986), and Gennotte (1986) who show that you can separate the filtering problem from the pricing problem.

David (1997) consider learning about the level of firm profitability, which is assumed to switch between two states. Moore and Shaller (1996) consider consumption/dividend based Markov switching models with state learning and power utility. Veronesi (2004) studies the implications of learning about a peso state in a Markov switching model with power utility. David and Veronesi (2010) consider a Markov switching model with learning about states. Pastor and Veronesi (2003, 2006) study uncertainty about a fixed dividend-growth rate or profitability levels with an exogenously specified pricing kernel, in part motivated in order to derive cross-sectional implications.

In the case of Epstein-Zin utility, Brandt, Zeng, and Zhang (2004) study alternative rules for learning about an unknown Markov state, assuming all parameters and the model is known. Chen and Pakos (2008), Lettau, Ludvigson, and Wachter (2008) and Boguth and Kuehn (2012) consider economic agents who know parameter values, but learn about states in a Markov switching consumption based asset pricing model. Ai (2010) studies learning in a production-based long-run risks model with Kalman learning about a persistent latent state variable. Bansal and Shaliastovich (2008) and Shaliastovich (2010) consider learning about the persistent component in a Bansal and Yaron (2004) style model with sub-optimal Kalman learning.

Alternative preferences with a preference for early resolution of uncertainty will exhibit similar effects to those we document with parameter learning and Epstein-Zin preferences. The quantitative effects will of course depend on the utility specification and parameter assumptions. Examples include general Kreps-Porteus preferences and smooth ambiguity aversion preferences of Klibanoff, Marinacci, and Mukerji (2009) and Ju and Miao (2012), as well as the fragile beliefs setup of Hansen and Sargent (2010).⁵ Strzalecki (2011) discusses of the relation between ambiguity attitudes and the preference for the timing of the resolution of uncertainty. Learning under ambiguity (e.g., Epstein and Schneider (2007)) differs from Bayesian learning as learning under ambiguity depends on the sets of priors entertained by the agent, with higher weight being given to more pessimistic prior beliefs when forming predictive distributions.

⁵Benzoni, Collin-Dufresne, Goldstein and Helwege (2010) investigate the implications for credit spreads of learning under fragile beliefs.

3 The simplest case: learning about the mean

Assume that aggregate log consumption growth is i.i.d. normal:

$$y_{t+1} = \Delta c_{t+1} = \mu + \sigma \varepsilon_{t+1}, \quad (5)$$

where σ and the shock distribution, $\varepsilon_{t+1} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$, are known. μ is not known, and the agent posits the conjugate prior $\mu \sim \mathcal{N}(\mu_0, A_0 \sigma^2)$. Rational beliefs sequential update upon observing consumption growth rates using Bayes rule, which implies that $\mu|y^t \sim \mathcal{N}(\mu_t, A_t \sigma^2)$ where $A_{t+1}^{-1} = A_t^{-1} + 1$. Defining $\omega_t \equiv (A_t^{-1} + 1)^{-1}$, beliefs have the familiar shrinkage form:

$$E[\mu_{t+1}|y^t] = \omega_t \Delta c_{t+1} + (1 - \omega_t) \mu_t. \quad (6)$$

The conditional sufficient statistics μ_t and A_t are state variables in the economy.

From the agent's perspective, predictive consumption dynamics evolve via

$$\Delta c_{t+1} = \mu_t + \sqrt{1 + A_t} \sigma \tilde{\varepsilon}_{t+1}, \quad (7)$$

where $\tilde{\varepsilon}_{t+1} \sim \mathcal{N}(0, 1)$ and the conditional mean evolves via

$$\mu_{t+1} = \mu_t + \frac{A_t}{\sqrt{1 + A_t}} \sigma \tilde{\varepsilon}_{t+1}. \quad (8)$$

In words, the agent thinks that consumption growth is normally distributed, but the moments are time-varying and expected consumption growth has a unit root. Compared to the consumption dynamics in Bansal and Yaron (2004), learning induces truly long-run consumption risks, as the agent perceives expected consumption growth shocks to be permanent versus Bansal and Yaron's persistent, but still transitory, shocks.⁶ The consumption growth process does not explode, however, as the posterior variance declines over time and will eventually (at $t \rightarrow \infty$) go to zero.

Since actual consumption growth as in Equation (5) is unpredictable, it (trivially) cannot be predicted by, e.g., the price-dividend ratios or risk-free rates. This difference between the objective long-run risks assumed by Bansal and Yaron, which imply a high degree of consumption predictability, and *subjective* long-run risks arising endogenously through pa-

⁶In Bansal and Yaron, conditional expected consumption growth follows an AR(1) with a monthly autoregression coefficient of 0.979.

parameter learning, is empirically relevant. One critique of long-run risk models is that they imply an implausible degree of consumption predictability (see Beeler and Campbell (2012)). From the example presented here it is clear that such critique is not valid for long-run risks induced by parameter learning.

3.1 Asset price implications when $EIS = 1$

When $\psi = 1$ and in continuous-time, there is an analytical solution for the value function, the details of which are given in the Online Appendix.⁷ This generates simple expressions for central asset pricing quantities and provides intuition for understanding the general equilibrium effects of structural parameter uncertainty with Epstein-Zin preferences.

The continuous-time equivalent of Equation (5) is

$$dc_t = \mu_t dt + \sigma dz_t,$$

where dz_t are innovations to a standard Brownian motion under the agent's filtration and the hyperparameters, μ_t and A_t (the state variables in this economy), evolve via $d\mu_t = A_t \sigma dz_t$ and $dA_t = -A_t^2 dt$. In continuous-time, the volatility of consumption growth (short-run risk) is the same as in a full-information economy. The log value function (v_t) is (see Online Appendix):

$$v_t = c_t + (1 - \gamma)^2 \sigma^2 \frac{1 + A_t/\tilde{\beta} - \exp\left(\tilde{\beta}/A_t\right) \text{Ei}\left(-\tilde{\beta}/A_t\right)}{2\tilde{\beta}} + \frac{1 - \gamma}{\tilde{\beta}} \mu_t, \quad (9)$$

where $\text{Ei}(z) = \int_{-z}^{\infty} \frac{e^{-t}}{t} dt$ is the exponential integral function and $\tilde{\beta} \equiv -\ln \beta$, where β is the discrete-time time preference parameter from Equation (3).

The maximal conditional Sharpe ratio (SR, conditional volatility of the log pricing kernel) and risk premium (RP) on the consumption claim are given by

$$SR = \gamma\sigma + \frac{\gamma - 1}{\tilde{\beta}} A_t \sigma \quad \text{and} \quad RP = \gamma\sigma^2 + \frac{\gamma - 1}{\tilde{\beta}} A_t \sigma^2, \quad (10)$$

respectively. The first terms in each expression are the familiar power utility terms, and the

⁷In subsequent models, it will be necessary to resort to numerical solutions and therefore we move back to a discrete-time setting shortly. The results do not hinge on the distinction between discrete and continuous time.

second terms are generated by learning. The quantitative impact of learning is therefore a function of (i) the preference for the timing of resolution of uncertainty, $\gamma - \psi^{-1}$, (ii) the duration of the belief shock in terms of its effect on utility, $1/\tilde{\beta}$, and (iii) the size of shocks to beliefs, $A_t\sigma$. Intuitively, a preference for early resolution of uncertainty ($\gamma > 1/\psi$) is needed for learning to increase risk. The extra risks arising from parameter learning and a preference for early resolution of uncertainty come from *updating beliefs* and not from a fat-tailed conditional distribution of consumption growth—the subjective distribution is normal. This is different from Geweke (2002) and Weitzman (2007), who note that learning about σ in discrete-time induces a fat-tailed predictive distribution for consumption.

3.1.1 Asset pricing implications and the speed of learning

Equation (10) implies that when $\psi = 1$ important moments like pricing kernel volatility and the risk premium have a constant loading on the amount of parameter uncertainty. With a preference for an early resolution of uncertainty, parameter learning clearly increases the Sharpe ratio and risk premium. However, since Bayesian learning is efficient, one might think that parameter learning effects are small and disappear quickly. Indeed, since $dA_t = -A_t^2 dt$ and assuming $A_0 = 1$, we obtain $A_t = (1 + t)^{-1}$. Thus, the *ex ante* magnitude of the shock to beliefs about the mean growth rate ($A_t\sigma$), declines at a rate $(1 + t)^{-1}$.

However, with parameter learning, even if the size of the belief shock is small, the effect is permanent and so the effect on continuation utility can still be large. As an example, consider a quarterly calibration with time-preference and risk aversion as in Bansal and Yaron (2004): $\tilde{\beta} = -\ln 0.994$ and $\gamma = 10$. The multiplier on the amount of parameter uncertainty is extremely large, $(\gamma - 1)/\tilde{\beta} = 1,495$. This implies that even after learning for 100 years, the annualized Sharpe ratio and risk premium are 1.37 times or 37% larger than the corresponding quantities in the full-information or known parameters case.⁸ With a 200 (300) years long training sample, the increase is 19% (12%). Thus, there is a *quantitatively large and long-lasting magnification* of macroeconomic risks. This is one of our primary results. In contrast, in the case of $\gamma = 1/\psi = 1$ (log utility), the agent is indifferent to the timing of resolution of uncertainty and there are *no* effects of parameter learning on the risk-premium or Sharpe ratio.

It is also important to note that the speed of learning slows over time. Measuring time t in quarters (consistent with the quarterly model calibration), the conditional volatility of

⁸The relative magnitude are obtained by dividing the equations in (10) by $\gamma\sigma$ and $\gamma\sigma^2$, respectively.

shocks to the mean beliefs about μ declines by a factor of 400 over the first one hundred years of learning, but the rate is not uniform. In the first 10 years of learning, volatility declines by a factor of 40. From year 10 to 50 by a factor of 5, and from years 50 to 100 and years 100 to year 200, the volatility declines only by a factor of 2. Thus the speed of learning slows. This is why the asset pricing impact of parameter learning persists for a very long time, even in this simple model.

At this stage, note that when learning about a fixed parameter, the amount of parameter uncertainty will typically, and in this case certainly, decline with time, implying time-trends in asset prices. While one can argue that the market Sharpe ratio and risk premium overall has declined over the available historical sample (see, e.g., Fama and French (2000)), one can reasonably rule out declines greater than, say, a factor of 5.⁹ Thus, the reasonability of a prior can be assessed in part by whether it implies excessive learning (time-trends) over samples such as those we have available relative to observed asset price behavior.

3.1.2 The term structure

The real risk-free rate in this economy is:

$$r_{f,t} = \tilde{\beta} + \mu_t + \frac{\sigma^2}{2} - \gamma\sigma^2 - \frac{\gamma - 1}{\tilde{\beta}}\sigma^2 A_t, \quad (11)$$

which are driven by μ_t , a martingale, and A_t , which decreases deterministically over time. Thus, future risk-free rates are always expected to be higher than current risk-free rates, suggesting an upward-sloping term structure. However, the risk premium on bonds is negative, as low realized consumption growth decreases μ_t , which, in turn, decreases the risk-free rate and increases bond prices. The slope of the term structure depends on the relative magnitude of the risk premium and the increase in future expected short-rates from the decreasing precautionary savings (A_t).

The price of a zero-coupon, default-free τ -year bond is

$$P(t, \tau) = (A_t \tau + 1)^{\frac{\sigma^2(2\gamma(A_t \tau + 1) - 1)}{2A_t}} e^{-\tau \left(\frac{1-\gamma}{\tilde{\beta}} A_t \sigma^2 + \tilde{\beta} + \mu_t \right)}. \quad (12)$$

⁹This is obviously not an exact statement. Readers are invited to make their own judgment about the data on this point.

The slope of the term structure at maturity τ is then:

$$y_{t,\tau} - y_{t,0} = -\frac{1}{\tau} \frac{\sigma^2 (2\gamma (A_t\tau + 1) - 1)}{2A_t} \ln(1 + A_t\tau) - \frac{\sigma^2}{2} + \gamma\sigma^2, \quad (13)$$

where $y_{t,0} = r_{f,t}$. Based on the preference parameters from above, the 10-year slope is between 0 and -1.3 basis points when $A_t \leq 1$, with σ set to the same value as in Bansal and Yaron (2004). The combination of a negative bond risk premium and increasing expected future short-rates roughly nets to zero, and the yield curve is flat. In contrast, the Bansal and Yaron (2004) model generates a strongly downward-sloping term structure, which Beeler and Campbell (2012) argue is counter-factual.

3.2 Asset pricing implications when $EIS > 1$

The case of $EIS \neq 1$ is discussed in detail in the Online Appendix, and we briefly summarize the implications. When the substitution effect dominates the wealth effect (i.e., when $EIS > 1$), the price-consumption ratio increases upon a positive revision of the beliefs about the growth rate. Overall, the primary effect of increasing the EIS is an increase in excess return volatility, which, in turn, increases the risk premium, both of which are important for matching historical asset price data. Further, the impact of parameter learning on the pricing kernel changes over time: the volatility of the pricing kernel decreases at a slower rate over time when the EIS is high than when it is low. This is due to an endogenous increase in the sensitivity of the price-consumption ratio to belief updates. This occurs as discount rates decrease over the sample, which makes the price-consumption ratio more sensitive to updates in the expected growth rate (see Pastor and Veronesi (2004)).

Importantly, these effects combine to imply that the risk premium on the consumption claim after 100 years of learning is almost twice as the risk premium in a full-information, known parameter economy when $EIS = 2$, while the price of risk is almost 1.5 times higher and the return volatility is 1.24 times higher than in the known parameter economy. Further, as highlighted in Lewellen and Shanken (2002), learning generates excess return predictability in standard, in-sample forecasting regressions.

3.3 Asset pricing implications of unknown volatilities

The Online Appendix, for completeness, also analyzes learning about σ^2 in the simple i.i.d. consumption growth case in a discrete time economy. Bakshi and Skoulakis (2010) note that

uncertain variance with reasonable truncation bounds for the support of the distribution leads to negligible effects on the price of risk in a power utility setting. Since learning about the variance parameter also generates shocks to continuation utility, it is not clear this result holds when agents have a preference for early resolution of uncertainty. However, as learning about a constant volatility parameter is more rapid than about a mean parameter, and since volatility is a second-order effect in terms of utility, the asset pricing effects of learning about the volatility of shocks become quickly very small with Epstein-Zin preferences.

3.4 Discussion

The analytical solution cleanly shows how parameter uncertainty in conjunction with a preference for early resolution of uncertainty can be a powerful amplification mechanism for the pricing of macro shocks. Parameter uncertainty generates quantitatively large and long-lasting effects and helps in understanding many of puzzling observations such as the high equity premium and Sharpe ratio, as well as the shape of the yield curve. Furthermore, the subjective nature of the long-run risks induced by parameter learning means that neither the risk-free rate nor valuation ratios (e.g., the price-dividend ratio) forecast future consumption growth. In particular, the high EIS and time-varying risk-free rate in this model are entirely consistent with estimates of the EIS close to zero obtained from Hall (1988)–type regressions, as actual consumption growth is i.i.d. and thus unpredictable.¹⁰

Of course, the i.i.d. normal model is overly simplistic as a description of actual consumption dynamics. We consider next parameter uncertainty in more realistic models, where learning is likely to take a long time and where asset pricing implications of parameter uncertainty are large. In particular, we consider learning about rare events such as the Great Depression, and also a case of model uncertainty where two models with very different asset pricing implications are hard to differentiate using available macroeconomic data.

4 Learning about rare events

Markov switching models have been widely used in consumption based asset pricing, both for their flexibility and their analytical tractability (see, e.g., Mehra and Prescott (1986) and

¹⁰The fact that the Hall-regression does not uncover the EIS of economic agents is also a feature in Garleanu and Panageas (2012), who show that long-run risk in individual consumption growth rates, with corresponding time-variation in the risk-free rate, are generated from optimal consumption sharing with heterogeneous agents even when aggregate consumption is i.i.d.

Rietz (1988)). Since the financial crisis, there has been a renewed interest in using these models to capture particularly bad periods economic periods like the Great Depression, commonly called consumption disasters. These models provide a particularly useful laboratory, as it is hard to learn about the parameters governing rare events and these parameters have particularly strong asset pricing implications.

It is difficult to estimate the frequency, severity, and length of consumption disasters or depressions. In fact, even using centuries of data and a broad panel of countries, Barro, Nakamura, Steinsson, and Ursua (BNSU, 2011) report significant uncertainty in parameter estimates in formal models. They estimate a consumption disaster frequency of 2.8% per year and a probability of exiting a disaster is 13.5% per year. The standard errors are high: e.g., a 2 standard-error bound for the average duration of the bad state is between 4.5 and 9 years. There is also a large amount of uncertainty over the size (mean and variance) of consumption disasters.

To investigate the impact of parameter learning in rare events models, we consider a two-state Markov switching model:

$$\Delta c_t = \mu_{s_t} + \sigma_{s_t} \varepsilon_t,$$

where $\varepsilon_t \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$, s_t is a 2-state observed Markov chain with transition matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & 1 - \pi_{11} \\ 1 - \pi_{22} & \pi_{22} \end{bmatrix}.$$

Without any loss of generality, we label $s_t = 1$ the ‘good’ or normal state and $s_t = 2$ the ‘bad’ or rare event state. With $i, j \in \{1, 2\}$ and $i \neq j$, the unconditional or ergodic probability of state i is

$$P[s_t = i] = \frac{1 - \pi_{jj}}{2 - \pi_{ii} - \pi_{jj}}. \tag{14}$$

We separately consider the cases of unknown transition probabilities (Section 4.1) and uncertain means/variances (Section 4.2) to understand the role of different types of parameter uncertainty while keeping the size of the state-space manageable.

4.1 Uncertain transition probabilities

4.1.1 The Learning Problem

Assume the transition matrix is unknown and conjugate, Beta-distributed priors $\pi_{ii} \sim \beta(a_{i,0}, b_{i,0})$, for $i = 1, 2$. Under the assumption that s_t is observed, the posterior distribution for the transition probabilities depends only on the state counts, which makes the learning problem particularly tractable.¹¹ Defining the observed states up to time t as $s^t = (s_1, \dots, s_t)$, the posterior distribution is $\pi_{ii}|s^t \sim \beta(a_{i,t}, b_{i,t})$, where

$$a_{i,t} = a_{i,0} + \sum_{k=1}^t \mathbf{1}(s_k = i, s_{k-1} = i) \text{ and } b_{i,t} = b_{i,0} + \sum_{k=1}^t \mathbf{1}(s_k = j, s_{k-1} = i), \quad (15)$$

for $i \neq j$ and $i, j \in \{1, 2\}$. The posterior mean and variance are respectively

$$E_t[\pi_{ii}] = \frac{a_{i,t}}{a_{i,t} + b_{i,t}} \text{ and } var_t(\pi_{ii}) = \frac{a_{i,t}b_{i,t}}{(a_{i,t} + b_{i,t})^2 (a_{i,t} + b_{i,t} + 1)}.$$

In calibrating the priors, we consider a range of historical experiences that are easy to incorporate given the conjugate structure. Our priors are unbiased, as our focus is on the effects of priced parameter uncertainty and not biased beliefs (see, e.g., Cogley and Sargent (2008) for biased priors). Thus, the priors parameters imply the number of prior transitions coincides with the true ergodic probability of the corresponding regime from equation (14). Given the true values of π_{11} and π_{22} , the priors and posteriors are functions of numbers of years of observations, T_0 .

We consider priors corresponding to 100, 200, or 300 years of quarterly observations. After 200 years of observations starting from a flat prior, the posterior standard deviation of π_{22} is very close to the corresponding standard error reported in BNSU. Since BNSU arrive at this standard error after having used the last 100 years of data in the estimation, we choose the prior based on 100 years of observations as our benchmark for understanding asset prices over this past century. The prior based on 300 years of learning is added as a very conservative case and would assume that the agent began learning early in the 1600s, close to the opening year for the world's first stock exchange – the Amsterdam Stock Exchange (1611).

¹¹If both the parameters and state are unobserved, the learning problem becomes intractable as the parameter posteriors require computing every possible combination of observed states and their probabilities.

4.1.2 Calibration

The remaining model parameters are calibrated to match the U.S. consumption data over the century. The bad state is calibrated to the Great Depression, when real, per capita log consumption declined -4.6% per year from 1929 to 1933 with 2.94% volatility per year ($\mu_2 = -1.15\%$ and $\sigma_2 = 1.47\%$ at a quarterly frequency). We set $\pi_{11} = 0.9975$ and $\pi_{22} = 0.9375$, corresponding to one 4-year depression per century. In the normal growth state, $\mu_1 = 0.54\%$ and $\sigma_1 = 0.98\%$, generating a time-averaged, annual log consumption growth mean and standard deviation of 1.8% and 2.2% , respectively, matching the observed values (from NIPA) from 1929 to 2011.

We price a dividend claim to compare to market returns and assume

$$\Delta d_{t+1} = \bar{\mu} + \lambda(\Delta c_{t+1} - \bar{\mu}) + \sigma_d \eta_{t+1}, \quad (16)$$

where d_t is the log of dividends, $\bar{\mu}$ is the unconditional mean consumption growth rate, λ is a leverage parameter, and η_t is an i.i.d. standard normal shock (independent of ε_t). This ensures the long-run growth rate of dividends and consumption is the same, while the short-run response of dividends to consumption shocks is higher than that of consumption. Using the 1929 to 2011 sample, we estimate the leverage parameter to be 2.5 by regressing annual real dividend growth (constructed from CRSP data) on annual consumption growth.

In terms of exposure to parameter uncertainty, our dividend assumption is conservative relative to the more standard specification, $\Delta d_{t+1} = \lambda \Delta c_{t+1} + \sigma_d \eta_{t+1}$.¹² Alternatively, one could assume consumption and dividends are cointegrated, which introduces another state variable and is computationally costly. We set the idiosyncratic volatility σ_d such that annual dividend volatility is 11.5% , as in Bansal and Yaron (2004). The model is solved numerically using a backwards recursion method where the known parameters economies are used as boundary conditions (see the Online Appendix for more details).¹³

We consider a range of preference parameters, but a preference for an early resolution of uncertainty, $\gamma > 1/\psi$. In our main calibration, we choose values for ψ and β commonly used in the long-run risks literature and set γ to match the level of the risk-free rate. This generates $\psi = 2$, $\beta = 0.994$ (as in Bansal and Yaron (2004)), and $\gamma = 3.9$. We consider

¹²In particular, the uncertainty about the long-run growth rate is the same for consumption and dividend, and $\bar{\mu} = E(s_1)\mu_1 + (1 - E(s_1))\mu_2$.

¹³The case of learning about transition probabilities when the regimes are observed can be solved particularly fast. While the Appendix gives more details as to why, we note here that this model therefore is well suited as a workhorse model for macro-finance applications.

robustness to these values.

4.1.3 Results

Given these priors, we compute standard asset pricing moments for a ‘typical’ long sample (100 years) by averaging across 20,000 simulated sample moments. We also feed the regime transitions corresponding to the U.S. historical experience from 1911 to 2010 into the model to understand how conditional asset pricing moments (such as the conditional risk premium and return volatility) respond to the Great Depression and, later, to the Great Recession.

Unconditional Moments

Panel A of Table 1 reports the average risk premium, return volatility, and Sharpe ratio for a year sample, as well as the level and volatility of the real risk-free rate (all in logs). The ‘Data’ column contains corresponding observed equity market moments for the U.S. from 1929 to 2011 (from CRSP). The real risk-free rate moments are from BY (2004).

As mentioned earlier, the $T_0 = 100$ years prior is our benchmark, which implies that the level of remaining parameter uncertainty after 100 years of observed data is consistent with the uncertainty in BNSU’s parameter estimates. The average risk premium is 5.7%, somewhat higher than its historical counterpart (5.1%), achieved with a 16% return volatility, compared to 20% in the data. The Sharpe ratio of simple annual excess returns is 0.39, slightly higher than in the data (0.36). The known parameters case ($T_0 = \infty$) generates a risk premium of 1.1% and a Sharpe ratio of 0.14, both well below their observed counterparts, and a much higher risk-free rate. For the alternative cases with 200 and 300 years of prior learning, the risk premium only falls slightly to 4.9% and 4.3%, respectively. In fact, the model with the tightest prior ($T_0 = 300$ years) can also match the risk premium if β is increased to 0.9952 to match the risk-free rate level (right column of Panel A Table 1).

In sum, learning is slow and the quantitative effect of parameter learning on the risk premium and Sharpe ratio are very large for a range of reasonable priors. It is remarkable that the model can match the equity premium and Sharpe ratio, as well as the risk-free rate, with a level of risk aversion of only 3.9 and an unconditional consumption volatility of 2.2%, consistent with U.S. consumption data. By comparison, Bansal and Yaron (2004) match the risk premium with relative risk aversion and consumption volatility calibrated to 10 and 2.7%, respectively.

Panel B of Table 1 reports results for $\psi = 1.1$. The risk premium with 100 years of prior observations decreases from 5.7% to 4.1%. The Sharpe ratio falls only slightly from 0.39 to

Table 1 - 100 year sample moments
Learning about the probability and persistence of a Great Depression
Unconditional Moments

Table 1: This table gives average sample moments from 20,000 simulations of 400 quarters of data from the 2-state switching regime model of consumption growth, where the transition probabilities are unknown. The bad state is calibrated to correspond to the U.S. consumption data over the Great Depression, as explained in the main text. $E_T[x]$ denotes the average sample mean of x , $SR_T[x]$ denotes the average sample Sharpe ratio of x , and $\sigma_T[x]$ denotes the average sample standard deviation of x . R_m and R_f denote the gross market return and real risk-free rate. Lower case letters denote log of upper case variable. All statistics are annualized and, except for the Sharpe ratio, given in percent. The relative risk aversion is 3.9 in all cases. Panel A shows the case of a high IES ($\psi = 2$), while Panel B shows the case of $\psi = 1.1$. The time-preference parameter β is set to 0.994, except for the case in the rightmost column which has $\beta = 0.9952$. The 'data' column shows the historical excess market return moments for the U.S. from 1929 to 2011, as given in CRSP. The real risk-free rate moments are taken from a similar sample as reported in Bansal and Yaron (2004).

Panel A:		$(\beta = 0.994)$				$(\beta = 0.9952)$	
$\psi = 2, \gamma = 3.9$	Data	$T_0 = 100\text{yrs}$	$T_0 = 200\text{yrs}$	$T_0 = 300\text{yrs}$	$T_0 = \infty$	$T_0 = 300\text{yrs}^*$	
	$E_T[r_m - r_f]$	5.10	5.67	4.87	4.34	1.06	5.07
	$\sigma_T[r_m - r_f]$	20.21	16.23	16.56	16.67	15.07	16.61
	$SR_T[R_M - R_f]$	0.36	0.39	0.33	0.31	0.14	0.39
	$E_T[r_f]$	0.86	0.91	1.48	1.77	2.97	0.91
	$\sigma_T[r_f]$	0.97	0.63	0.65	0.67	0.61	0.65

Panel B:		$(\beta = 0.994)$				$(\beta = 0.9952)$	
$\psi = 1.1, \gamma = 3.9$	Data	$T_0 = 100\text{yrs}$	$T_0 = 200\text{yrs}$	$T_0 = 300\text{yrs}$	$T_0 = \infty$	$T_0 = 300\text{yrs}^*$	
	$E_T[r_m - r_f]$	5.10	4.07	3.68	3.13	0.64	3.92
	$\sigma_T[r_m - r_f]$	20.21	14.54	15.05	15.30	14.02	15.12
	$SR_T[R_M - R_f]$	0.36	0.33	0.31	0.28	0.11	0.33
	$E_T[r_f]$	0.86	3.44	3.58	3.65	3.95	3.07
	$\sigma_T[r_f]$	0.97	0.86	0.87	0.88	0.86	0.87

0.35, but the level of the risk-free rate increases significantly with a lower EIS. The volatility of the price-consumption and price-dividend ratios are also lower. Thus, a high level of the EIS is helpful in terms of matching the standard moments in models with parameter uncertainty. For the more precise priors ($T_0 = 200$ years and $T_0 = 300$ years), there is a modest decline in the risk premiums and increase in the risk free rate relative to the case

when $T_0 = 100$ years, similar to the dynamics from the high EIS case.

Conditional Moments using regime changes from 1911 to 2010

To mimic the U.S. historical experience, we feed into the model a path of states corresponding to the 1911 to 2010 sample. We designate the NBER dates for the Great Depression and the Great Recession as realizations of the bad state, with the remaining quarters are assumed to be draws from the good state. The Great Recession was not as severe as the Great Depression, though there were extensive fears in its early stages that it may become a depression-like event.¹⁴

Figure 1 reports the mean beliefs about the transition probabilities for this case. From 1910 to 1929, the probability of remaining in the good state (π_{11}) increased slowly, while the belief about the persistence of the Depression state (π_{22}) was not updated as there were no observations from which to learn. When the Depression starts in 1929, $E_t[\pi_{11}]$ is sharply revised downwards, and $E_t[\pi_{22}]$ increases during the Depression. At the end of the Depression, $E_t[\pi_{22}]$ is revised downwards as the length of this Depression now has been resolved. From 1934 and onwards the pattern repeats as $E_t[\pi_{11}]$ increases slowly until the onset of the Great Recession. Since the Great Recession was quite short, $E_t[\pi_{22}]$ is revised downwards upon exit from the recession. Figure 2 shows how these belief dynamics are reflected in conditional asset price moments. Abrupt shifts and then trending beliefs are hallmarks of learning about rare events.

The wealth-consumption ratio (upper left panel of Figure 2) decreases strongly when the bad state is realized, and continues to decrease until the bad state is exited. Given a preference for early resolution of uncertainty and the stochastic discount factor, this event is very risky, strongly increasing marginal utility. This is due both to a lower consumption growth rate in a Depression, as well as increased risk coming from updates about the persistence of the bad state. The latter is reflected in the difference between the dashed blue line, the case with parameter uncertainty, and the solid red line, the benchmark case of known parameters. The wealth-consumption ratio is lower in the case with parameter uncertainty, reflecting higher discount rates, and it also falls more conditional on the bad state.

The real risk-free rate level (upper right panel of Figure 2) in both the models with

¹⁴The sample also contain 400 shocks to quarterly consumption growth (ε 's), which are random, antithetic draws from a standard normal, normalized to have unit variance. We do not use actual consumption data here as the pre-WW2 sample only has annual consumption data. In any case, the realized consumption shocks have no impact on the evolution of either the wealth-consumption or the price-dividend ratios in this model.

Figure 1 - Mean beliefs about transition probabilities

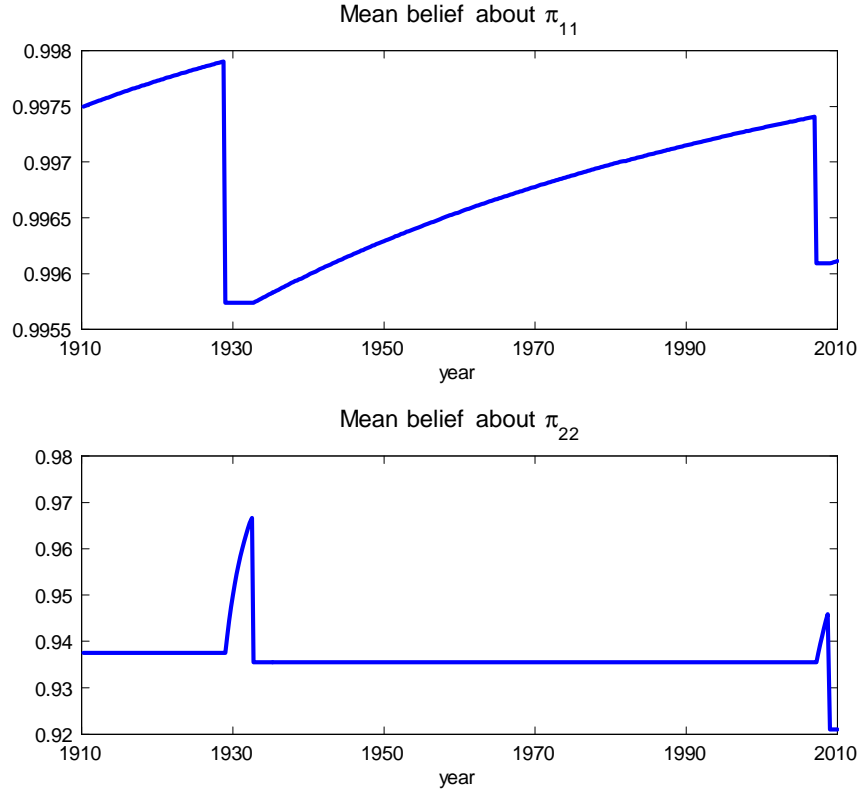


Figure 1: The top plot shows the mean beliefs about the probability of staying in the good state, π_{11} for the 2-state switching regime model where the transition probabilities are unknown and where the regimes are based on the U.S. macro data from 1911 to 2010. The lower plot shows the mean beliefs about the probability of staying in the bad state π_{22} .

and without parameter uncertainty decreases by about 5% in Depressions, due to the low expected growth rate. In comparison, the real risk-free rate, measured as the nominal 3-month T-bill rate minus the median inflation expectation from the Survey of Professional Forecasters, decreased by about 5.5% from right before the Financial Crisis to the end of the event. In the Great Depression the nominal rate did decrease by about 6%, but inflation expectations are not available for this period. Realized inflation was, however, at some points negative, indicating that the real rate decreased less than the nominal rate or possibly even increased. On the other hand, the nominal rate was hitting the zero lower bound at this point, so it is unclear how to relate this data to the frictionless economy presented here.

Figure 2 - Sample path of the risk premium for case of learning about the disaster probability

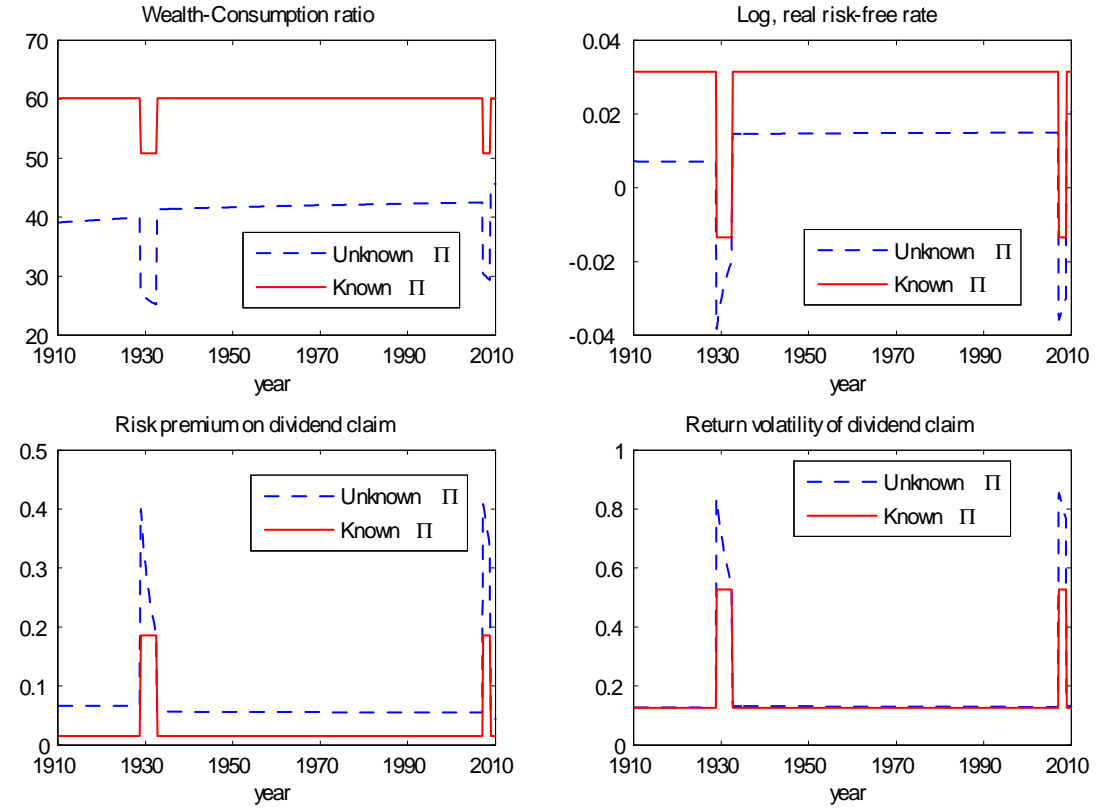


Figure 2: The figure shows conditional moments from the 2-state switching regime model where the transition probabilities are unknown and where the regimes are based on the U.S. macro data from 1911 to 2010. The dashed line corresponds to the case with unknown transition probabilities, while the solid line corresponds to the case with known transition probabilities. The preference parameters are $\beta = 0.994$, $\gamma = 3.9$, $\psi = 2$.

The two bottom panels of Figure 2 display the conditional, annualized risk premium and return volatility of the dividend claim. With parameter uncertainty, the risk premium and return volatility increase to more than 40% and 80%, respectively, at the onset of the bad state. Thereafter, both quantities decreased during the Depression, as the agent learns about the persistence of the bad state. This decreasing return volatility is consistent with recent experience over the financial crisis, both looking at realized volatility and the VIX index. When the parameters are known, the conditional risk premium and return volatility

stay constant through the depression at 18% and 52%, respectively. For comparison, Martin (2012) argues that the risk premium at the onset of the financial crises exceeded 55%. Both realized market volatility and the VIX index showed volatility in excess of 80% at the same time. Realized volatility in the fall of 1929 also exceeded 80%.¹⁵

Table 2
Learning about the probability and persistence of a Great Depression
Additional moments using regimes from 1911 – 2010

Table 2: This table gives the annualized sample moments from feeding the 2-state regime switching model with unknown transition probabilities 400 quarters of regime switches based on U.S. macro data from 1911 to 2010. $E_T[x]$ denotes the average sample mean of x , $SR_T[x]$ denotes the average sample Sharpe ratio of x , and $\sigma_T[x]$ denotes the average sample standard deviation of x . Lower case letters denote log of upper case counterparts. The subscript m refers to the dividend claim (the 'market' portfolio), while the subscript f refers to the real risk-free rate. All statistics are annualized and, except for the Sharpe ratio, are given in percent. The superscript 'A' refers to annual data. In particular, Δc^A is annual log consumption growth where, for the model simulated data, the annual consumption data is time-averaged based on the quarterly model calibration. The preference parameters used are $\beta = 0.994$, $\gamma = 3.9$, and $\psi = 2$. The 'data' column shows the historical excess market return moments for the U.S. from 1929 to 2011, as given in CRSP. The real risk-free rate moments are taken from a similar sample as reported in Bansal and Yaron (2004).

$\psi = 2, \gamma = 3.9$	Model		Data
	Unknown π 's	Known π 's	
$E_T[r_m - r_f]$	5.51	0.98	5.10
$\sigma_T[r_m - r_f]$	20.96	18.70	20.21
$SR_T[R_M - R_f]$	0.32	0.14	0.36
$E_T[r_f]$	1.10	2.90	0.86
$\sigma_T[r_f]$	1.02	1.02	0.97
$Corr(\Delta c^A, \Delta d^A)$	0.59	0.59	0.38
$Corr(\Delta c^A, r_m^A)$	0.30	0.36	0.65
$\frac{PD_{Depression}}{PD_{Normal}}$	0.45	0.50	0.21/0.50
$E \left[R_m \Big _{\text{depression}}^{\text{Beginning of}} \right]$	40	18	> 55
$\sigma \left[R_m \Big _{\text{depression}}^{\text{Beginning of}} \right]$	85	52	> 80

Table 2 reports sample moments feeding in realized states. The return volatility with parameter uncertainty is over this period 20.96%, relative to the observed volatility of 20.21%.

¹⁵Here realized volatility is calculated as the annualized value of the square root of realized daily variance over a month.

It is higher than the average reported across simulated economies in Table 1 due to the presence of two crises in the sample. Table 2 also reports the correlation between annual log consumption growth and market returns. With parameter uncertainty, this correlation is 0.30, while with known transition probabilities this correlation is 0.36. In the data, from 1929 to 2011 as available from the BEA, this correlation is 0.65.¹⁶

The drop in the price-dividend ratio of the dividend claim at the onset of the bad state is 55% in the model with parameter uncertainty and 50% in the model with known transition probabilities. In the Great Depression the drop in the price-dividend ratio from the beginning of the Depression in 1929 to its lowest point in 1932 was 79%. For the recent Financial Crisis, the corresponding drop was 50%.

In sum, the model with a depression state calibrated to consumption dynamics under the U.S. Great Depression performs well along a number of dimensions when compared to relevant moments from the data. In particular, it predicts a high unconditional risk premium and a low risk-free rate, along with a very high risk premium and return volatility in the crisis period, and only requires a relative risk aversion coefficient of 3.9. Relative to the benchmark case with known transition probabilities, parameter uncertainty increases the Sharpe ratio and the risk premium by factors of about 2.5 and 5, respectively.

Alternative Calibrations of Consumption Dynamics in the Depression State.

Table 3 shows results for alternative calibrations of the consumption dynamics in the Depression state. The original calibration used the NBER dating of the Great Depression (1929 – 1933) and the annual, real, per capita consumption data from BEA as the basis for the calibration of the persistence of the bad state, as well as the mean and volatility of consumption growth in this state. The 2-state regime switching model as calibrated here does not account for any reversal phase with strong growth after a Depression has ended. The high consumption growth of 5.4% in 1934 is consistent with such a reversal. The total drop in consumption in the years 1929 – 1933 was 18%, while the decrease from 1929 to 1934 was 13%, and from 1929 to 1935 it was "only" 9%.

To assess the robustness of the results with respect to the calibration of the consumption dynamics in this bad state, we also consider calibrations based on a 5- and 6-year Depression,

¹⁶This correlation is obtained using the beginning-of-period timing for consumption, given the time-averaging of the consumption data (see Working (1960) and Campbell (1999)). That is, reported consumption growth for year t is correlated with returns for year $t - 1$. The correlation between consumption growth reported for year t and returns in year t , corresponding to using the end-of-period timing when calculating consumption growth, is 0.18.

with mean growth rate and volatility based on the years 1929 – 1934 and 1929 – 1935, respectively. For ease of comparison, we keep the parameters in the good state the same. The true quarterly persistence of the bad state is then 0.95 or 0.9583, with corresponding quarterly means of -0.65% and -0.39% . The corresponding quarterly volatilities (where volatility is as before adjusted for the time-averaging of the consumption data) are 3.01% and 3.11% , respectively.

Table 3
Learning about the probability and persistence of a Great Depression
Alternative calibrations of Depression consumption dynamics

Table 3: This table gives average annualized sample moments from 20,000 simulations of 400 quarters of data from the 2-state switching regime model of consumption growth, where the transition probabilities are unknown. The bad state in the model is calibrated to the U.S. consumption data for alternative choices for the duration of the Great Depression, as explained in the main text. $E_T[x]$ denotes the average sample mean of x , $SR_T[x]$ denotes the average sample Sharpe ratio of x , and $\sigma_T[x]$ denotes the average sample standard deviation of x . Lower case letters denote log of upper case counterparts. The subscript m refers to the dividend claim (the 'market' portfolio), while the subscript f refers to the real risk-free rate. All statistics are annualized and, except for the Sharpe ratio, are given in percent. The preference parameters used are given in the table below. The preference parameters used are given for each column.

	4yr depression	5yr depression	6yr depression	6yr dep. $\beta = 0.996$
$\psi = 2, \gamma = 3.9$	$\mu_D = -1.15\%$ $\sigma_D = 1.1\%$	$\mu_D = -0.65\%$ $\sigma_D = 3.01\%$	$\mu_D = -0.39\%$ $\sigma_D = 3.11\%$	$\mu_D = -0.39\%$ $\sigma_D = 3.11\%$
$E_T[r_m - r_f]$	5.67	4.05	3.03	4.12 (0.85)
$\sigma_T[r_m - r_f]$	16.23	16.13	15.84	15.73 (14.77)
$SR_T[R_M - R_f]$	0.43	0.30	0.25	0.32 (0.13)
$E_T[r_f]$	0.91	1.72	2.19	0.80 (2.22)
$\sigma_T[r_f]$	0.63	0.61	0.56	0.54 (0.54)

Table 3 shows that the risk premium and Sharpe ratio are both decreasing in the alternative calibrations, as the mean state is less severe. For the 4-year benchmark calibration, the risk premium is 5.7% , whereas it is 4.1% and 3.0% in the 5- and 6-year calibrations, respectively. The risk-free rate increases in these calibrations for the same reason. The final column of Table 3 shows a calibration of the model with a 6-year Depression state where the time-preference parameter (β) has been increased from 0.994 to 0.996 such that the level of the risk-free rate is on par with the data. In this case, the risk premium and Sharpe

ratio of the least risky, 6-year calibration are increased to 4.1% and 0.32, respectively. The Sharpe ratio is on par with that observed historically. The corresponding case with known parameters is shown in parentheses. Here the risk premium and Sharpe ratio are 0.9% and 0.13, respectively. We conclude from this that the overall implications of the parameter uncertainty model are robust to alternative, reasonable calibrations of the Depression state, in particular the strong increase in relevant asset pricing quantities such as the risk premium and the Sharpe ratio relative to the benchmark known parameters case.

4.2 Uncertain mean and variance of the Depression state

In an experiment detailed in the online Appendix, we consider the case where the transition probabilities are known, but where instead the mean and variance parameters of the Depression state (μ_2 and σ_2) are unknown. To summarize the results, learning about the mean growth rate in the Depression state increases the risk premium by a factor of about 2 relative to the known mean benchmark case. Learning about the variance in the bad state has only negligible impact on asset pricing moments.¹⁷

We next turn to a case of model uncertainty that generates interesting dynamics in the price of risk and risk premium at the business cycle frequency. Since business cycles are observed more frequently, learning can only be slow if the two models are very hard to distinguish using available macro samples, which is exactly the case in the following analysis.

5 Learning about competing model specifications

In this section, we investigate an economy where agents face uncertainty over the model specification. As discussed earlier, model uncertainty can be viewed as uncertainty about an ‘indicator’ parameter, M , that equals one for the true model and zero for an alternative model. Here, we consider an agent who is uncertainty between the Bansal and Yaron (BY, 2004) model ($M = 0$) with homoskedastic shocks and a normally distributed i.i.d model ($M = 1$).¹⁸ This is a natural framework to evaluate model uncertainty for two reasons: (1)

¹⁷We truncate the support of the mean and the variance parameters to ensure existence of equilibrium. See the online Appendix for details.

¹⁸A similar problem was considered in Hansen and Sargent (2010), though their alternative model is not the iid case, but a case where there is still positive, but less autocorrelation in consumption growth than in the long-run risk model. Also, our focus is on the quantitative implications of rational learning for long-horizon claims when the agent has Epstein-Zin preferences. Ju and Miao (2012) and Collard, Mukerji, Sheppard, and Tallon (2011) consider different cases of model uncertainty under smooth ambiguity aversion.

the predictable component in the BY model is highly persistent and difficult to detect over even modestly sized samples; and (2) the asset pricing implications of the models are quite different.

Consumption growth evolves via

$$\Delta c_{t+1} = M \{ \mu + \sigma_{iid} \varepsilon_{t+1}^{iid} \} + (1 - M) \{ \mu + x_t + \sigma_{BY} \varepsilon_{t+1}^{BY} \},$$

where $x_{t+1} = \rho x_t + \varphi \sigma_x \eta_{t+1}$ and $\varepsilon_{t+1}^{iid}, \varepsilon_{t+1}^{BY}, \eta_t \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$. The agent knows neither M nor x_t and learns about them using Bayes rule. The steady state Kalman filter solves the the filtering problem for x_t . Under the BY model, the subjective consumption dynamics are given by

$$\begin{aligned} \Delta c_{t+1} &= \mu + \hat{x}_t + \hat{\sigma}_{BY} \hat{\varepsilon}_{t+1}, \\ \hat{x}_{t+1} &= \rho \hat{x}_t + \hat{\varphi} \hat{\sigma}_{BY} \hat{\varepsilon}_{t+1}, \end{aligned} \tag{17}$$

where $\hat{x}_t = E[x_t | y^t, \hat{x}_0]$ and $\hat{\varepsilon}_{t+1}$, $\hat{\sigma}_{BY}$, and $\hat{\varphi}$ have simple closed form expressions.

The agent initially assumes $p_0 = \mathbb{P}(M = 0)$, and updates beliefs via Bayes rule:

$$p_{t+1} = \mathbb{P}(M = 0 | y^{t+1}) \propto p(y_{t+1} | y^t, M = 0) p_t. \tag{18}$$

The probabilities are martingales. Letting $p(y_{t+1} | y^t, M = 0) = p_{BY}(y_{t+1} | y^t)$ and $p(y_{t+1} | y^t, M = 1) = p_{iid}(y_{t+1})$, the belief recursion is

$$p_{t+1} = \frac{p_{BY}(y_{t+1} | y^t) p_t}{p_{BY}(y_{t+1} | y^t) p_t + p_{iid}(y_{t+1}) (1 - p_t)}, \tag{19}$$

where $p_{BY}(y_{t+1} | y^t) \sim \mathcal{N}(\mu + \hat{x}_t, \sigma_{BY}^2 + \sigma_x^2)$ and $p_{iid}(y_{t+1}) \sim N(\mu, \sigma_{iid}^2)$. The value function normalized by consumption is a function of p_t and \hat{x}_t and is computed numerically using value function iteration, with boundary values given by the cases $p_t = 0$ and $p_t = 1$. See the Online Appendix for more details on the numerical solution methodology.

5.1 Results and calibration

We calibrate the consumption dynamics based on Bansal and Yaron (2004) at a quarterly frequency, since we will later feed actual consumption realizations. In particular, $\mu = 0.45\%$, $\sigma_{iid} = 1.65\%$, $\rho = 0.979^3$, $\sigma_{BY} = \sigma_{iid}$, and $\varphi = 0.089$. This implies that $\hat{\sigma}_{BY} = 1.706\%$ and

$\hat{\varphi} = 0.044$. The values for ρ and $\hat{\varphi}$ are the same as the values for ρ and φ assumed in Bansal and Yaron (2004). In other words, the amount of long-run risk as perceived by the agent learning about x_t from consumption growth is the same as that for the agent in Bansal and Yaron (2004) who observes x_t . Given the discussion in the previous section, we assume an EIS greater than one. In our main calibration, we set $\psi = 2$ and calibrate $\beta = 0.9963$ and $\gamma = 9$ such that we match the level of the equity market risk premium and risk-free rate.

5.1.1 The Price Of Risk

Figure 3 displays the annualized conditional price of risk $(\sigma(M_{t+1}|p_t, \hat{x}_t) / E(M_{t+1}|p_t, \hat{x}_t))$ in this economy plotted against the state variables in the economy, p_t and \hat{x}_t .

Figure 3 - Price of risk for case of model uncertainty

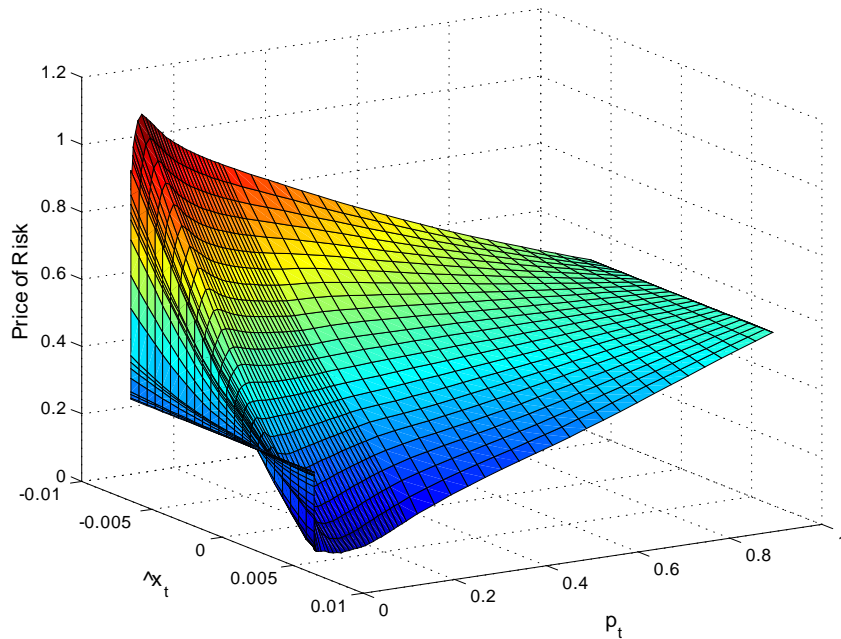


Figure 3: The figure shows the annualized, conditional price of risk in the economy where the agent is unsure whether true consumption growth is iid or follows the dynamics in Case 1 in Bansal and Yaron (2004) – the homoskedastic case. The state variables are the current belief about the model p_t , where $p_t = 1$ means the agent is certain the BY model is the true model, and \hat{x}_t – the current belief about expected consumption growth, conditional on the BY model being the correct model.

When $p_t = 1$, the agent is certain the BY economy is the true specification, in which

case, the annualized price of risk is constant and equal to 0.51.¹⁹ In the i.i.d. case, $p_t = 0$, the price of risk is 0.30. For $p_t \in (0, 1)$ the price of risk is different than a simple weighted average of the two boundary case economies—a crucial feature of priced model uncertainty. In particular, at $\hat{x}_t = 0$, the price of risk remains close to, and in fact slightly higher than, 0.51 even for values of p_t close to zero. Thus, even if the BY model is very unlikely, the agent still perceives the economy to be much riskier than the i.i.d. case for two reasons. First, shocks to model beliefs are permanent and therefore have a large impact on the continuation utility, provided the two models have different continuation utilities (which is clearly the case here). Second, as p_t declines, the event that the BY model is the true model has an impact similar to a ‘disaster’ scenario. This occurs because the distribution of continuation utility becomes increasingly negatively skewed as p_t decreases, and such negative skewness is disliked by risk averse agents with preferences for early resolution of uncertainty.

Unconditionally, the BY model generates lower utility than the iid model, and this difference increases when $\hat{x}_t < 0$, as future expected consumption growth rates are lower than the i.i.d. model. Thus, model uncertainty is ‘worse’ in these states of the world, generating a higher conditional price of risk. Further, the two risks in the economy, the shock to consumption and the update in the model probability, reinforce each other in these states. A low consumption growth is bad also in the iid model. However, when $\hat{x}_t < 0$, a low consumption growth realization also increases the likelihood of the BY model. In fact, Figure 3 shows that when $p_t = 0.05$ and \hat{x}_t is three standard deviations below its mean, the price of risk is about 1.10, almost twice that of the riskiest alternative model of the world. On the other hand, when $\hat{x}_t > 0$ the updates in beliefs hedge the consumption shock in the following sense. A low consumption realization is bad (which is also the case in the iid model), but since $\hat{x}_t > 0$, the low consumption growth increases the likelihood that the iid model is the true model, which is a good in a utility sense. Therefore, the total price of risk in these states can drop below that of either of the limiting economies. In fact, when $p_t = 0.05$ and when x_t is three standard deviations above its mean, the annualized price of risk is only 0.06.

Model uncertainty generates counter-cyclical risk prices, as \hat{x}_t tends to be low/high in recessions/expansions. Figure 3 also show the tenuous nature of the rational expectations assumption (see also Hansen (2007)). It is not until p_t gets lower than 0.01% that the asset pricing implications of model uncertainty are negligible, which, would take agents on

¹⁹In the exactly solved Bansal and Yaron model, the price of risk actually varies a tiny amount with \hat{x}_t , but to the third decimal it is constant, as in the approximate solution for the homoskedastic case given in Bansal and Yaron (2004).

average about 800 years to learn starting from the initial prior $p_0 = 0.5$. In sum, one cannot outright dismiss a model as unimportant even though it is rejected by the data at conventional significance levels—a point commonly made in the robustness literature. This conclusion does depend on the agent having a preference for the timing of the resolution of uncertainty, but as we show below, it does not rely on a high EIS. A model with an EIS close to one, for instance, delivers similar dynamics. Power utility preferences, however, are unaffected by model uncertainty and the price of risk is constant at $\gamma\sigma$.²⁰

5.1.2 Asset price moments

As in the previous case, the endogenous sensitivity to the learning dynamics generates strong and long-lasting asset pricing implications even though agents are rational Bayesian learners. Figure 4 shows the conditional risk premium, Sharpe ratio, and return volatility of the dividend claim (see Equation(16)), as well as the model probability (p_t) plotted against time passed since the initial prior. The figure shows these moments averaged across 20,000 simulated economies where the initial beliefs are set to $p_0 = 0.5$ and $\hat{x}_0 = 0$ and where the true model is assumed to be the i.i.d. model (i.e., $p_{t=\infty} = 0$). The corresponding moments from each of the boundary economies are plotted as well.

After 100 years of learning the average model probability decreases from 0.5 to 0.17. Though the agent updates in the direction of the true model, learning is slow as it is difficult to discriminate between the two models. Despite downweighting the likelihood of the BY model, the asset pricing moments are close to the corresponding values in the BY economy. In fact, the risk premium and Sharpe ratio are only very slightly lower than in the BY model, and barely decrease with time. In contrast to the initial case discussed earlier (learning about the mean growth rate with EIS = 1), the asset pricing moments are highly nonlinearly related to the variance of beliefs, which is $p_t(1 - p_t)$ and which decline substantially over time, starting at 0.25 when $p_t = 0.5$ and ending at 0.146 when $p_t = 0.17$. The intuition is similar to that for the price of risk, as given in Figure 3. The endogenous sensitivity of continuation utility to belief shocks increases as p_t declines, due to the increased negative skewness in model risk.

Table 4 shows average 100-year sample moments for four different calibrations corresponding to combinations of the EIS and initial model probability: $\psi \in \{1.1, 2\}$ and

²⁰Basically, it is $\gamma - 1/\psi$ that matters for the pricing of shocks to the continuation utility. With $\gamma = 9$ this magnitude is 8.5 if $\psi = 2$, but only falls to 7 if $\psi = 0.5$. Thus, the main implications shown here are robust to the level of the IES, as long as ψ is not very close to $1/\gamma$.

Figure 4 - Average conditional moments for case of model uncertainty

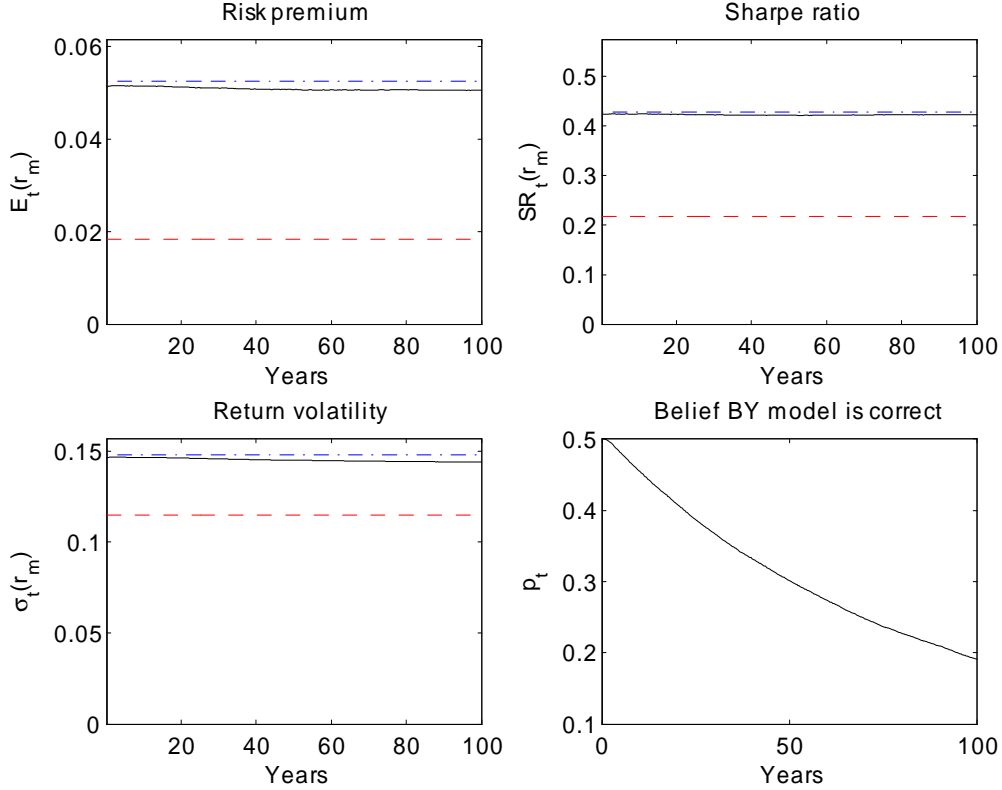


Figure 4: The figure shows a the annualized conditional risk premium, Sharpe ratio, and return volatility of the aggregate dividend claim averaged across 20,000 simulated economies over a 100 year period. The solid line corresponds to the case of model uncertainty, where the agent is unsure whether true consumption growth is iid or follows the dynamics in Case 1 in Bansal and Yaron (2004), the dashed line corresponds to the iid consumption growth model, and the dash-dotted line corresponds to the case of the BY model. The bottom right plot shows the model probability (p_t) for each quarter over the 100 year samples, averaged across the 20,000 simulations.

$p_0 \in \{0.1, 0.5\}$. The ‘Data’ column contains equity market moments and real risk-free rate moments as described above. Table 4 also reports the correlation between annual log consumption growth and market returns, which in the data, from 1929 to 2011 is 0.65.²¹

²¹This correlation is computed using the beginning-of-period timing for consumption, given the time-averaging of the consumption data (see Working (1960) and Campbell (1999)). That is, reported consumption growth for year t is correlated with returns for year $t - 1$. The correlation between consumption growth reported for year t and returns in year t , corresponding to using the end-of-period timing when calculating consumption growth, is 0.18.

**Table 4 – Model Uncertainty
100 year sample moments**

Table 4: This table gives average sample moments from 20,000 simulations of 400 quarters of data when the true model is that consumption growth is i.i.d., but the agent is unsure whether the true model is Bansal-Yaron or the i.i.d. model. The prior probability p_0 and the intertemporal elasticity of substitution ψ are given for each column. In all cases, the time-preference β is 0.9963 and the relative risk aversion coefficient γ is 9, set to match the risk-free rate and the risk premium in 'Model I'. $E_T[x]$ denotes the average sample mean of x , $SR_T[x]$ denotes the average sample Sharpe ratio of x , and $\sigma_T[x]$ denotes the average sample standard deviation of x . Lower case letters denote log of upper case counterparts. The subscript m refers to the dividend claim (the 'market' portfolio), while the subscript f refers to the real risk-free rate. The parameters governing the dividend dynamics are $\lambda = 2.5$ and σ_d , as discussed in the main text. All statistics are annualized and, except for the Sharpe ratio, are given in percent. The 'data' column shows the historical excess market return moments for the U.S. from 1929 to 2011, as given in CRSP. The real risk-free rate moments are taken from a similar sample as reported in Bansal and Yaron (2004).

$\beta = 0.9963$ $\gamma = 9$	Data	Model I $\psi=2, p_0=0.5$	Model II $\psi=1.1, p_0=0.5$	Model III $\psi=2, p_0=0.1$	BY Model $\psi=2, p_0=1$	iid Model $\psi=2, p_0=0$
$E_T[r_m - r_f]$	5.10	5.09	4.65	4.90	5.25	1.84
$\sigma_T[r_m - r_f]$	20.21	14.44	13.94	14.25	14.82	11.49
$SR_T[R_M - R_f]$	0.36	0.42	0.40	0.41	0.42	0.21
$E_T[r_f]$	0.86	0.86	1.56	0.84	0.84	1.66
$\sigma_T[r_f]$	0.97	0.44	0.44	0.50	0.32	0.00
$Corr_T(\Delta c^A, r_m^A)$	0.65	0.54	0.54	0.54	0.61	0.54

The 'Model I' column summarizes the case with initial model uncertainty, p_0 , set to 0.5 and $\psi = 2$. The model undershoots volatility somewhat (14.44% versus 20.21% in the data), but otherwise matches these unconditional moments quite well. In fact, the moments are very similar to the moments in the BY model, despite the true model here being the i.i.d. model. In contrast, the i.i.d. model does a poor job in asset pricing. The 'Model II' column keeps the prior the same, but decreases ψ to 1.1. The risk premium and return volatility both decrease somewhat (to 4.65% and 13.94%, respectively), but overall the model does quite well. In fact, as the risk-free rate here is a little too high, the model admits a calibration with a higher β , which as seen in Equation (10) increases the risk premium and Sharpe ratio when there is parameter uncertainty. Thus, the implications of model uncertainty as considered here are robust to the level of the EIS, as long as it is above 1.²²

²²With an IES < 1, the return volatility decreases faster, making excess volatility too small. Also, as

The column 'Model III' in Table 4 shows the asset pricing moments for the case when the prior belief that the Bansal-Yaron model is the true model is low, $p_0 = 0.1$. The asset pricing moments look largely the same, although volatility is increased slightly. This is in line with the discussion regarding the price of risk (Figure 1), which shows that as the probability declines, the sensitivity of the continuation utility to updates in beliefs is higher, even though the variance of beliefs is decreased substantially in this case: 0.09 versus 0.25 in the case when $p_0 = 0.5$.

5.1.3 Feeding the model actual consumption data

Finally, we consider the impact of model uncertainty on the post-WW2 sample using the corresponding time series of U.S. quarterly, real, per capita consumption growth. To be consistent with the model, we first remove autocorrelation of 0.25 induced by time-averaging of the data (see Working (1960)) and then normalize the sample mean and variance of this modified consumption growth series to have mean and variance as assumed in the model calibration.²³

The solid line in the top graph in Figure 5 shows the posterior probability of the Bansal and Yaron model, $\mathbb{P}(M = 0|y_{t+1})$, from 1947Q3 to 2010Q4, starting with an initial probability of 0.5. The model probabilities vary quite a bit, from about 0.25 to 0.9. Periods with long runs of either high consumption growth (late 1960's) or low consumption growth (the Great Recession) increase the probability of the Bansal and Yaron model relative to the iid model. At the end of the sample, the likelihood of the Bansal and Yaron model is 0.9 at its maximum. The dashed line shows the case where the initial model probability is set to 0.1. The dynamics are largely the same, but the probability is of course shifted down relative to the earlier case. In this case, the probability of the Bansal-Yaron model at the end of the sample is about 0.45.

The middle plot of Figure 5 shows the conditional price of risk, which varies substantially and is overall counter-cyclical. However, there are cases where this is not the case. For instance, in the expansion of the late 1960's the price of risk *increases* as the Bansal and Yaron model becomes more likely. Through the recession of 2001, on the other hand, the price of risk decreases as the Bansal and Yaron model becomes more likely. This is due

emphasized in Drechsler and Yaron (2010), and $IES > 1$ is needed to explain the variance risk premium.

²³We first construct $y_t = \Delta c_t - 0.25 \times \Delta c_{t-1}$, using actual real per capita quarterly consumption growth data from Q2 in 1947 to Q4 in 2010. The modified consumption growth series is then constructed as $\tilde{\Delta}c_t \equiv \mu + \sigma_{iid} \times \frac{y_t - E_T[y_t]}{\sigma_T[y_t]}$, where $E_T[\cdot]$ and $\sigma_T[\cdot]$ denote the sample mean and variance, respectively.

to the then current high value of \hat{x}_t , arising from high growth in the 1990's. As can be seen from Figure 3, the high current \hat{x}_t makes the prospect of facing the Bansal and Yaron consumption dynamics a conditionally less risky prospect as the agent then can enjoy higher expected consumption growth than in the iid case. For the case with $p_0 = 0.5$, the price of risk in this sample reaches its maximum of roughly 0.7 during the Great Recession, and its lowest point close to 0.4 in the mid 1960s. Corresponding to the dynamics apparent from Figure 3, the price of risk in fact becomes more volatile with $p_0 = 0.1$. Now, the price of risk is about 0.9 during the financial crisis about 0.2 in the mid 60s. The bottom graph of Figure 5 shows that the conditional risk premium on the levered consumption claim largely inherits the dynamics of the price of risk. The conditional, annualized risk premium varies substantially throughout the sample, from about 4% to 8% when $p_0 = 0.5$ and 1% to 9% when $p_0 = 0.1$.

Overall, model learning leads to interesting risk price and equity premium dynamics, even though both candidate models are homoskedastic and exhibit constant risk premiums and Sharpe ratios. As in the earlier example, model learning has long-lasting, quantitatively significant implications for standard asset pricing moments. This is due to the martingale shocks, as well as the large difference in the utility continuation values implied by the two models, and to the endogenously increased sensitivity of continuation utility to updates in beliefs as p_t decreases (as evident from Figure 3).

When feeding the agent the realized consumption growth from the post-WW2 sample, the learning problem itself is quite hard and model beliefs for most of the sample are close to 0.5. As before, the learning mechanism that gives rise to a potential resolution of standard asset pricing puzzles does not rely on the predictability of any moment of consumption growth in the data.

6 Conclusion

This paper finds that uncertainty about fixed parameters governing the exogenous aggregate endowment process of the economy can have long-lasting, quantitatively significant asset pricing implications. This conclusion relies on rational learning, which implies that posterior probabilities regarding fixed quantities are martingales, and that agents have a preference for early resolution of uncertainty. For such agents the updating of beliefs, with its associated permanent shocks to the conditional distribution of future consumption growth, constitute an additional risk that can serve as a tremendous amplification mechanism for the impact

Figure 5
Model uncertainty: post-WW2 sample conditional moments

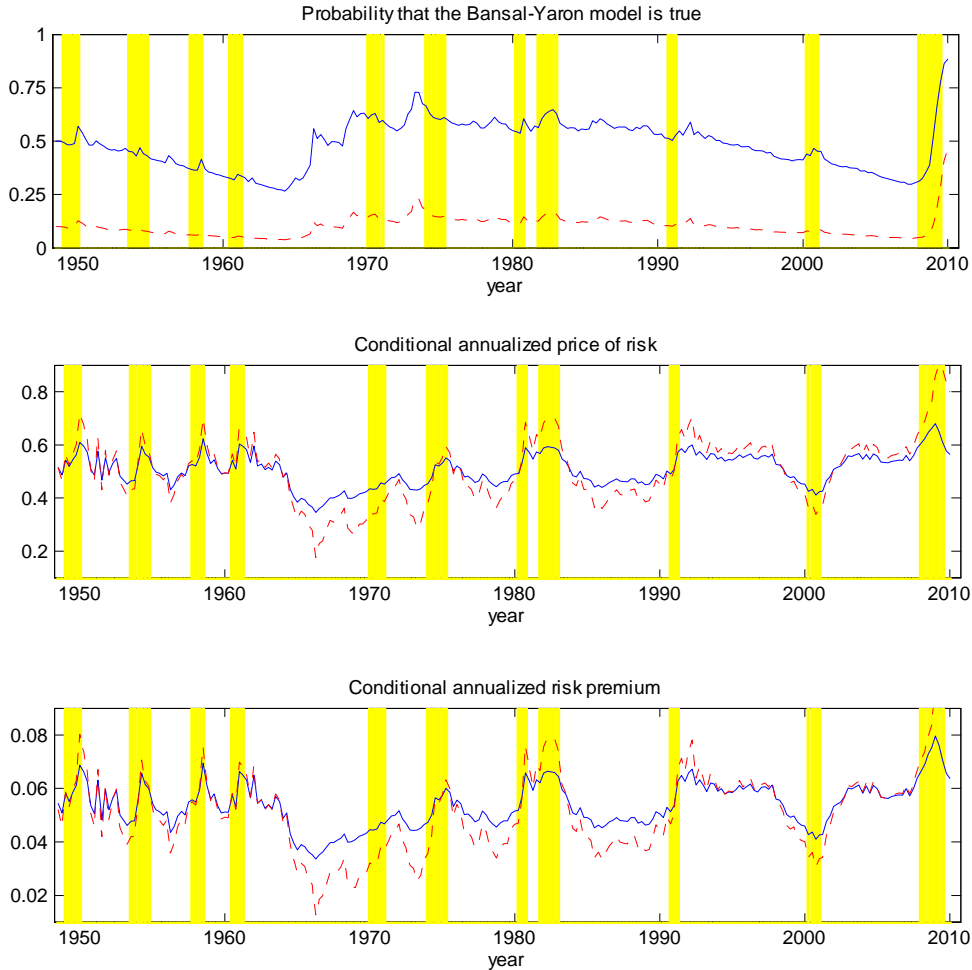


Figure 5: The figure shows sample paths of the model probability (p_t), the annualized conditional price of risk, and the annualized conditional risk premium for the case of model uncertainty, where the agent is unsure whether true consumption growth is iid or follows the dynamics in Case 1 in Bansal and Yaron (2004), where the shocks are taken from the post-WW2 real per capita consumption growth as explained in the main text. The solid line corresponds to the case where the initial subjective probability that the Bansal-Yaron model being the true model is set to 0.5. The dashed line corresponds to the case where the initial probability of the Bansal and Yaron model is set to 0.1. The yellow bars correspond to NBER recessions.

of macro shocks on marginal utility.

In several cases of parameter uncertainty we document an endogenous interaction between

the increased precision of beliefs and an increased sensitivity of marginal utility to shocks to beliefs. Thus, in a general equilibrium setting the asset pricing impact of parameter uncertainty does not in general decline one for one with the variance of beliefs. For instance, in a particular case of model uncertainty, we document that the price of risk in the economy in fact can increase as the variance of parameter uncertainty declines due to a strong concurrent increase in the skewness of beliefs.

We evaluate the quantitative impact of many different forms of parameter uncertainty and show that learning about the persistence of bad states has the most dramatic asset pricing implications. In particular, a model with a bad state calibrated to aggregate consumption data from the U.S. during the Great Depression, where the agents learn about the persistence of this state, yields a high equity premium (6%), a low risk-free rate (1%), with low relative risk aversion (slightly less than 4), and low consumption volatility as in the data. In contrast, the otherwise identical economy with known parameters yields an equity premium of 1% with a risk-free rate of 3%. Further, this model can match the extremely high equity return volatility observed at the onset of the Depression. We also emphasize a particular case of model uncertainty, where agents learn whether consumption growth has a small persistent component or not. This case, when fed the actual consumption growth realizations of U.S. post-WW2 data, produces strongly counter-cyclical price of risk and equity risk premium at the business cycle frequency despite the fact that both alternative models have homoskedastic fundamentals and that preferences are isoelastic.

In sum, with Epstein-Zin preferences, which allow a separation of the intertemporal elasticity of substitution and the relative risk aversion, parameter learning provides a significant source of additional macro risk that can help explain both a high level and counter-cyclical time-variation in the volatility of the marginal utility of the representative agent, as needed to explain asset market data.

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Online Appendix – Not Intended For Publication

7 Derivations of analytical solutions for the learning about the mean case

There are several formal treatments of stochastic differential utility and its implications for asset pricing (see, e.g., Duffie and Epstein (1992a,b), Duffie and Skiadas (1994), Schroder and Skiadas (1999, 2003), and Skiadas (2003)). In this Appendix we offer a simple derivation of the pricing kernel that obtains in an exchange economy where the representative agent has a KPEZ recursive utility with unit EIS and where he is learning about the constant growth rate of aggregate consumption. We first recall a few well-known results about stochastic differential utility.

7.1 Representation of Preferences and Pricing Kernel

We assume the existence of a standard filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ on which there exists a vector $z(t)$ of d independent Brownian motions.

Aggregate consumption in the economy is assumed to follow a continuous process, with stochastic growth rate and volatility:

$$d \log C_t = \mu_C(X_t) dt + \sigma_C(X_t) dz(t), \quad (20)$$

$$dX_t = \mu_X(X_t) dt + \sigma_X(X_t) dz(t), \quad (21)$$

where X_t is a n -dimensional Markov process (we assume sufficient regularity on the coefficient of the stochastic differential equation (SDE) for it to be well-defined, e.g., Duffie (2001) Appendix B). In particular μ_X is an $(n, 1)$ vector, σ_X is an (n, d) matrix.

Following Epstein and Zin (1989), we assume that the representative agent's preferences over a consumption process $\{C_t\}$ are represented by a utility index $U(t)$ that satisfies the following recursive equation:

$$U(t) = \left\{ (1 - e^{-\beta dt}) C_t^{1-\rho} + e^{-\beta dt} E_t (U(t+dt)^{1-\gamma})^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}}. \quad (22)$$

With $dt = 1$, this is the discrete time formulation of KPEZ, in which $\Psi \equiv 1/\rho$ is the EIS and γ is the risk-aversion coefficient.

To simplify the derivation let us define the function

$$u_\alpha(x) = \begin{cases} \frac{x^{1-\alpha}}{(1-\alpha)} & 0 < \alpha \neq 1 \\ \log(x) & \alpha = 1. \end{cases}$$

Further, let us define

$$g(x) = u_\rho(u_\gamma^{-1}(x)) \equiv \begin{cases} \frac{((1-\gamma)x)^{1/\theta}}{(1-\rho)} & \gamma, \rho \neq 1 \\ u_\rho(e^x) & \gamma = 1, \rho \neq 1 \\ \frac{\log((1-\gamma)x)}{(1-\gamma)} & \rho = 1, \gamma \neq 1, \end{cases}$$

where

$$\theta = \frac{1-\gamma}{1-\rho}.$$

Then, defining the ‘normalized’ utility index J as the increasing transformation of the initial utility index $J(t) = u_\gamma(U(t))$, Equation 22 becomes:

$$g(J(t)) = (1 - e^{-\beta dt})u_\rho(C_t) + e^{-\beta dt} g(E_t [J(t + dt)]). \quad (23)$$

Using the identity $J(t + dt) = J(t) + dJ(t)$ and performing a simple Taylor expansion we obtain:

$$0 = \beta u_\rho(C_t)dt - \beta g(J(t)) + g'(J(t)) E_t [dJ(t)]. \quad (24)$$

Slightly rearranging the above equation, we obtain a backward recursive stochastic differential equation which could be the basis for a formal definition of stochastic differential utility (see Duffie Epstein (1992), Skiadas (2003)):

$$E_t [dJ(t)] = -\frac{\beta u_\rho(C_t) - \beta g(J(t))}{g'(J(t))} dt. \quad (25)$$

Indeed, let us define the so-called ‘normalized’ aggregator function:

$$f(C, J) = \frac{\beta u_\rho(C) - \beta g(J)}{g'(J)} \equiv \begin{cases} \frac{\beta u_\rho(C)}{((1-\gamma)J)^{1/\theta-1}} - \beta \theta J & \gamma, \rho \neq 1 \\ (1-\gamma)\beta J \log(C) - \beta J \log((1-\gamma)J) & \gamma \neq 1, \rho = 1 \\ \frac{\beta u_\rho(C)}{e^{(1-\rho)J}} - \frac{\beta}{1-\rho} & \gamma = 1, \rho \neq 1. \end{cases} \quad (26)$$

We obtain the following representation for the normalized utility index:

$$J(t) = E_t \left(\int_t^T f(C_s, J(s)) ds + J(T) \right). \quad (27)$$

Note the well-known fact that when $\rho = \gamma$ (i.e., $\theta = 1$) then $f(C, J) = \beta u_\rho(C) - \beta J$ and a simple application of Itô’s lemma shows that

$$J(t) = E_t \left(\int_t^T e^{-\beta(s-t)} \beta u_\rho(C_s) ds + e^{-\beta(T-t)} J(T) \right). \quad (28)$$

Further, Duffie-Epstein (1992b) show that the pricing kernel $(\Pi(t))$ for this economy has the following form (if

there exists an ‘interior’ solution to the optimal consumption portfolio choice problem of the representative agent):

$$\Pi(t) = e^{\int_0^t f_{J(C_s, J_s)} ds} f_C(C_t, J_t). \quad (29)$$

It is the Riesz representation of the gradient of the normalized utility index at the optimal consumption (See Chapter 10 of Duffie (2001) for further discussion.) We now consider the case of unitary EIS ($\rho = 1$) and give an expression for the utility index and for the pricing kernel in this economy.

7.2 Equilibrium Prices when $\rho = 1$

Assuming the equilibrium consumption process given in equations (20)-(21) above, we obtain an explicit characterization of the utility index J and the corresponding pricing kernel Π .

For this we define, respectively, the operator

$$\mathcal{D}h(x) = h_x(x) \mu_X(x) + \frac{1}{2} \text{trace} \left(h_{xx} \sigma_X(x) \sigma_X(x)^T \right)$$

where h_x is the $(n, 1)$ Jacobian vector of first derivatives and h_{xx} denotes the (n, n) Hessian matrix of second derivatives. With these notations, we find:

Proposition 1 *Suppose $I(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ solves the following equation:*

$$0 = I(x) \left((1 - \gamma) \mu_C(x) + (1 - \gamma)^2 \frac{\|\sigma_C(x)\|^2}{2} \right) + \mathcal{D}I(x) + (1 - \gamma) \sigma_C(x) \sigma_X(x)^T I_x(x) - \beta I(x) \log I(x) \quad (30)$$

and satisfies the transversality condition $\lim_{T \rightarrow \infty} E \left[e^{-\beta T} \log C_T + e^{-\beta T} \log I(X_T) \right] = 0$ then the value function is given by:

$$J(t) = u_\gamma(C_t) I(x_t) \quad (31)$$

The corresponding pricing kernel is:

$$\Pi(t) = e^{-\int_0^t \beta(1 + \log I(x_s)) ds} (C_t)^{-\gamma} I(x_t) \quad (32)$$

Proof. From its definition in equations 27 and 26 we obtain:

$$\frac{dJ(t)}{J(t)} = (-(1 - \gamma)\beta \log C(t) + \beta \log((1 - \gamma)J(t))) dt + \sigma_J(t) dz(t) \quad (33)$$

for some \mathcal{F}_t -measurable diffusion process σ_J . An application of Itô’s lemma to $e^{-\beta t} \log((1 - \gamma)J(t))$ shows that its

solution satisfies the following integral equation for any $T > t$:

$$\log((1-\gamma)J(t)) = \mathbb{E} \left[\int_t^T e^{-\beta(s-t)} \left(\beta(1-\gamma) \log C(s) + \frac{1}{2} |\sigma_J(s)|^2 \right) ds + e^{-\beta(T-t)} \log((1-\gamma)J(T)) \right] \quad (34)$$

Further, if it satisfies the transversality condition $\lim_{T \rightarrow \infty} \mathbb{E}[e^{-\beta T} \log((1-\gamma)J(T))] = 0$, then $J(t)$ solves

$$\log((1-\gamma)J(t)) = \mathbb{E} \left[\int_t^\infty e^{-\beta(s-t)} \left(\beta(1-\gamma) \log C(s) + \frac{1}{2} |\sigma_J(s)|^2 \right) ds \right] \quad (35)$$

Now suppose $I(x)$ satisfies the ODE given in equation 30. Applying Itô's lemma to $e^{-\beta t} \log((1-\gamma)J(t))$ using our guess $J(t) = u_\gamma(C_t)I(x_t)$ we find that

$$e^{-\beta T} \log((1-\gamma)J(T)) - e^{-\beta t} \log((1-\gamma)J(t)) = - \int_t^T e^{-\beta s} \left(\beta(1-\gamma) \log C(s) + \frac{1}{2} |\sigma_J(s)|^2 \right) ds + \int_t^T e^{-\beta s} ((1-\gamma)\sigma_C(X_s) + \sigma_I(X_s)) dz(s)$$

where

$$\sigma_I(x)^\top = \frac{\sigma_X(s)^\top I_X(x)}{I(x)}$$

and

$$\sigma_J = (1-\gamma)\sigma_C + \sigma_I.$$

Suppose that (a) the stochastic integral is a martingale (sufficient conditions are $\mathbb{E} \left[\int_t^T e^{-2\beta s} (|\sigma_C(X_s)|^2 + |\sigma_I(X_s)|^2) ds \right] < \infty$) and (b) the transversality condition listed in the proposition is satisfied, then taking expectations and the limit when $T \rightarrow \infty$ in equation 35 above we obtain:

$$\log((1-\gamma)J(t)) = \mathbb{E} \left[\int_t^\infty e^{-\beta(s-t)} \left(\beta(1-\gamma) \log C(s) + \frac{1}{2} |\sigma_J(s)|^2 \right) ds \right] \quad (36)$$

This shows that our candidate solution satisfies the recursive backward stochastic differential equation we are trying to solve. Uniqueness follows from the appendix in Duffie, Epstein, Skiadas (1992) (under some additional technical conditions listed therein).

The expression for the pricing kernel follows from its definition in equation 29, the expression for the aggregator in equation 26 and the expression for the value function just derived. ■

The next result investigates the property of equilibrium prices.

Proposition 2 *The risk-free interest rate is given by:*

$$r(x_t) = \beta + \mu_C(x_t) + \frac{\|\sigma_C(x_t)\|^2}{2} - \gamma \|\sigma_C(x_t)\|^2 + \sigma_C(x) \sigma_I(x) \quad (37)$$

Further, if the following transversality condition is satisfied $\lim_{T \rightarrow \infty} \mathbb{E} [\Pi_T C_T] = 0$, then the value of the claim to aggregate consumption is given by:

$$V(t) = \frac{C(t)}{\beta} \quad (38)$$

It follows that

$$\frac{dV_t}{V_t} = (\mu_C(x_t) + \frac{1}{2}\|\sigma_C(x_t)\|^2)dt + \sigma_C(x_t)dz(t) \quad (39)$$

The risk premium on the claim to aggregate consumption is given by

$$\mu_V(x) + \beta - r(x) = (\gamma\sigma_C(x) - \sigma_I(x))^\top \sigma_C(x) \quad (40)$$

Proof. To prove the result for the interest rate, apply the Itô-Doebelin formula to the pricing kernel. It follows from $r(t) = -E[\frac{d\Pi(t)}{\Pi(t)}]/dt$ that:

$$r(x_t) = \beta + \beta \log I(x_t) + \gamma\mu_c(x_t) - \frac{1}{2}\gamma^2\|\sigma_c(x_t)\|^2 - \frac{\mathcal{D}I(x_t)}{I(x_t)} + \gamma\sigma_I^\top \sigma_C. \quad (41)$$

Now substitute the expression for $\log I(x)$ from the ODE for $I(x)$ in Proposition 1 to obtain the result.

To prove the result for the consumption claim, define $V(t) = \frac{C_t}{\beta}$. It follows from Itô's lemma that:

$$\frac{dV_t + C_t dt}{V_t} = (\mu_V(X_t) + \beta)dt + \sigma_C(X_t)dz_t$$

with $\mu_V(x) = (\mu_C(x_t) + \frac{1}{2}\|\sigma_C(x_t)\|^2)$. Then, using the definition of the risk-free rate we obtain:

$$\frac{dV_t + C_t dt}{V_t} = \left(r(X_t) + (\gamma\sigma_C(X_t) - \sigma_I(X_t))^\top \sigma_C(X_t) \right) dt + \sigma_C(X_t)dz_t$$

In turn since the state price density has dynamics:

$$\frac{d\Pi_t}{\Pi_t} = -r(X_t)dt - (\gamma\sigma_C(X_t) - \sigma_I(X_t)) dz_t$$

an application of the Itô's formula shows that

$$\Pi_T V_T - \Pi_t V_t + \int_t^T \Pi_s C_s ds = \int_t^T \Pi_s V_s ((1 - \gamma)\sigma_C(X_s) + \sigma_I(X_s)) dz_s,$$

which is a martingale (under appropriate regularity conditions for the stochastic integral to be a Martingale). Taking an expectation we thus obtain

$$\Pi_t V_t = E_t[\int_t^T \Pi_s C_s ds + \Pi_T V_T]$$

If, furthermore, the solution satisfies the transversality condition listed in the proposition (i.e., $\lim_{T \rightarrow \infty} E[\Pi_T C_T] = 0$), then we can let $T \rightarrow \infty$ and have indeed proved that $V_t = \frac{C_t}{\beta}$ is the value of the claim to aggregate consumption.

■

7.3 Application to learning

Suppose now that log consumption follows the following process:

$$d \log C_t = \mu dt + \sigma dz(t) \quad (42)$$

but we assume further that μ has to be estimated by the representative agent based on observing past consumption. Suppose that he starts with some Gaussian prior $\mu \sim N(m_0, \Sigma_0)$. Then it is well-known that his posterior is also Gaussian with mean and variance given by (m_t, Σ_t) with dynamics:

$$dm_t = \lambda_t (d \log C_t - m_t dt) \quad (43)$$

$$= \lambda_t \sigma d\tilde{z}(t) \quad (44)$$

where the second equation defines the innovation process \tilde{z}_t , a Brownian motion in the observation filtration of the agent, in terms of which the consumption process can be rewritten as

$$d \log C_t = m_t dt + \sigma d\tilde{z}(t). \quad (45)$$

Further, the posterior variance is:

$$d\Sigma_t = -\lambda_t^2 \sigma^2 dt \quad (46)$$

and the regression coefficient is given by:

$$\lambda_t = \frac{\Sigma_t}{\sigma^2}. \quad (47)$$

Therefore, note the dynamics of λ :

$$d\lambda_t = -\lambda_t^2 dt. \quad (48)$$

The solution of which is simply

$$\frac{1}{\lambda(t)} = \frac{1}{\lambda_0} + t.$$

Now we see that the state-vector in the information filtration of the agent is $X(t) = [m_t, t]$ (or equivalently, $[m_t, \Sigma_t]$).

7.3.1 The pricing kernel with EIS = 1

We now derive an expression for the $I(\cdot)$ function from proposition 1 above (for the case unitary EIS $\rho = 1$ and arbitrary risk-aversion γ). The ode given in equation 30 simplifies to (we drop arguments for simplicity):

$$0 = I((1 - \gamma)m + (1 - \gamma)^2 \frac{\sigma^2}{2}) + \frac{1}{2} I_{mm} \lambda(t)^2 \sigma^2 + (1 - \gamma) \sigma^2 \lambda(t) I_m - \beta I \log I + I_t. \quad (49)$$

We guess a solution of the form

$$\log I(m, t) = a(t) + b(t)m$$

Plugging into the pde and setting coefficients in m to zero we obtain two odes which are:

$$b'(t) - \beta b(t) + 1 - \gamma = 0 \quad (50)$$

$$a'(t) - \beta a(t) + \frac{(1-\gamma)^2 \sigma^2}{2} + \frac{1}{2} b(t)^2 \lambda(t)^2 \sigma^2 + (1-\gamma) \sigma^2 \lambda(t) b(t) = 0 \quad (51)$$

Now for the boundary conditions we note that (since $\lim_{t \rightarrow \infty} m_t = \mu$ and $\lim_{t \rightarrow \infty} \lambda(t) = 0$):

$$\lim_{t \rightarrow \infty} \log I(t) = \frac{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2 \sigma^2}{\beta}$$

thus we find the boundary conditions:

$$\lim_{t \rightarrow \infty} b(t) = \frac{1-\gamma}{\beta} \quad (52)$$

$$\lim_{t \rightarrow \infty} a(t) = \frac{(1-\gamma)^2 \sigma^2}{2\beta} \quad (53)$$

Now a solution satisfying this is (uniqueness follows from the result on BSDE):

$$b(t) = \frac{1-\gamma}{\beta} \quad (54)$$

$$a(t) = (1-\gamma)^2 \sigma^2 \int_t^\infty e^{-\beta(s-t)} \left(\frac{1}{2} + \frac{\lambda(s)}{\beta} + \frac{\lambda(s)^2}{2\beta^2} \right) ds \quad (55)$$

$$= (1-\gamma)^2 \sigma^2 \frac{1 + \frac{\lambda_t}{\beta} - e^{\beta/\lambda_t} \text{Ei}\left(-\frac{\beta}{\lambda_t}\right)}{2\beta} \quad (56)$$

where we have used the definition of the exponential integral function (the principal value of the integral $\text{Ei}(z) = \int_{-z}^\infty \frac{e^{-t}}{t} dt$). It is straightforward to verify that the transversality condition of proposition 1 is satisfied.

Now, using the expression for the pricing kernel in Proposition 1, we can obtain the interest rate in closed form using Proposition 2.

Specifically we find.

$$r(t) = \beta + m_t + \frac{\sigma^2}{2} - \gamma \sigma^2 + \sigma^2 b \lambda(t) \quad (57)$$

and the dynamics of the pricing kernel:

$$\frac{d\Pi(t)}{\Pi(t)} = -r(t)dt - (\gamma\sigma - b\sigma\lambda(t))d\tilde{z}_t. \quad (58)$$

Note that interestingly the function $a(t)$ plays no role in the expression for the interest rate and the risk-premium. We can also solve for long-term zero-coupon bond prices (and hence yields in this model). Note that the risk-free

zero coupon bond prices has price:

$$P(0, T) = E^Q[e^{-\int_0^T r_t dt}] \quad (59)$$

$$= E^Q[e^{-\int_0^T \left(\beta + m_t + \frac{\sigma^2}{2} - \gamma\sigma^2 + \sigma^2 b\lambda(t)\right) dt}] \quad (60)$$

where under the risk-neutral measure m has dynamics:

$$dm_t = \lambda_t \sigma \left(d\tilde{z}^Q(t) - (\gamma\sigma - b\lambda_t) dt \right) \quad (61)$$

$$= -\sigma^2 (\gamma\lambda_t - b\lambda_t^2) dt + \lambda_t \sigma d\tilde{z}^Q(t). \quad (62)$$

Since λ_t is deterministic, m_t is a Gaussian process and the solution to the risk-free zero coupon bond is immediate.

$$P(0, T) = e^{-\int_0^T \left(\beta + m_0 + \frac{\sigma^2}{2} - \gamma\sigma^2 + \sigma^2 b\lambda(t)\right) dt} E^Q[e^{-\int_0^T \left(\int_0^t \sigma^2 (\gamma\lambda_u - b\lambda_u^2) du - \int_0^t \lambda_u \sigma d\tilde{z}_u^Q\right) dt}]. \quad (63)$$

Now, note that:

$$E^Q[e^{-\int_0^T \int_0^t \lambda_u \sigma d\tilde{z}_u^Q dt}] = E^Q[e^{-\int_0^T \int_u^T \lambda_t \sigma dt d\tilde{z}_u^Q}] \quad (64)$$

$$= e^{\frac{1}{2} \int_0^T \left(\int_u^T \lambda_t \sigma dt\right)^2 du}. \quad (65)$$

Thus, we get the final solution for the zero-coupon bond price:

$$P(0, T) = e^{-\int_0^T \left(\beta + m_0 + \frac{\sigma^2}{2} - \gamma\sigma^2 + \sigma^2 b\lambda(t)\right) dt + \int_0^T \int_0^t \sigma^2 (\gamma\lambda_u - b\lambda_u^2) du dt + \frac{1}{2} \int_0^T \left(\int_t^T \lambda_u \sigma du\right)^2 dt}. \quad (66)$$

Now, we can do a lot of the integrals explicitly. By plugging in the expression for λ

$$\frac{1}{\lambda(t)} = \frac{1}{\lambda_0} + t$$

we find

$$P(0, T) = (\lambda_0 T + 1)^{\frac{\sigma^2(2\gamma(\lambda_0 T + 1) - 1)}{2\lambda_0}} e^{-T(b\lambda_0\sigma^2 + \beta + m_0)} \quad (67)$$

and corresponding yield curve:

$$Y(0, T) = \frac{\sigma^2(\lambda_0 T(2b\lambda_0 + 2\gamma - 1) + (1 - 2\gamma(\lambda_0 T + 1)) \log(\lambda_0 T + 1))}{2\lambda_0 T} + \beta - \gamma\sigma^2 + m_0 + \frac{\sigma^2}{2}. \quad (68)$$

8 Further discussion of the "learning about the mean"-economy

Here we discuss in detail the effect of increasing the EIS above 1 in the case when agents are learning about the mean growth rate in the economy discussed in Section 3. The upshot is that increasing the EIS increases return volatility, which in turn increases the risk premium. In addition, there is an interesting interaction effect between the endogenous discount rate and the effect of shocks to the mean growth rate to the wealth-consumption ratio which makes the pricing kernel more sensitive over time to shocks to beliefs. This is different from the case where the EIS = 1, where the sensitivity of the pricing kernel to updates in beliefs is constant, and means that the asset pricing implications of learning decrease over time at a slower rate than the posterior variance of beliefs.

8.1 The Effect of the Intertemporal Elasticity of Substitution

Increasing the EIS from 1 to, say, 2 has in typical calibrations only a minor effect on the sensitivity of the pricing kernel to shocks to the continuation utility. For instance, if $\gamma = 10$, as in Bansal and Yaron (2004), increasing the EIS from 1 to 2, means that this sensitivity, $\gamma - 1/\psi$, only increases from 9 to 9.5. However, when the substitution effect dominates the wealth effect, the wealth-consumption ratio increases upon a high realization of consumption growth as the expected mean growth rate is revised upwards. These dynamics create excess return volatility, which in turn increases the risk premium. Further, interestingly, the endogenous sensitivity of the continuation utility to updates in beliefs in this case increases over time, which leads to a slower decline in asset return Sharpe ratios over time than in the cases with a lower EIS. Before we discuss these effects, we first explain how the model is solved.

8.1.1 Model solution

In the case when EIS $\neq 1$, it is necessary to resort to numerical solution of the model. Further, since the wealth and substitution effects now no longer cancel, it is necessary to bound the support for the beliefs about the mean growth rate, μ .²⁴ Therefore, we consider a truncated normal prior distribution for μ , where the truncation bounds $\underline{\mu}$ and $\bar{\mu}$ are set such that equilibrium exists for all possible 'boundary' ($t = \infty$) economies, where the agents have learned the true mean. This ensures the existence of equilibrium. The updating equations for the hyperparameters, $\mu_{t+1} = \mu_t + \frac{A_t}{\sqrt{1+A_t}} \sigma \tilde{\varepsilon}_{t+1}$ and $A_{t+1}^{-1} = A_t^{-1} + 1$, remain the same. The reason for this is easiest to see by considering Bayes' rule when the truncation of a general, untruncated prior, $p(\theta)$, is achieved by multiplying by an indicator function which takes the value 1 if $\theta \in [\underline{\theta}, \bar{\theta}]$:

$$p(\theta|y^t) \mathbf{1}_{\underline{\theta} \leq \theta \leq \bar{\theta}} \propto p(y^t|\theta) p(\theta) \mathbf{1}_{\underline{\theta} \leq \theta \leq \bar{\theta}}. \quad (69)$$

²⁴The reason is that a positive probability of an arbitrarily high (low) μ when the IES is greater than (smaller than) 1, leads to a violation of a transversality condition and the equilibrium does not exist (i.e., the wealth-consumption ratio is infinite). This is easiest to see by considering a deterministic economy with a constant growth rate. If the growth rate is higher than the risk-free rate, the wealth-consumption ratio, and thus utility, is infinite.

The conjugacy of the prior comes from the functional forms of the original prior and the likelihood, $p(\theta)$ and $p(y^t|\theta)$. Thus, if the likelihood function is normal and the prior is truncated normal, the posterior is truncated normal with the same truncation limits as the prior. The hyperparameters, μ_t and A_t , along with the truncation limits, completely describe the solution. Of course, the hyperparameter μ_t , for instance, no longer in general corresponds to the subjective conditional mean of consumption growth.

Note that since the posterior variance is deterministic ($A_{t+1}^{-1} = A_t^{-1} + 1$), we can replace the state variable A_t with time, t . Thus, the parameter uncertainty model is nonstationary. Building on Johnson (2007), who consider the simpler case of power utility, we develop a solution methodology for the case of parameter uncertainty and Epstein-Zin preferences where we solve the model using a backwards recursion from the known-parameters ($t = \infty$) boundary economies.²⁵ In particular, the log wealth-consumption ratio, pc , is found for each t on a grid for $\mu_t \in [\underline{\mu}, \bar{\mu}]$ using the recursion:

$$e^{pc(\mu_t, t)} = E \left[\beta^\theta e^{-\gamma \Delta c_{t+1} + (\theta-1) \ln(\exp(pc(\mu_{t+1}, t+1)) + 1)} \middle| \mu_t, t; A_0, \underline{\mu}, \bar{\mu} \right], \quad (70)$$

where the evolution equations of the state variables are $\mu_{t+1} = \mu_t + \frac{A_t}{\sqrt{1+A_t}} \sigma \tilde{\varepsilon}_{t+1}$ and $A_{t+1}^{-1} = A_t^{-1} + 1$.²⁶

8.1.2 Asset pricing moments versus amount of parameter uncertainty

To assess the asset pricing implications of varying degrees of parameter uncertainty in cases where $EIS \neq 1$, it is necessary to calibrate the consumption dynamics as the model relies on a numerical solution. For this exercise, we choose the true consumption dynamics to match the mean and volatility of time-averaged annual U.S. log, per capita consumption growth, as reported in Bansal and Yaron (2004): $E_T[\Delta c] = 1.8\%$ and $\sigma_T(\Delta c) = 2.72\%$. This implies true (not time-averaged) quarterly mean and standard deviation of 0.45% and 1.65%, respectively, which are the numbers we use in the quarterly calibration. The lower bound for the quarterly growth rate μ is set at -0.3% , while the corresponding upper bound is set at 1.2%. The prior beliefs are assumed to be unbiased and the maximum level of prior uncertainty is $A_0 = 1$, as before. The market claim we consider is a levered consumption claim. In particular, we simply multiply the excess returns on the consumption claim with 1.5, which is consistent with the average aggregate leverage ratio in the U.S. stock market.²⁷

The left plot in Figure 6 shows that the conditional, annualized price of risk (that is, $\sigma_t(M_{t+1})/E_t(M_{t+1})$; see Hansen and Jagannathan (1992)) is constant at 0.33 in the known parameters benchmark economy (solid line). This is true for any level of the EIS since consumption growth is i.i.d. In the two economies with parameter uncertainty, however, the price of risk decreases from about 1.15 to about 0.45. Thus, as in the $EIS = 1$ case discussed earlier, the

²⁵See Vazquez-Grande (2009) for a similar solution technique.

²⁶This backwards recursion is fast and very accurate. The only additional requirement is continuity at the boundary when going from the boundary solutions where $t = \infty$ to a large t (we use $t = 5000$ as the point in time before the known-parameter boundary is reached). This general solution method is used throughout the paper, with detailed explanations relegated to the Appendix.

²⁷Note that we do not add idiosyncratic dividend growth shocks. Thus, the return volatility of the market claim ought not be compared directly to equity market return volatility. The risk premium, however, which is a function of the covariance of dividend and consumption growth, can reasonably be compared to the equity market risk premium.

APPENDIX: Figure 6 - Price of Risk for case of unknown mean growth rate when IES $\neq 1$

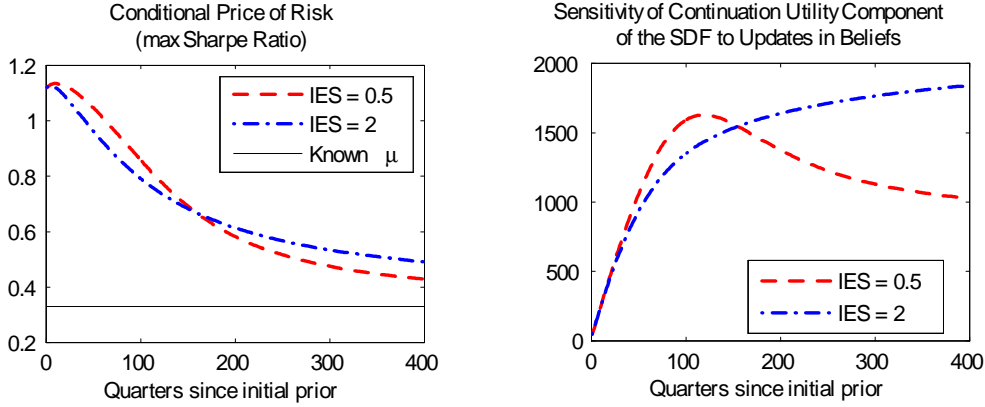


Figure 6: The graph shows the conditional price of risk as well as the sensitivity of the continuation utility component of the stochastic discount factor (SDF), as defined in the main text, to updates in the mean beliefs about the growth rate of the economy (μ). The horizontal axis gives the number of quarters passed since the initial prior. For both graphs the relevant statistics are evaluated for unbiased beliefs (i.e., at $E_t[\mu] = \mu$). The red dashed line gives the case when the intertemporal elasticity of substitution (IES) is 0.5, the blue dash-dotted line gives the case when the IES is 2, while the black solid line gives the case for which the mean is known (note that in the latter case the IES does not matter for the moments shown here).

price of risk decreases with the decreasing amount of parameter uncertainty (see Equation (10)). However, whereas the component of the price of risk due to shocks to beliefs in the EIS = 1 case decreases by a factor of 200 over the first 50 years, it only decreases by a little less than a factor of 3 in the cases shown in Figure 6.²⁸

This much slower decline in the price of risk at the beginning of the sample is mainly due to the effect of truncation bounds, which alters the prior distribution. In particular, for high A_t the prior is close to a uniform distribution with upper and lower bounds $\underline{\mu}$ and $\bar{\mu}$. Thus, extreme values are ruled out, which decreases the price of risk, but at the same time, learning is slower as the signal to noise ratio in this case is decreased, relative to the untruncated case.

After about 100 quarters the effect of the truncation is negligible as, for $\mu_t = \mu$, the truncation bounds are 5 standard deviations away given the tighter prior at this time. At this point, there is an interesting relation between the price of risk and the level of the EIS. In particular, the price of risk decreases more slowly with time with a higher EIS (the red, dashed line has EIS = 0.5, while the blue, dashed-dotted line has EIS = 2). This is the result of an endogenous interaction between the amount of parameter uncertainty, the effect of such uncertainty on the volatility of the pricing kernel, and the level of the EIS.

The rightmost plot of Figure 6 shows the sensitivity of the continuation utility component of the log pricing

²⁸In the plot, the price of risk decreases from about 1.15 to about 0.6 over the first 200 quarters. Since the component due to learning is given by what is in excess of 0.33 (the case of known parameters), the decrease is from about 0.8 to about 0.3, which is a little less than a factor of 3.

kernel to shocks to beliefs as a function of time (prior variance) and for the two levels of the EIS (0.5 and 2). This sensitivity is calculated numerically from each model as $(\theta - 1) \frac{dpc_t}{d\mu_t} |_{\mu_t=\mu}$ (see Equation (4)). From the analytical EIS = 1 case (see Equation (9)), the relevant sensitivity is given by $\frac{\gamma-1}{\beta} = 1,495$ and is thus constant. From the rightmost plot of Figure 6, focusing on the dynamics after 100 quarters of learning has taken place and the effects of truncation are negligible, it is clear that this sensitivity is increasing for the high EIS case, but decreasing for the low EIS case. In fact, this time-varying sensitivity is due to an endogenous interaction between discount rates and the size of the shock to the growth rate, μ_t . As is well understood, the price-consumption ratio is more sensitive to shocks to the growth rate if discount rates are low (see, e.g., Pastor and Veronesi (2002)). When the EIS is greater than one and the substitution effect dominates, the price-consumption ratio increases when uncertainty decreases (see also Bansal and Yaron (2004)) and so, holding $\mu_t = \mu$, the price-consumption ratio in this case increases over time, meaning that discount rates decrease and, thus, the sensitivity to belief shocks increases. When the EIS is less than one, however, the price-consumption ratio is decreasing over time as discount rates now increase when uncertainty decreases, and this makes the pricing kernel less sensitive to shocks to beliefs over time. These endogenous dynamics are why after 100 years of learning the component of the price of risk that is due to belief shocks is almost twice as big for EIS = 2 versus EIS = 0.5, even though the direct risk price of shocks to the continuation utility ($\gamma - 1/\psi$) when the EIS = 2 is only 1.19 times the same when the EIS = 0.5.

Figure 7 shows the conditional, annualized market risk premium, log return volatility, as well as the log real yield spread, for different economies versus the amount of parameter uncertainty, as measured by quarters passed since the initial prior, $A_0 = 1$. The conditional moments are always calculated assuming an unbiased prior, $\mu_t = \mu$.²⁹ The yield spread is defined as the difference between the annualized 10-year yield on a real, default-free zero-coupon bond minus the current (quarterly), annualized real risk-free rate. The economies with parameter uncertainty have EIS $\psi = \{0.5, 2\}$, and as before we let $\gamma = 10$ and $\beta = 0.994$. As a benchmark, the black solid line depicts the case where μ is known (in which the EIS does not matter for any of the reported moments, given the assumed i.i.d. consumption growth process).

For the high EIS case ($\psi = 2$), the risk premium is initially about 12%, declining as posterior variance decreases, and after 100 years of learning about 3%. For comparison, the case where μ is known yields a risk premium of 1.7% (for any level of the EIS). Thus, there is again a significant effect of parameter uncertainty even after 100 years of learning, even though at this point the standard deviation of beliefs about μ is only 0.09%.

For the low EIS case ($\psi = 0.5$), however, the risk premium is initially negative and increasing as posterior variance decreases. When the EIS is less than one, the wealth effect dominates, and therefore a positive revision in beliefs about μ , resulting from high realized consumption growth, is accompanied by a *decrease* in the price-dividend ratio of this claim. For high levels of parameter uncertainty, the return on this claim is in fact negatively correlated with realized consumption growth, which leads to a negative risk premium. As the posterior variance tightens over time, the updates in the price-dividend ratio become smaller and the realized dividend dominates, which restores the positive correlation between the return to this claim and consumption growth.

²⁹See, e.g., Cogley and Sargent (2008) for a discussion of the asset pricing implications of biased priors.

APPENDIX: Figure 7 - Conditional Moments for case of unknown mean growth rate when $IES \neq 1$

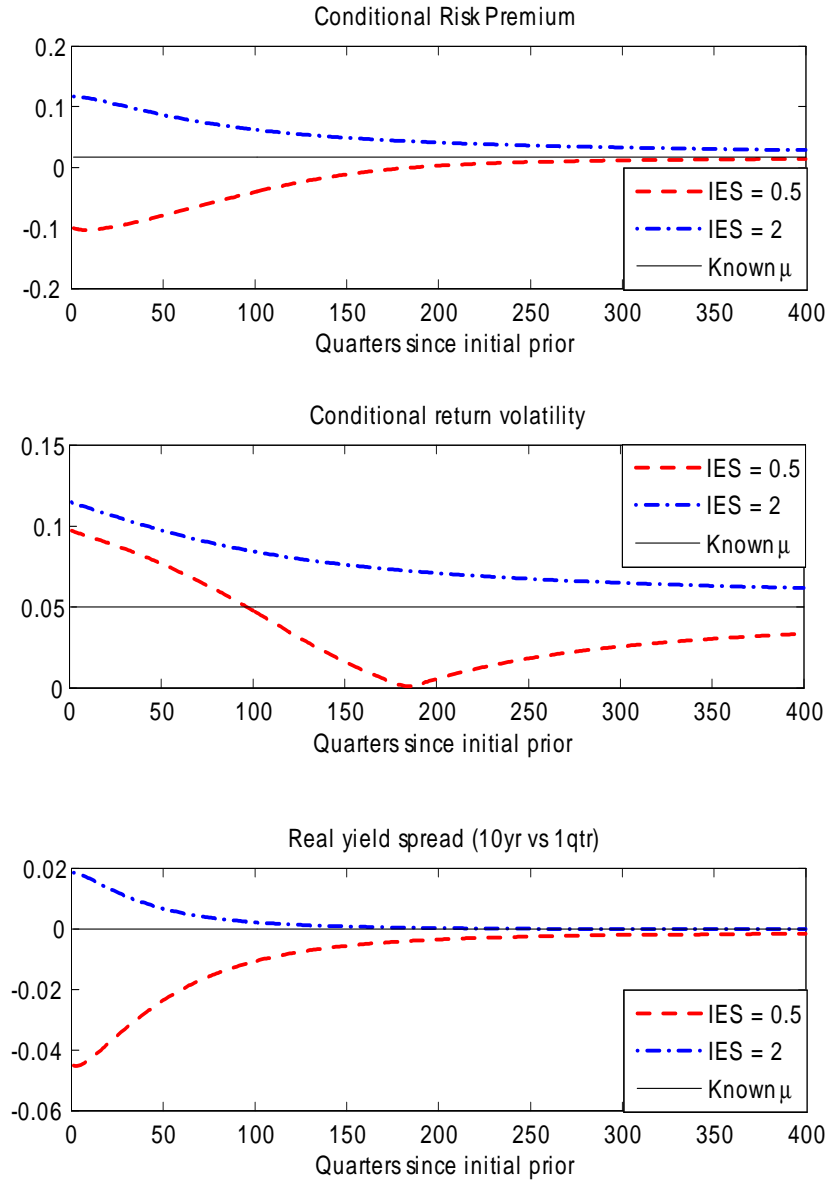


Figure 7: The graph shows the conditional risk premium, return volatility and real yield spread for the economy with unknown mean growth rate (μ). The horizontal axis gives the number of quarters passed since the initial prior. For all graphs the relevant statistics are evaluated for unbiased beliefs (i.e., $E_t[\mu] = \mu$). The red dashed line gives the case when the intertemporal elasticity of substitution (IES) is 0.5, the blue dash-dotted line gives the case when the IES is 2, while the black solid line gives the case for which the mean is known (note that in the latter case the IES does not matter for the moments shown here).

The middle plot in Figure 7 shows how these dynamics play out for the volatility of returns. With a high EIS, when the substitution effect dominates, a positive shock leads to both high dividends and an increase in the price-dividend ratio of the claim, which in turn means return volatility is high (excess volatility). After 100 years of learning, the volatility is 6.2% versus 5% in the benchmark, known parameters economy. With a low EIS, the response of the price-dividend ratio and realized dividend growth offset to some extent and volatility is therefore always lower than in the case with an $EIS > 1$. The lowest level of volatility occurs when the level of parameter uncertainty is such that the two offset almost exactly, as seen at around 180 quarters of learning.

The bottom plot in Figure 7 shows that the yield spread is positive initially and then basically zero for the case of high EIS, but negative initially for the case of a low EIS. With a low EIS, the volatility of the risk-free rate is higher, which leads to more volatile bond returns. The negative risk premium on real bonds, which are hedges in this economy as discussed earlier, is therefore higher in the case if a low EIS, which leads to a negative yield spread. With a high EIS, however, the effect of expected future higher short-term rates, as precautionary savings decreases over time, dominates initially.

In sum, with an EIS greater than one, the model delivers excess return volatility and a higher risk premium than in the case when $EIS = 1$, as analyzed earlier. Further, the real term structure is, except for in cases with very high parameter uncertainty, essentially flat. Again, this is in sharp contrast to the objective long-run risks assumed in Bansal and Yaron (2004), which implies a strongly downward-sloping yield curve. A low EIS generates a counter-factual negative risk premium and also tends to deliver return volatility that's lower than dividend volatility, which is strongly counter-factual (see, e.g., Shiller (1980), Campbell and Shiller (1987)).

8.2 Unknown variance

In the preceding, the variance parameter σ^2 was assumed known to investors. It is straightforward to relax this assumption, though as pointed out in Weitzmann (2007) and Bakshi and Skouliakis (2010), it is necessary to truncate also the support for σ^2 in order to ensure finite utility. Weitzmann (2007) argues that learning about the variance parameter can lead to arbitrarily high risk premiums as the subjective distribution for consumption growth becomes fat-tailed. He further argues that learning about the mean, as in the preceding section, does not increase the fatness of the tails of the conditional consumption growth distribution and therefore cannot help in explaining asset pricing puzzles. Clearly, the latter intuition does not hold when considering a utility function that allows for a preference for early resolution of uncertainty.³⁰

Bakshi and Skouliakis (2010) argue that Weitzmann's results, which are developed under power utility, are not robust to reasonable truncation limits for σ^2 . However, given that we focus primarily not on the fatness of the tails, but on permanent shocks to the conditional consumption growth distribution induced by the learning process itself, uncertain variance can potentially still have important asset pricing implications. In the following, we show that

³⁰In fact, with a truncated normal as the prior, the tails of the subjective distribution are actually less fat than for a normal distribution with the same dispersion, but due to the updating that generates long-run risks, the asset pricing implications were shown to be nontrivial.

quantitatively large asset pricing implications of learning about the variance parameter indeed can arise, but that interesting asset pricing effects of learning about the variance parameter are shorter-lived than those documented for the uncertain mean case.

We assume that the joint prior over the mean μ and the variance σ^2 is Normal-Inverse-Gamma:

$$p(\mu, \sigma^2 | y^t) = p(\mu | \sigma^2, y^t) p(\sigma^2 | y^t), \quad (71)$$

where

$$p(\sigma^2 | y^t) \sim IG\left(\frac{b_t}{2}, \frac{B_t}{2}\right), \quad (72)$$

$$p(\mu | \sigma^2, y^t) \sim N(a_t, A_t \sigma^2). \quad (73)$$

Given that log consumption growth is normally distributed, these prior beliefs lead to posterior beliefs that are of the same form (conjugate priors). The updating equations for investors' beliefs are:

$$A_{t+1}^{-1} = 1 + A_t^{-1}, \quad (74)$$

$$\frac{a_{t+1}}{A_{t+1}} = \frac{a_t}{A_t} + y_{t+1}, \quad (75)$$

$$b_{t+1} = b_t + 1, \quad (76)$$

$$B_{t+1} = B_t + \frac{(y_{t+1} - a_t)^2}{1 + A_t}. \quad (77)$$

In terms of pricing, note that this system can be reduced to three state-variables: a_t , B_t , and t , given initial priors. We solve the model numerically and, as before, use the closed-form solution for the known parameters cases as the boundary values in a recursion that is solved backwards in time on a grid for a_t and B_t . In order for the Inverse Gamma distribution to have a finite mean and variance, which is convenient, we set the maximum prior uncertainty as $b_0 = 5$. As mentioned, we need to truncate the distribution for σ^2 and we choose wide bounds: $\bar{\sigma}^2 = 100 * \sigma^2$, $\underline{\sigma}^2 = \sigma^2/100$. As before, the true quarterly variance is calibrated as $\sigma^2 = (1.65\%)^2$, and the model is solved at the quarterly frequency. The other parameters of the model are the same as in the case where the mean was the only unknown parameter: $a_0 = \mu = 0.45\%$, $A_0 = 1$, $\gamma = 10$, $\psi = 2$, and $\beta = 0.994$. We set $b_0 = 5$ and $\frac{B_0}{b_0 - 2} = \sigma^2$. The latter implies that the initial truncated prior for the variance is unbiased, with a standard deviation of $(1.85\%)^2$.

Figure 8 shows the conditional annualized volatility of the log pricing kernel as the average per quarter across 20,000 simulated economies over a 100 year sample. We plot three cases. Learning about the mean only, as discussed in the previous section, learning about the variance only, and learning about the mean and the variance parameters. First, consider the dashed line, which shows the case when learning about the variance only. The volatility of the pricing kernel is very high in the first decade, but then comes down quite quickly towards the benchmark, known parameter value of 0.33.³¹ Pretty much all of this pattern comes from the continuation utility component of the

³¹The somewhat uneven line for the variance cases in the 5 first years is due to the truncation bounds

**APPENDIX Figure 8 - Conditional Volatility of the Pricing Kernel:
Cases with unknown variance**

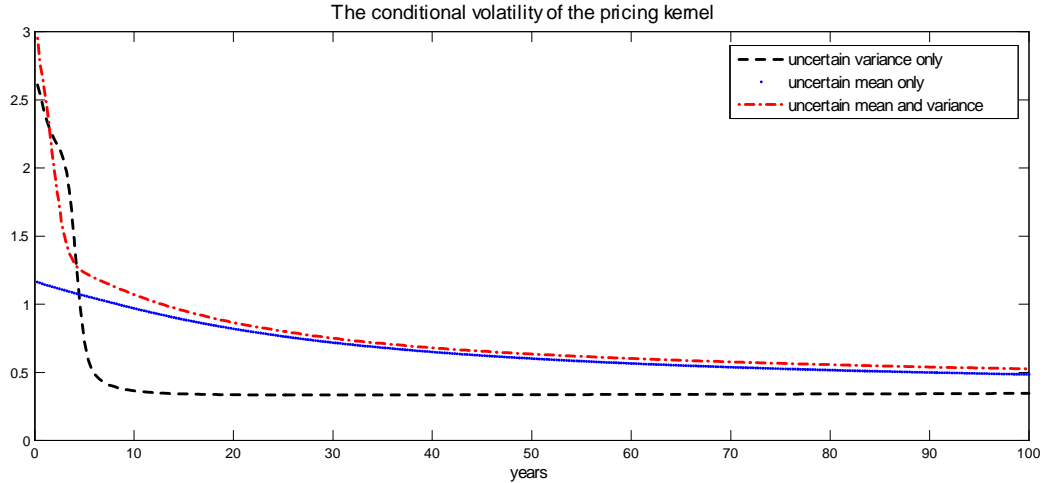


Figure 8: The graph shows the subjective conditional annualized volatility of the Epstein-Zin stochastic discount factor with preference parameters $\gamma = 10$, $\psi = 2$ and $\beta = 0.994$ over a 100 year sample period, averaged across 20,000 simulated economies at each time t . The dashed lined corresponds to the case of unknown variance only, the dotted line corresponds to the case of unknown mean only, while the dash-dotted line corresponds to the case of unknown mean and variance.

pricing kernel and not from the power utility component. Thus, we confirm the results in Bakshi and Skouliakis (2010), the fatness of the tails given reasonable bounds on the variance parameter is not sufficient to strongly affect asset prices. However, the large updates in beliefs about the variance that occur in the first 10 years does have significant impact on the volatility of the pricing kernel through the effect on the continuation utility. After this, the impact of shocks to beliefs about the variance parameter have a very small impact. The dash-dotted line shows the case of unknown mean *and* variance. Here, we see that adding unknown variance yields a pricing kernel that is on average always more volatile than in the known variance, unknown mean case. However, there are only large differences in the first decade, relative to the case with only unknown mean (dotted line).

The risk premium for a 100 year long sample that start with priors corresponding to tossing out the 10 first years plotted in Figure 8, is 1.8% for the case of unknown variance but known mean, relative to 1.7% for the benchmark known parameters economy. In the case of unknown mean and variance, the average risk premium over this sample slightly affecting the form of the subjective distribution for the variance parameters when the level of uncertainty is very high.

is 4.9% compared to 4.4% for the case of unknown mean and known variance.

In sum, unknown variance has more of a second-order effect on asset pricing moments, unless uncertainty is very large, as would be the case in the decade after a structural break for instance. There are two reasons for this more short-lived effect. First, Bayesian learning implies that learning about variance is much faster than learning about the mean. Second, the variance is a second order moment, so generally less important for the continuation utility than changes in the mean.

9 Uncertain mean and variance of the Depression state

Here, we consider the case where the transition probabilities are known, but where instead the mean and variance parameters of the Depression state (μ_2 and σ_2) are unknown. The true model parameters are as in the previous case.³²

We assume that the joint prior over the unknown parameters is Normal-Inverse-Gamma:

$$p(\mu_2, \sigma_2^2 | \Delta c^\tau) = p(\mu_2 | \sigma_2^2, \Delta c^\tau) p(\sigma_2^2 | \Delta c^\tau), \quad (78)$$

where

$$p(\sigma_2^2 | \Delta c^\tau) \sim IG\left(\frac{b_\tau}{2}, \frac{B_\tau}{2}\right), \quad (79)$$

$$p(\mu_2 | \sigma_2^2, \Delta c^\tau) \sim N(a_\tau, A_\tau \sigma_2^2), \quad (80)$$

and where τ counts time spent in state 2 and where Δc^τ denotes the history of consumption growth realizations in the Depression state. Obviously, there is no learning about μ_2 and σ_2^2 from consumption growth in the good state. Given that log consumption growth is normally distributed, these prior beliefs lead to posterior beliefs that are of the same form (conjugate priors). The updating equations for investors' beliefs are:

$$A_{\tau+1}^{-1} = 1 + A_\tau^{-1}, \quad (81)$$

$$\frac{a_{\tau+1}}{A_{\tau+1}} = \frac{a_\tau}{A_\tau} + \Delta c_{\tau+1}, \quad (82)$$

$$b_{\tau+1} = b_\tau + 1, \quad (83)$$

$$B_{\tau+1} = B_\tau + \frac{(\Delta c_{\tau+1} - a_\tau)^2}{1 + A_\tau}. \quad (84)$$

In terms of pricing, note that this system can be reduced to three state-variables: a_τ , B_τ , and τ , given initial priors. We solve the model numerically and, as before, use the closed-form solution for the known parameters cases as the

³²In a related paper, Lu and Siemer (2011) consider an economy where agents use an adaptive learning rule to learn about whether there is a disaster or not, as well as the mean growth rate in the disaster state. This mean growth rate is drawn at the beginning of each disaster and so it is not a fixed parameter as in the case we consider here.

boundary values in a recursion that is solved backwards in time, where time again is counted in terms of time spent in the Depression state, τ , on a grid for a_τ and B_τ .

To ensure existence of equilibrium, it is necessary to truncate the distribution for the unknown parameters. For example, the normal distribution for the mean in the disaster state implies that there is a positive probability that the disaster state has, in fact, an arbitrarily high mean growth rate. As is well known, the growth rate of the economy has, in conjunction with preference parameters, to satisfy a transversality condition, so an unbounded support for μ_2 is inadmissible. Further, as pointed out by Geweke (2002), Weitzman (2007), and Bakshi and Skoulakis (2010), it is also necessary to truncate the support for σ^2 . The updating equations for the state variables are not affected by the truncation, although of course the numerical integration will take the truncation into account. More details on the model solution technique are provided later in this Appendix.

As discussed earlier, our focus in this paper is on unbiased priors and we choose wide truncation limits relative to the initial prior to limit the effects of truncation on our results.³³ In particular, we let the truncation limits on the mean growth rate be ± 4 standard deviations away from the true mean, where the standard deviation in question is that which arises after an initial prior learning period of 100 years. Thus, $\underline{\mu}_2 = \mu_2 - 4 \times \sigma_2 \times \sqrt{A_0}$, $\overline{\mu}_2 = \mu_2 + 4 \times \sigma_2 \times \sqrt{A_0}$, where $A_0 = 16$ (given the assumption of one previously observed Depression in the 100 year training period) and $a_{t=0} = \mu_2$. This implies that the prior standard deviation of beliefs about the Depression mean is 0.3675%. Similarly, we set $\overline{\sigma^2} = 9 * \sigma^2$, $\underline{\sigma^2} = 1e - 6$. We set $b_0 = 16$, reflecting the 100 year prior learning prior, and $\frac{B_0}{b_0 - 2} = \sigma^2$ so the prior is unbiased.³⁴ Simulating data from the truncated Inverse-Gamma distribution, we find that this implies that the prior standard deviation of beliefs about the variance σ_2^2 is $(0.91\%)^2$, while the standard deviation of beliefs about the standard deviation σ_2 is 0.27%.

Table 5 shows average 100-year sample asset pricing moments from this economy for $\gamma = \{3.9, 5\}$, $\beta = 0.994$, and $\psi = 2$, as well as for the 100-, 200-, and 300-year prior training sample periods. The table also gives the asset pricing implications of assuming that only the mean or only the variance are unknown. This is achieved by setting $A_0 = 0$ and $b_0 = \infty$, respectively.

Panel A of Table 5 shows the case where only the disaster mean μ_2 is unknown. When $\gamma = 3.9$ the average sample equity premium is 1.80% versus 1.05% in the known parameters benchmark case. This compares to 5.67% for the case of unknown transition probabilities. Thus, while an uncertain disaster mean adds risk to this calibration, the risk amplification is much less than in the case of unknown persistence parameters. The Sharpe ratio is 0.19, while the risk-free rate is somewhat high at 2.75%, again giving a much poorer fit to the data than the case of unknown transition probabilities.

Figure 9 shows as an example a simulated path of the annual wealth-consumption from this economy, which

³³At the same time, we do not choose truncation limits that are very close to violating transversality conditions. Bakshi and Skoulakis (2010) show that the results in Weitzman (2007) are due exactly to this, i.e. that the quantitatively significant results come from the tiny probability of extreme levels of consumption volatility.

³⁴Through simulation we verify that the truncation bounds are such that $E[\sigma_2^2 | b_0, B_0] = \frac{B_0}{b_0 - 1}$ is very close to a correct expression also for the *truncated* Inverse-Gamma distribution.

shows how updates in beliefs only happen in the Depression state. In this case, the updating leads to time-varying beliefs about μ_2 (see Equation (82)), which is reflected in time-variation in the wealth-consumption ratio.³⁵ Thus, the learning about the mean economy generates somewhat more interesting dynamics in valuation ratios during a disaster than the case when learning about the transition probabilities.

APPENDIX: Figure 9 - Mean beliefs about disaster mean and the wealth-consumption ratio

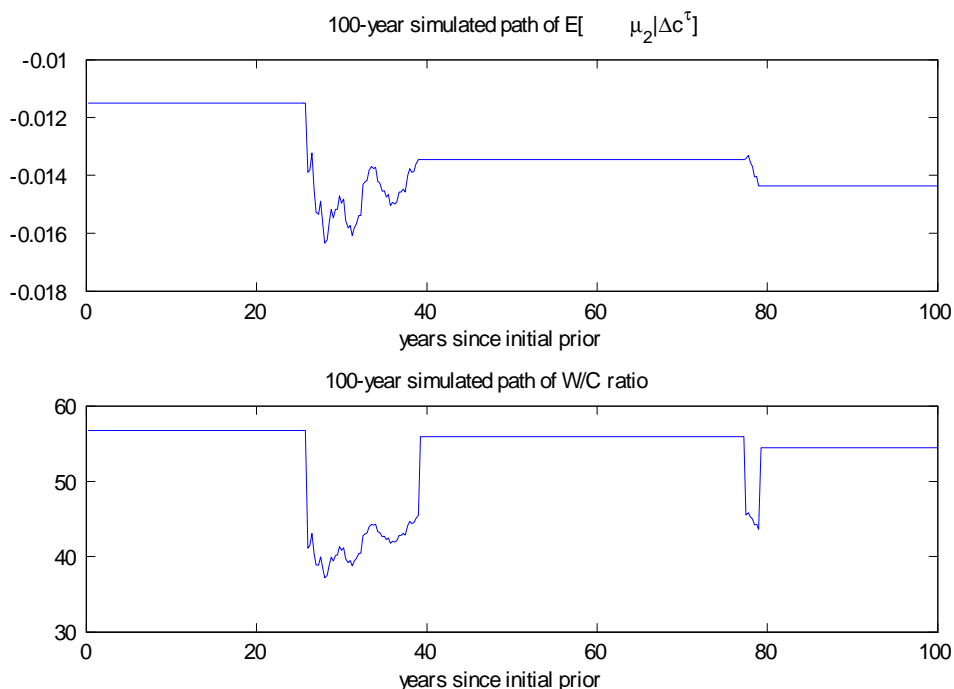


Figure 9: The figure shows a representative simulated 100-year sample path from the 2-state regime switching model, where the mean consumption growth rate in the bad state is unknown. Top plot shows the mean belief about this parameter, while the lower plot shows the annualized wealth-consumption ratio (P/C), given a 100-year training period for the prior.

In Barro, Nakamura, Steinsson, and Ursua (2011), the standard error about the annual disaster mean is reported to be 0.7%, which corresponds to about 0.18% for a quarterly mean. This level of uncertainty corresponds roughly to a 200-year prior learning period before the last 100-years of data these authors base their estimates on. In the 200-year prior case, the equity premium is 1.45% and the Sharpe ratio is 0.17, while for the 300-year prior the risk premium is 1.33% and the Sharpe ratio is 0.16. Thus, while the effects of parameter uncertainty are decreasing over time, the decrease is very slow simply because one can only learn about a parameter that governs dynamics in a rare event when the event occurs.

³⁵We did not feed actual regimes and consumption shocks into this model as quarterly consumption data is not available for the pre-WW2 period (including, of course, the Great Depression).

The rightmost set of columns shows the case when the risk aversion coefficient is increased from 3.9 to 5. Now, the 100-year prior yields a risk premium of 4.95% for the case of unknown mean with a 100-year prior, decreasing to 3.57% for the 300-year prior, which is still substantially higher than the 2.74% in the known mean benchmark case. Note that the modest increase in risk aversion in this experiment has a large impact on the asset pricing moments. This nonlinearity is due to the negative skewness and ensuing non-normality of the consumption dynamics the Depression calibration implies (see, e.g., Rietz (1988)). To summarize Panel *A* of Table 5, unknown Depression mean can almost double the risk premium and increase the Sharpe ratio by up to a factor of 1.5 relative to the known parameter case. This is in contrast to the previous case with unknown transition probabilities which increased the risk premium by up to a factor of 5 and the Sharpe ratio by almost a factor of 3.

Figure 9 shows a 100-year path of the mean belief about the Depression mean, a_t , as well as the wealth-consumption ratio, when the model is fed regimes from the last century of U.S. macro data, as discussed earlier for the case of unknown transition probabilities. The calibration has $\gamma = 3.9$ and a 100-year prior and uncertainty over only the mean parameter. The wealth-consumption ratio falls at the onset of the Great Depression, as before, but now the updating about the mean growth rate in the Depression state leads to more dynamics in the wealth-consumption ratio while in the Depression state. Once the normal state re-emerges, the wealth-consumption ratio is constant as there is nothing to learn outside of the Depression event for the uncertain parameters considered here.

Panel *B* of Table 5 shows the case where only the variance in the bad state is unknown. This case is quickly summarized. For either risk aversion assumptions, the asset pricing implications arising from the 100-, 200-, and 300-year priors are in all cases the same as those for the known parameters benchmark model. As is well-known, second moments are much more precisely estimated than first moments, and this is reflected in the speed of learning about the variance parameter. Note that this result is robust to small changes in the truncation bounds for σ_2^2 .

Panel *C* of Table 5 shows the case where both the mean variance in the bad state are unknown. In this case, the risk premium and Sharpe ratio are slightly higher than for the case where only the mean was unknown. For instance, for $\gamma = 3.9$, the risk premium increases from 1.80% to 1.83%, while for the $\gamma = 5$ case, the risk premium increases from 4.95% to 5.13%. Thus, the unknown variance does interact with the unknown mean, in particular to create a little higher return volatility, but the effect is not very large. In sum, in the case considered here where the priors are unbiased, uncertainty about a truncated volatility parameter does not have large asset pricing impact given a reasonably calibrated prior. Of course, with biased priors, a drifting variance parameter can affect both ex ante and ex post risk premiums considerably (see Johannes, Lochstoer, and Mou (2010)). The exercise in this paper, however, is to establish the properties of the risk pricing of parameter uncertainty.

10 Numerical solution method for model uncertainty case

The model is given in the main text from Equation (10). The state variables are $\hat{x}_t = E[\Delta c_{t+1} - \mu | \Delta c^t, M = 0]$, the conditional level of long-run risk given the Bansal-Yaron model, and the probability that the Bansal-Yaron model is

the true model, relative to the iid consumption growth model, p_t .

The boundary cases, $p_t = 0$ and $p_t = 1$, are solved as follows. For iid consumption growth, the price-consumption ratio, PC_t , is found using the fact that this ratio is constant in this case. Thus:

$$\left(\frac{PC(p_t = 0)}{1 + PC(p_t = 0)} \right)^\theta = E_t \left[\beta^\theta e^{(1-\gamma)\Delta c_{t+1}} \right] = \beta^\theta e^{(1-\gamma)\mu + \frac{1}{2}(1-\gamma)^2\sigma^2}. \quad (85)$$

For the case where $p_t = 1$, we have that:

$$PC(p_t = 1, \hat{x}_t)^\theta = E_t \left[\beta^\theta e^{(1-\gamma)\Delta c_{t+1}} (PC(p_t = 1, \hat{x}_{t+1}) + 1)^\theta \right], \quad (86)$$

where

$$\Delta c_{t+1} = \mu + \hat{x}_t + \hat{\sigma}_{BY} \hat{\varepsilon}_{t+1}, \quad (87)$$

$$\hat{x}_{t+1} = \rho \hat{x}_t + \hat{\varphi} \hat{\sigma}_{BY} \hat{\varepsilon}_{t+1}, \quad (88)$$

$$p_{t+1} = \frac{p_{BY}(y_{t+1}|y^t)p_t}{p_{BY}(y_{t+1}|y^t)p_t + p_{iid}(y_{t+1})(1-p_t)}, \quad (89)$$

where the parameters are defined in the main text. The price-consumption ratio is found on a grid for \hat{x} by iterating on Equation (86) given an initial guess of $PC(p_t = 1, \hat{x}_t)$.

Given these boundary solutions, the price-consumption ratio for the general cases where $0 < p_t < 1$ are found by iterating on the equation:

$$PC(p_t, \hat{x}_t)^\theta = E_t \left[\beta^\theta e^{(1-\gamma)\Delta c_{t+1}} (PC(p_{t+1}, \hat{x}_{t+1}) + 1)^\theta \right], \quad (90)$$

on a grid for \hat{x} and p , given an initial guess of $PC(p_t, \hat{x}_t)$. We note that it is important to have an incredibly fine grid as p approaches zero in order to ensure a nicely behaved PC ratio at this boundary. We use 200 grid points for p with a strongly nonlinear grid for the reason just mentioned. The exact grid used can be obtained upon request.

Once the solution for the price-consumption ratio is found, we solve for the price-dividend ratio of the claim to the exogenous dividend stream:

$$\Delta d_t = \mu + \lambda(\Delta c_t - \mu) - \frac{1}{2}\sigma_d^2 + \sigma_d \varepsilon_{d,t}, \quad (91)$$

where $\varepsilon_{d,t}$ is iid standard normal and uncorrelated with the shock to consumption. In particular, we solve in a similar manner as that just described for the consumption claim, the recursion:

$$PD(p_t, \hat{x}_t) = E_t \left[\beta^\theta e^{-\gamma\Delta c_{t+1} + \Delta d_{t+1}} ((PC(p_{t+1}, \hat{x}_{t+1}) + 1) / PC(p_t, \hat{x}_t))^{\theta-1} (1 - PD(p_{t+1}, \hat{x}_{t+1})) \right]. \quad (92)$$

We set, as the initial guess, the price-dividend ratio equal to the price-consumption ratio.

11 Numerical solution method for 2-state switching regime model with unknown parameters

Aggregate consumption growth is given by:

$$\Delta c_{t+1} = \mu(s_{t+1}) + \sigma(s_{t+1}) \varepsilon_{t+1}, \quad (93)$$

where $\varepsilon_{t+1} \stackrel{i.i.d.}{\sim} N(0, 1)$ and where $s_t \in \{1, 2\}$ follows a 2-state observable Markov chain with constant transition probabilities:

$$\Pi = \begin{bmatrix} \pi_{11} & 1 - \pi_{11} \\ 1 - \pi_{22} & \pi_{22} \end{bmatrix}, \quad (94)$$

with $\pi_{ii} \in (0, 1)$. The regime changes are assumed to be independent of the Gaussian shocks.

The agent knows the parameters within each state $(\mu_1, \mu_2, \sigma_1, \sigma_2)$, but does not know the transition probabilities (π_{11} and π_{22}). At $t = 0$, the agent is given an initial, Beta-distributed prior over each of these parameters and thereafter updates beliefs sequentially upon observing the time-series of realized regimes, s_t . We denote the history of realized regimes up until time t as s^t . The prior Beta-distribution coupled with the realization of regimes, which are governed by constant probabilities, leads to a conjugate prior and so posterior beliefs are also Beta-distributed.

The probability density function of the Beta-distribution is:

$$p(\pi|a, b) = \frac{\pi^{a-1} (1 - \pi)^{b-1}}{B(a, b)}, \quad (95)$$

where $B(a, b)$ is the Beta function (a normalization constant). The parameters a and b govern the shape of the distribution. Of particular interest is the expected value:

$$E[\pi|a, b] = \frac{a}{a + b}. \quad (96)$$

In our case, there is one uncertain probability corresponding to each regime and a standard application of Bayes rule shows that the updating equations basically count the number of times state i has been followed by state i versus the number of times state i has been followed by state j . Given this sequential updating, we let the a and b parameters have a subscript for the relevant state (1 or 2), as well as a time subscript. In particular:

$$a_{i,t} = a_{i,0} + \#(\text{state } i \text{ has been followed by state } i), \quad (97)$$

$$b_{i,t} = b_{i,0} + \#(\text{state } i \text{ has been followed by state } j). \quad (98)$$

When solving this problem numerically, we use the known parameters boundary economies (at $T = \infty$ when the parameters have been learned) as terminal values in a backwards recursion, following Johnson (2007).³⁶ We use the

³⁶Johnson uses this approach in a case with parameter learning and power utility. We extend this to the

following state variables in the numerical solution :

$$\tau_{1,t} = a_{1,t} - a_{1,0} + b_{1,t} - b_{1,0} \quad (99)$$

$$\lambda_{1,t} \equiv E_t [\pi_{11}] = \frac{a_{1,t}}{a_{1,t} + b_{1,t}} \quad (100)$$

$$\tau_{2,t} = a_{2,t} - a_{2,0} + b_{2,t} - b_{2,0} \quad (101)$$

$$\lambda_{2,t} \equiv E_t [\pi_{22}] = \frac{a_{2,t}}{a_{2,t} + b_{2,t}}, \quad (102)$$

where the initial prior beliefs $(a_{1,0}, b_{1,0}, a_{2,0}, b_{2,0})$ are given as parameter inputs to the economy.

The equilibrium, recursive expression for the wealth-consumption ratio (PC) is standard in the Epstein-Zin case and is (when $\psi \neq 1$) given by:

$$PC_t^\theta = E_t \left[\beta^\theta e^{(1-\gamma)\Delta c_{t+1}} (PC_{t+1} + 1)^\theta \right], \quad (103)$$

where the subscript t here denotes dependence on information known at time t and $E_t[\cdot]$ denotes the conditional expectation given all information available at time t . We note that the state variables are s_t and X_t , where $X_t \equiv [\tau_{11,t}, \lambda_{1,t}, \tau_{22,t}, \lambda_{2,t}]$ are sufficient statistics for the agent's priors. Further, note from Equations (97) through (102) that we can write $X_{t+1} = f(s_{t+1}, s_t, X_t)$. Given this, we write the recursion equation (Equation (103)) as:

$$\begin{aligned} PC(s_t, X_t)^\theta &= \\ &= \beta^\theta E \left[e^{(1-\gamma)(\mu(s_{t+1}) + \sigma(s_{t+1})\varepsilon_{t+1})} (PC(s_{t+1}, s_t, X_t) + 1)^\theta | s_t, X_t \right] \\ &= \beta^\theta E \left[E \left[e^{(1-\gamma)(\mu(s_{t+1}) + \sigma(s_{t+1})\varepsilon_{t+1})} (PC(s_{t+1}, s_t, X_t) + 1)^\theta | s_{t+1}, s_t, X_t \right] | s_t, X_t \right] \\ &= \beta^\theta E \left[e^{(1-\gamma)\mu(s_{t+1}) + \frac{1}{2}(1-\gamma)^2\sigma(s_{t+1})^2} (PC(s_{t+1}, s_t, X_t) + 1)^\theta | s_t, X_t \right] \\ &= \beta^\theta \sum_{s_{t+1}=1}^2 \Pr(s_{t+1}|s_t, X_t) e^{(1-\gamma)\mu(s_{t+1}) + \frac{1}{2}(1-\gamma)^2\sigma(s_{t+1})^2} (PC(s_{t+1}, s_t, X_t) + 1)^\theta. \end{aligned} \quad (104)$$

where the second to last equality uses the fact that regime changes and the Gaussian shocks to consumption growth (s_{t+1} and ε_{t+1}) are independent. Next, we need to find $\Pr(s_{t+1}|s_t, X_t)$. Denote the conditional density of π_{s_{t+1}, s_t} as $g(\pi_{s_{t+1}, s_t} | s_t, X_t)$. Then:

$$\begin{aligned} \Pr(s_{t+1}|s_t, X_t) &= \int_0^1 \pi_{s_{t+1}, s_t} g(\pi_{s_{t+1}, s_t} | s_t, X_t) d\pi_{s_{t+1}, s_t} \\ &= E[\pi_{s_{t+1}, s_t} | s_t, X_t]. \end{aligned} \quad (105)$$

Given the state variables described above, this expectation equals $\lambda_{s_t, t}$ or $1 - \lambda_{s_t, t}$.

case of Epstein-Zin utility. A similar approach has also been used by Vasquez-Grande (2009).

Thus, the numerical backward recursion will be as follows:

$$PC(s_t, X_t)^\theta = \beta^\theta \sum_{s_{t+1}=1}^2 E[\pi_{s_{t+1}, s_t} | s_t, X_t] e^{(1-\gamma)\mu(s_{t+1}) + \frac{1}{2}(1-\gamma)^2 \sigma(s_{t+1})^2} (PC(s_{t+1}, s_t, X_t) + 1)^\theta, \quad (106)$$

where the boundary values for the wealth-consumption ratio for this backwards recursion are given by the limiting economies at $\tau_{11, \infty}$ and/or $\tau_{22, \infty}$, where π_{11} and/or π_{22} are known. This backward recursion is solved on a grid for λ_1 and λ_2 . It is important to have very dense grid for λ of each state as it approaches 1 for the numerical solution to be accurate. We use 100 grid points for each λ and the exact grid used can be obtained upon request.

11.1 Solving for a dividend claim

Let exogenous dividend growth be given by:

$$\begin{aligned} \Delta d_{t+1} &= \mu + \lambda (\Delta c_{t+1} - \mu) - \frac{1}{2} \sigma_d^2 + \sigma_d \varepsilon_{d, t+1} \\ &= \lambda \Delta c_{t+1} + (1 - \lambda) \mu - \frac{1}{2} \sigma_d^2 + \sigma_d \varepsilon_{d, t+1}, \end{aligned} \quad (107)$$

where $\mu \equiv E(\Pr(s_\infty = 1 | \pi_{11}, \pi_{22})) \mu_1 + E(\Pr(s_\infty = 2 | \pi_{11}, \pi_{22})) \mu_2$ and $\Pr(s_\infty = i)$ is the ergodic probability of being in state i . Thus, the leverage factor λ affects (relatively) short-run movements, whereas the uncertainty about true long-run (unconditional) growth, which is a function of the uncertainty about the transition probabilities, is the same for the dividend claim as for the consumption claim. Note that the long-run mean under the agent's filtration is in fact random and its' $t + 1$ value can be expressed as $\mu(s_{t+1}, s_t, X_t)$. Finally, dividends have an idiosyncratic component given by the standard normal shock ε_d , which is assumed uncorrelated with any other shocks in the economy.

Solving for the price-dividend ratio of this claim is analogous to solving for the consumption claim. Note how the uncertainty about the infinite horizon dividend growth rate is the same as that for infinite horizon consumption growth rate as the exposure of dividend growth to $\mu(s_{t+1}, s_t, X_t)$ is always one, unaffected by the leverage parameter, λ . In particular:

$$PD(s_t, X_t) =$$

$$\begin{aligned}
& \beta^\theta E \left[\begin{aligned} & e^{(\lambda-\gamma)(\mu(s_{t+1})+\sigma(s_{t+1})\varepsilon_{t+1})+(1-\lambda)\mu(s_{t+1},s_t,X_t)} \left(\frac{PC(s_{t+1},X_{t+1})+1}{PC(s_t,X_t)} \right)^{\theta-1} \dots \\ & \times (PD(s_{t+1},s_t,X_t)+1) |s_t, X_t \end{aligned} \right] \\
= & \beta^\theta E \left[E \left[\begin{aligned} & e^{(\lambda-\gamma)(\mu(s_{t+1})+\sigma(s_{t+1})\varepsilon_{t+1})+(1-\lambda)\mu(s_{t+1},s_t,X_t)} \dots \\ & \left(\frac{PC(s_{t+1},X_{t+1})+1}{PC(s_t,X_t)} \right)^{\theta-1} (PD(s_{t+1},s_t,X_t)+1) \end{aligned} \middle| s_{t+1}, s_t, X_t \right] |s_t, X_t \right] \\
= & \beta^\theta E \left[\begin{aligned} & e^{(\lambda-\gamma)\mu(s_{t+1})+\frac{1}{2}(\lambda-\gamma)^2\sigma(s_{t+1})^2+(1-\lambda)\mu(s_{t+1},s_t,X_t)} \left(\frac{PC(s_{t+1},X_{t+1})+1}{PC(s_t,X_t)} \right)^{\theta-1} \dots \\ & \times (PD(s_{t+1},s_t,X_t)+1) |s_t, X_t \end{aligned} \right] \\
= & \beta^\theta \sum_{s_{t+1}=1}^2 \Pr(s_{t+1}|s_t, X_t) \left\{ \begin{aligned} & e^{(1-\gamma)\mu(s_{t+1})+\frac{1}{2}(1-\gamma)^2\sigma(s_{t+1})^2+(1-\lambda)\mu(s_{t+1},s_t,X_t)} \dots \\ & \left(\frac{PC(s_{t+1},X_{t+1})+1}{PC(s_t,X_t)} \right)^{\theta-1} (PD(s_{t+1},s_t,X_t)+1) \end{aligned} \right\}. \quad (108)
\end{aligned}$$

11.2 Limiting economies – boundary values for general case

11.2.1 All parameters known

The simplest limiting economy is given by the case where both π_{11} and π_{22} are known. Since the state is observed and all the parameters are known, s_t is the only state variable and thus the wealth-consumption ratio can only take on two values. Solving this limiting economy amounts to solving two nonlinear equations in two unknowns ($PC(s=1)$ and $PC(s=2)$):

$$PC(s=1)^\theta = \beta^\theta \left(\begin{aligned} & \pi_{11} e^{(1-\gamma)\mu_1 + \frac{1}{2}(1-\gamma)^2\sigma_1^2} (PC(s=1)+1)^\theta \dots \\ & + (1-\pi_{11}) e^{(1-\gamma)\mu_2 + \frac{1}{2}(1-\gamma)^2\sigma_2^2} (PC(s=2)+1)^\theta \end{aligned} \right), \quad (109)$$

$$PC(s=2)^\theta = \beta^\theta \left(\begin{aligned} & (1-\pi_{22}) e^{(1-\gamma)\mu_1 + \frac{1}{2}(1-\gamma)^2\sigma_1^2} (PC(s=1)+1)^\theta \dots \\ & + \pi_{22} e^{(1-\gamma)\mu_2 + \frac{1}{2}(1-\gamma)^2\sigma_2^2} (PC(s=2)+1)^\theta \end{aligned} \right). \quad (110)$$

These equations are relatively straightforward to solve numerically, imposing the requirement ex ante that the wealth-consumption ratio is positive and real. In our case, it was possible to verify that there is only one economically reasonable solution by plotting the function space for the grid for the π 's, given the other parameters as assumed in the main paper.

We solve these limiting equations for a grid on π_{11} and π_{22} with lower and upper bounds set to the 0.01% and 99.9% percentile values of the initial prior distribution for the general case we ultimately want to solve for, as given by $a_{i,0}$ and $b_{i,0}$, $i \in \{1, 2\}$.

11.2.2 One transition probability known, one unknown

Another set of boundary economies are given by the case where one of the transition probabilities is known and the other is unknown. This case corresponds to $\tau_{11,t} < \infty$, $\tau_{22,\infty}$ or $\tau_{11,\infty}$, $\tau_{22,t} < \infty$. We can find the wealth-

consumption for these cases using the backward induction as given by Equation (106):

$$PC(s_t, X_t)^\theta = \beta^\theta \sum_{s_{t+1}=1}^2 E[\pi_{s_{t+1}, s_t} | s_t, X_t] e^{(1-\gamma)\mu(s_{t+1}) + \frac{1}{2}(1-\gamma)^2 \sigma(s_{t+1})^2} (PC(s_{t+1}, s_t, X_t) + 1)^\theta, \quad (111)$$

where for the transition probability whose value is known, trivially $E[\pi_{s_{t+1}, s_t} | s_t, X_t] = \pi_{s_{t+1}, s_t}$. For instance, if π_{11} is known, then $E[\pi_{11} | s_t, X_t] = \pi_{11}$ and $E[1 - \pi_{11} | s_t, X_t] = 1 - \pi_{11}$ for all t . Also, in this case we have that $X_t = [\tau_{22,t}, \lambda_{2,t}]$.

From these boundary values, we iterate backwards in time (which here is a 2-dimensional concept, as we are recording time spent in each regime) to find the solution for the finite τ 's we ultimately are interested in.

11.3 Existence of equilibrium

The existence proof relies on the concavity of the value function and the fact that the value function is finite in all the boundary, known parameter cases. In particular, the value function is bounded above by either, depending on whether the value of the EIS is above or below 1, the case where there is only the good state ($\pi_{11} = 1$ and $\pi_{22} = 0$, and economy starts in good state), or the case where there is only the bad state ($\pi_{11} = 0$ and $\pi_{22} = 1$ and economy starts in bad state). Thus, the existence condition amounts to checking existence for cases of i.i.d. normal log consumption growth with known mean and variance. A similar approach can be used to prove existence for all the cases considered in this paper.

12 Unknown mean growth rate and variance of shocks in Depression state of 2-state regime switching model

Aggregate consumption growth is given by:

$$\Delta c_{t+1} = \mu(s_{t+1}) + \sigma(s_{t+1}) \varepsilon_{t+1}, \quad (112)$$

where $\varepsilon_{t+1} \stackrel{i.i.d.}{\sim} N(0, 1)$ and where $s_t \in \{1, 2\}$ follows a 2-state observable Markov chain with constant transition probabilities:

$$\Pi = \begin{bmatrix} \pi_{11} & 1 - \pi_{11} \\ 1 - \pi_{22} & \pi_{22} \end{bmatrix}, \quad (113)$$

with $\pi_{ii} \in (0, 1)$. The regime changes are assumed to be independent of the Gaussian shocks.

In this case, the transition probabilities are assumed known, as are the mean and volatility parameters in the good state, but the mean and volatility parameters of the bad state, μ_2 and σ_2 , are assumed unknown. As discussed in the main text, the conjugate prior for μ_2 and σ_2 is the Normal-Inverse-Gamma. In particular,

$$p(\mu_2, \sigma_2^2 | \Delta c^\tau) = p(\mu_2 | \sigma_2^2, \Delta c^\tau) p(\sigma_2^2 | \Delta c^\tau), \quad (114)$$

where

$$p(\sigma_2^2 | \Delta c^\tau) \sim IG\left(\frac{b_\tau}{2}, \frac{B_\tau}{2}\right), \quad (115)$$

$$p(\mu_2 | \sigma_2^2, \Delta c^\tau) \sim N(a_\tau, A_\tau \sigma_2^2), \quad (116)$$

and where τ counts time spent in state 2 and where Δc^τ denotes the history of consumption growth realizations in the Depression state. Obviously, there is no learning about μ_2 and σ_2^2 from consumption growth in the good state. Given that log consumption growth is normally distributed, these prior beliefs lead to posterior beliefs that are of the same form (conjugate priors). The updating equations for investors' beliefs are:

$$A_{\tau+1}^{-1} = 1 + A_\tau^{-1}, \quad (117)$$

$$\frac{a_{\tau+1}}{A_{\tau+1}} = \frac{a_\tau}{A_\tau} + \Delta c_{\tau+1}, \quad (118)$$

$$b_{\tau+1} = b_\tau + 1, \quad (119)$$

$$B_{\tau+1} = B_\tau + \frac{(\Delta c_{\tau+1} - a_\tau)^2}{1 + A_\tau}. \quad (120)$$

In terms of pricing, note that this system can be reduced to three state-variables: a_τ , B_τ , and τ , given initial priors. We solve the model numerically and, as before, use the closed-form solution for the known parameters cases as the boundary values in a recursion that is solved backwards in time, where time again is counted in terms of time spent in the Depression state, τ , on a grid for a_τ and B_τ . Both distributions have to be truncated in order to ensure existence of equilibrium.

Define the state vector $X_t = [a_\tau, B_\tau]$ as the state-variables in the economy with the exception of time in state 2, τ , and the state itself, s_t . The equilibrium recursion used to solve the model is then:

$$\begin{aligned} PC(s_t, X_\tau | s_t = 2)^\theta &= \beta^\theta \Pr(s_{t+1} = 2 | s_t = 2, X_t^s) \times \\ &\dots E \left[e^{(1-\gamma)\Delta c_{t+1}} (PC(s_{t+1}, s_t, X_{\tau+1}) + 1)^\theta | s_{t+1} = 2, s_t = 2, X_t \right] + \\ &\dots \beta^\theta \Pr(s_{t+1} = 1 | s_t = 2, X_t^s) \times \\ &\dots E \left[e^{(1-\gamma)\Delta c_{t+1}} (PC(s_{t+1}, s_t, X_\tau) + 1)^\theta | s_{t+1} = 1, s_t = 2, X_t \right]. \end{aligned} \quad (121)$$

$$\begin{aligned} PC(s_t, X_\tau | s_t = 1)^\theta &= \beta^\theta \Pr(s_{t+1} = 2 | s_t = 1, X_t^s) \times \\ &\dots E \left[e^{(1-\gamma)\Delta c_{t+1}} (PC(s_{t+1}, s_t, X_{\tau+1}) + 1)^\theta | s_{t+1} = 2, s_t = 1, X_t \right] + \\ &\dots \beta^\theta \Pr(s_{t+1} = 1 | s_t = 1, X_t^s) \times \\ &\dots E \left[e^{(1-\gamma)\Delta c_{t+1}} (PC(s_{t+1}, s_t, X_\tau) + 1)^\theta | s_{t+1} = 1, s_t = 1, X_t \right]. \end{aligned} \quad (122)$$

Note that $PC(s, X_\tau)$ appears in both sides of each equation as there is no updating of beliefs if the economy is in state 1 next period. Thus, the numerical recursion involves also solving the nonlinear 2 equations, 2 unknowns problem

implicit in these equations. We numerically integrate over the uncertainty in the mean and volatility parameters, as well as for the shock to consumption . In particular, we use quadrature weights for the truncated normal for μ_2 as well as for the inverse gamma for σ_2 , in addition to standard quadrature weights for the normal shock, ε . The price-dividend ratio for the claim to the exogenous dividend stream is found analogously given the solution for the price-consumption ratio (as in the earlier cases).

**APPENDIX: Table 5 - 100 year sample moments
Learning about the mean and variance in a Great Depression**

Table 5: This table gives average sample moments from 20,000 simulations of 400 quarters of data from the 2-state switching regime model of consumption growth, where the mean and/or variance parameters μ_2 and σ_2^2 are unknown. The bad state is calibrated to correspond to the U.S. consumption data over the Great Depression, as explained in the main text. $E_T[x]$ denotes the average sample mean of x , $SR_T[x]$ denotes the average sample Sharpe ratio of x , and $\sigma_T[x]$ denotes the average sample standard deviation of x . R_m and R_f denote the gross market return and real risk-free rate. Lower case letters denote log of upper case variable. All statistics are annualized and, except for the Sharpe ratio, given in percent. In all cases, the time-preference parameter β is set to 0.994. The relative risk aversion and intertemporal elasticity of substitution are given for each panel. The 'data' column shows the historical excess market return moments for the U.S. from 1929 to 2011, as given in CRSP. The real risk-free rate moments are taken from a similar sample as reported in Bansal and Yaron (2004).

Panel A: Learning about the disaster mean											
$\psi = 2, \gamma = 3.9$ Data											
	$T_0 = 100\text{yrs}$	$T_0 = 200\text{yrs}$	$T_0 = 300\text{yrs}$	$T_0 = \infty$	$\psi = 2, \gamma = 5$	$T_0 = 100\text{yrs}$	$T_0 = 200\text{yrs}$	$T_0 = 300\text{yrs}$	$T_0 = \infty$		
$E_T[r_m - r_f]$	5.10	1.80	1.45	1.33	1.05	4.95	3.92	3.57	2.74		
$\sigma_T[r_m - r_f]$	20.21	15.84	15.50	15.36	14.94	15.98	15.78	15.69	15.41		
$SR_T[R_M - R_f]$	0.36	0.19	0.17	0.16	0.14	0.37	0.31	0.29	0.24		
$E_T[r_f]$	0.86	2.75	2.89	2.92	2.98	1.61	2.15	2.31	2.60		
Panel B: Learning about the disaster variance											
$\psi = 2, \gamma = 3.9$ Data											
	$T_0 = 100\text{yrs}$	$T_0 = 200\text{yrs}$	$T_0 = 300\text{yrs}$	$T_0 = \infty$	$\psi = 2, \gamma = 5$	$T_0 = 100\text{yrs}$	$T_0 = 200\text{yrs}$	$T_0 = 300\text{yrs}$	$T_0 = \infty$		
$E_T[r_m - r_f]$	5.10	1.05	1.05	1.05	1.05	2.74	2.74	2.74	2.74		
$\sigma_T[r_m - r_f]$	20.21	14.94	14.94	14.94	14.94	15.41	15.41	15.41	15.41		
$SR_T[R_M - R_f]$	0.36	0.14	0.14	0.14	0.14	0.24	0.24	0.24	0.24		
$E_T[r_f]$	0.86	2.98	2.98	2.98	2.98	2.60	2.60	2.60	2.60		
Panel C: Learning about the disaster mean and variance											
$\psi = 2, \gamma = 3.9$ Data											
	$T_0 = 100\text{yrs}$	$T_0 = 200\text{yrs}$	$T_0 = 300\text{yrs}$	$T_0 = \infty$	$\psi = 2, \gamma = 5$	$T_0 = 100\text{yrs}$	$T_0 = 200\text{yrs}$	$T_0 = 300\text{yrs}$	$T_0 = \infty$		
$E_T[r_m - r_f]$	5.10	1.83	1.45	1.33	1.05	5.13	3.94	3.57	2.74		
$\sigma_T[r_m - r_f]$	20.21	15.83	15.48	15.35	14.94	15.97	15.77	15.68	15.41		
$SR_T[R_M - R_f]$	0.36	0.19	0.17	0.16	0.14	0.38	0.31	0.29	0.24		
$E_T[r_f]$	0.86	2.68	2.87	2.92	2.98	1.35	2.11	2.30	2.60		