firms without commitment are non-zero. In contrast, the credit spreads for almost zero leverage firms are zero for the other two benchmark cases.

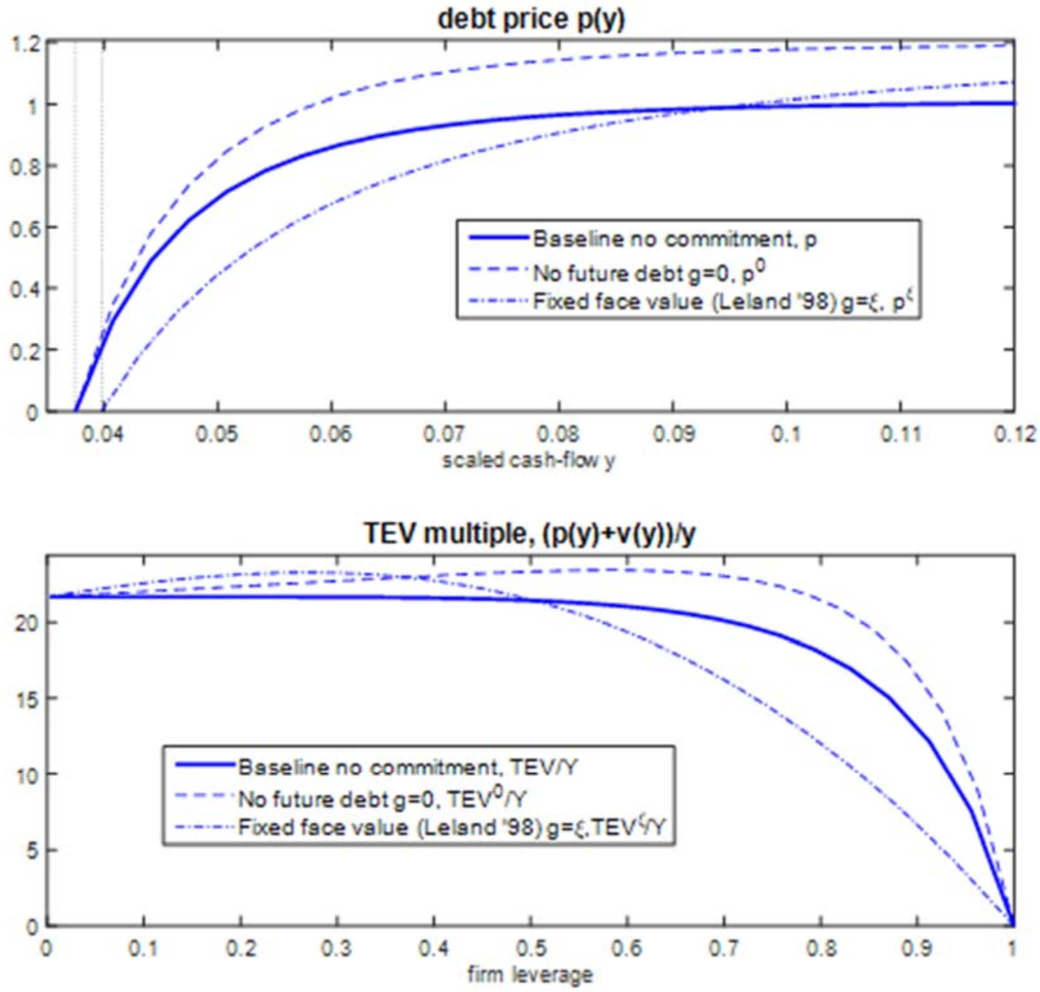

Figure 5: Debt Price and TEV Multiple for Alternative Debt Issuance Policies

Parameters are $r = 5\%$, $c = 8\%$, $\xi = 0.1$, $\mu = 2\%$, $\sigma = 25\%$, $\overline{\pi} = 35\%$.

Relative to our base case, the fixed face value (Leland 1998) case generates a lower debt price for low y but higher debt price for high y. This is due to the endogenous issuance policy g^* plotted in Figure 4. There, we observe that the debt issuance policy without commitment is increasing in y, and slower (faster) than the fixed face value policy when y is low (high), and investors price the debt in anticipation of these future leverage polices.

The next proposition summarizes the comparison of debt values across three models, depending on the firm's profitability state y. Figure 5 corresponds to the case of $y_b^\xi > y_b$, so for sufficiently low y, the debt price in the case of fixed face value p^ξ drops below the other two cases.

PROPOSITION 8. We always have $p(y) < p^0(y)$. For $y \to \infty$ we have

$$p(y) < p^\xi(y) < p^0(y)$$

For sufficiently low y so that $y \to \min(y_b, y_b^\xi)$, we have

$$\begin{aligned} &p^\xi(y) < p(y) < p^0(y) \text{ if } y_b^\xi > y_b \\ &p(y) < p^0(y) < p^\xi(y) \text{ if } y_b^\xi < y_b \end{aligned}.$$

PROOF: $p(y) < p^0(y)$ is implied by (36). When $y \to \infty$, $p^\xi(y) = \frac{c+\xi}{r+\xi}\left(1-\left(\frac{y}{y_b^\xi}\right)^{-\gamma_p^\xi}\right) \to \frac{c+\xi}{r+\xi}$ which exceeds $p(y) = \frac{(1-\bar{\pi})c+\xi}{r+\xi}\left(1-\left(\frac{y}{y_b}\right)^{-\gamma}\right) \to \frac{(1-\bar{\pi})c+\xi}{r+\xi}$. To show that $p^\xi(y) < p^0(y)$, we need to show that $\left(\frac{y}{y_b}\right)^{-\gamma} < \left(\frac{y}{y_b^\xi}\right)^{-\gamma_p^\xi}$ holds when $y \to \infty$, which is equivalent to $\gamma_p^\xi < \gamma$; but the latter holds by comparing (39) and (23). The second part of result is obvious as the debt price drops to zero at the default boundary. ■

Finally, as indicated in (29), in our no commitment case the firm's TEV multiple is strictly increasing in the scaled cash flow y. Consequently, holding the level of cash flows Y fixed, total firm value decreases with the debt face value F. In other words, in the no commitment equilibrium there is always a loss to total firm value from leverage – the tax benefits of debt more than offset the resulting bankruptcy costs due to future debt increases. This result is shown in the TEV-multiple-against-leverage plot in the bottom panel of Figure 5: there, the solid line (i.e., the no commitment case) achieves its maximum at zero leverage. In contrast, TEV multiples in both commitment cases have an interior maximum.[17] This interior maximum is often viewed as the

[17] This is evident in the bottom panel of Figure 5, as both dashed and dash-dotted lines have a positive slope at zero leverage, and drop to zero when leverage reaches 100%.

"optimal" leverage in the traditional trade-off theory, though of course in a dynamic context it may be far from optimal ex-post once shocks are realized (see, e.g., Fischer, Heinkel, Zechner (1989) and Strebulaev (2007)).

A Comparison of Leverage Dynamics

In two benchmark cases with commitment, the scaled cash-flows follow a geometric Brownian motion with exogenous drifts, i.e., $dy_t/y_t = (\mu+\xi)dt + \sigma dZ_t$ for the case of no future debt issuance, and $dy_t/y_t = \mu dt + \sigma dZ_t$ for the fixed face value case. In our model with no commitment, the equilibrium evolution of the firm's scaled cash-flows is:

$$dy_t = \left(\mu+\xi-g^*(y_t)\right)y_t dt + \sigma y_t dZ_t = \left[\mu+\xi-\frac{(r+\xi)\bar{\pi}cy_t^{\gamma}}{\gamma\left((1-\bar{\pi})c+\xi\right)y_b^{\gamma}}\right]y_t dt + \sigma y_t dZ_t. \quad (43)$$

The equilibrium debt issuance policy $g^*(y_t)$ in (28) is increasing in y_t, implying that y_t grows slower when y_t is higher. In fact, the firm's scaled cash-flows are mean-reverting towards the steady-state value (recall the definition of $\hat{y}_g$ in (30)).[18]

$$\hat{y}_{g=\mu+\xi} \equiv y_b\left(\frac{\gamma(\mu+\xi)\left(c(1-\bar{\pi})+\xi\right)}{(r+\xi)\bar{\pi}c}\right)^{1/\gamma} \quad (44)$$

[18] Strictly speaking, to ensure y_t to mean revert over its equilibrium region $y_t \in [y_b,\infty)$, one has to show that $\hat{y}_{\mu+\xi} > y_b$ so that the drift of y_t is positive when $y_t = y_b$, which is indeed the case for our baseline parameters. However, $\hat{y}_{\mu+\xi} < y_b$ could occur for sufficiently large σ (so $\gamma \to 0$ in (44)).

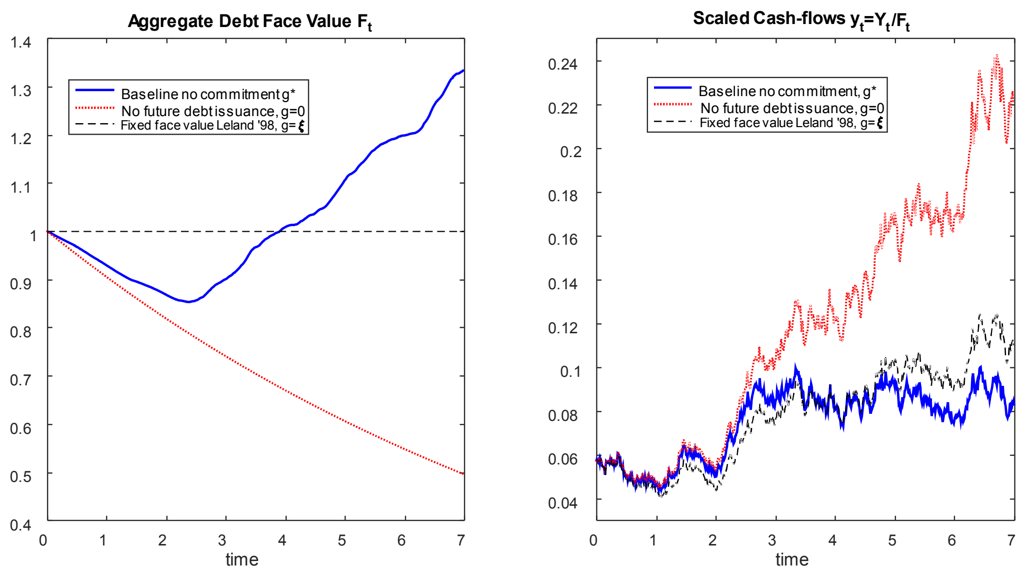

Figure 6: Aggregate debt face values (left panel) and scaled cash-flows dynamics (right panel) for three models.

With fixed cash-flow shocks $\{dZ\}$ and $r = 5\%$, $c = 8\%$, $\xi = 0.1$, $\mu = 2\%$, $\sigma = 25\%$, $\bar{\pi} = 35\%$.

We are interested in the firm leverage dynamics implied by three different models. For given underlying cash-flow shocks $\{dZ_t\}$, the left panel of Figure 6 plots the debt face value F_t, while the right panel shows the dynamics of scaled cash-flow y_t, which tracks one-to-one to the firm's interest-coverage-ratio (or book leverage). Because the underlying shocks are the same, the differences across these three different models are purely due to their different debt issuance policies. In this sample path, negative shocks in the early years cause the firm in our baseline no commitment case to issue less debt compared to the fixed face value case which commits to $g = \xi$, but of course since $g^* > 0$ the firm has more debt than it would in the no issuance case. Later, when the firm has positive shocks, the firm issues debt even faster than it matures and the debt level grows. As a result, y_t in the no commitment case (blue solid line) has a larger upward drift initially, but this reverses near the end of the sample path.

3.6. Debt Maturity Structure

In our model, the firm commits to a constant debt maturity structure, i.e., all debt has an expected maturity of $1/\xi$. This assumption is common in much of the dynamic capital structure literature which treats the debt maturity structure as a parameter.[19] It is beyond the scope of this paper to relax the full commitment assumption on the firm's debt maturity structure policy; for some recent research, see He and Milbradt (2016).

We can consider, however, the consequences of the initial maturity choice. What if the firm can choose the debt maturity ξ to commit to for the future? In this section, we first show if the firm does not need to borrow a fixed amount upfront, then it is indifferent between all debt maturity choices. We then show that if the firm is restricted to borrow a fixed amount initially, then debt with shortest maturity becomes optimal. Finally, we analyze the role of leverage commitment with ultra-short-term debt which matures instantaneously.

Optimal Debt Maturity without Fixed Borrowing

We have seen that from (29) that the firm's TEV multiple is decreasing in the scaled cash-flows $y = Y / F$. Consequently, value-maximizing firms should set the optimal initial debt face value to be $F_0^* = 0$. Given this choice, (29) implies that the firm's TEV multiple no longer depends on the debt maturity structure ξ. This indifference result is deeper than it appears: Although the firm starts with no initial debt, recall that (32) says the firm will begin issuing debt immediately, and the debt maturity ξ does affect those future debt contracts. Nevertheless, in our model this dynamic consideration has no bite on the optimal maturity choice by equity holders: Although different maturity structures lead to very different future leverage dynamics, any gains from tax savings are offset by increased bankruptcy costs.

In fact, this irrelevance result is fairly general in our model. Imagine the following thought experiment, in which equity holders -- facing the state pair (Y_t, F_t) and the existing debt maturity structure ξ -- are offered with a one-time chance of choosing ξ' for the firm's future debt. That is, the firm's existing debts continue to retire at the old speed ξ, but the newly issued debts are with the new maturity and hence will retire at the new speed ξ'. The following proposition shows that equity holders are indifferent among all choices of ξ'.

[19] To mention a few, Leland (1998), Leland and Toft (1996), He and Xiong (2012), and Diamond and He (2014).

Proposition 9. In a no-commitment equilibrium with smooth debt issuance, the firm's equity value is independent of the maturity structure ξ' of new debt.

Proof: For equilibria in which equity holders are taking smooth debt issuance polices, equity holders obtain zero profit by issuing future debt, and their value will be the same as if equity does not issue any future debt. As a result, the equity value depends on the maturity structure ξ of existing debt, but not on the maturity structure ξ' of future debt. ■

The logic of Proposition 9 and hence the indifference result can be further generalized to a setting in which the firm is free to choose any maturity structure for its newly issued debt any time. Again, the equity value only depends on the maturity structure of existing debt.

Note, however, that while equity holders are indifferent between alternative maturity structures, different maturity choices will lead to very different patterns and levels of future leverage. For example, **Figure 7** shows the total enterprise value, debt amount, and leverage (debt value/TEV) for the firm given identical productivity shocks, but financed either using five-year or one-year debt. In both cases the initial TEV and equity value is the same, but leverage evolves very differently. With longer-term debt, debt changes gradually, as the firm issues debt more slowly, and leverage is lower on average. With shorter-term debt, the firm issues debt more rapidly knowing it can decrease debt quickly by not rolling over maturing debt. Because it can adjust debt more quickly, the firm has higher leverage on average.[20]

[20] Of course, in our model we have assumed away transactions costs associated with issuing or rolling over debt. Such considerations would make long-term debt less costly, as in Dangl and Zechner (2016).

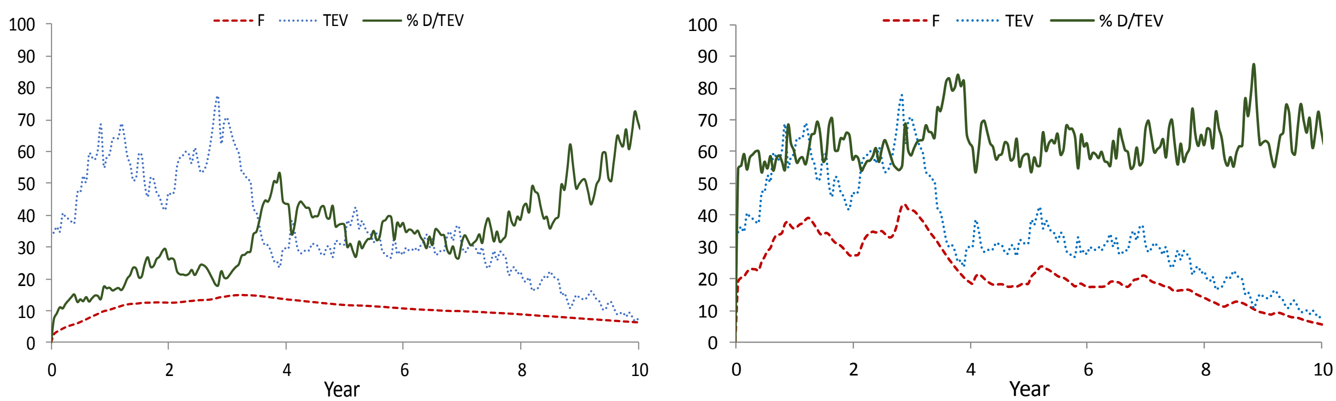

Figure 7: Debt and Leverage with Differing Maturities.
Left panel shows TEV, debt face value, and market leverage with 5-year average debt maturity. Right panel shows 1-yr average debt maturity. In either case, initial firm value is unchanged, but leverage is higher and adjusts more quickly with shorter-term debt. Parameters are $\mu = 2\%$, $\sigma = 40\%$, $\bar{\pi} = 30\%$, $c(1-\bar{\pi}) = r = 5\%$, $\xi = 0.2$ (5-year debt) or 1 (1-year debt).

Optimal Debt Maturity with Fixed Borrowing

Suppose the firm *must* raise some amount of funds initially through debt. Issuing a discrete amount of debt in our model is not optimal – shareholders would be better off issuing debt gradually – but suppose the firm must raise funds quickly and equity capital is not available in the short run. In that case we can show that short-term debt maximizes not only the firm value, but also the debt capacity, i.e., the maximum amount of debt that the firm can raise.

Our model highlights one advantage of short-term debt which allows firms to adjust their leverage burden in response to fundamental shocks in a more flexible way. This point has been neglected in the Leland-style literature which often assumes the firm is committing to a future leverage policy with fixed debt face value. For instance, He and Xiong (2012) show that the longest possible debt maturity structure minimizes the rollover risk. As we will explain, the difference is driven by different assumptions on leverage policies.

Given initial cash-flow Y_0, the firm sets the initial debt face value F_0 to raise D_0 from debt holders. From (29) we know that the firm value is (recall $y_0 = Y_0 / F_0$)

$$Y_0 \frac{v(y_0)+p(y_0)}{y_0} = Y_0 \frac{1-\bar{\pi}}{r-\mu}\left[1-\left(\frac{y}{y_b}\right)^{-\gamma-1}\right] < Y_0 \frac{1-\bar{\pi}}{r-\mu} \tag{45}$$

Hence, both the firm value TEV_0 and the debt value D_0 cannot exceed the upper bound $Y_0 \frac{1-\bar{\pi}}{r-\mu}$.

PROPOSITION 10. Suppose the firm starts with initial cash flows Y_0.

i) For any target debt value $D_0 < Y_0 \frac{1-\bar{\pi}}{r-\mu}$, the optimal debt maturity structure that maximizes the levered firm value (and hence the equity value) is $\xi^* = \infty$.

ii) The debt capacity $\sup D_0$, which is the highest debt value that the firm is able to raise, equals $Y_0 \frac{1-\bar{\pi}}{r-\mu}$ by setting $F_0 \uparrow \frac{Y_0}{y_b}$ and $\xi^* \to \infty$.

PROOF: The first claim follows by showing that $\xi = \infty$ always achieves the upper bound of $Y_0 \frac{1-\bar{\pi}}{r-\mu}$. Because

$$\gamma = \frac{\mu + \xi - 0.5\sigma^2 + \sqrt{\left(0.5\sigma^2 - \mu - \xi\right)^2 + 2\sigma^2 \left(r + \xi\right)}}{\sigma^2} \to \infty \text{ as } \xi \to \infty$$

So given $y > y_b$ we have $\left(y / y_b\right)^{-\gamma-1}$ to vanish in (45) as $\xi \to \infty$, which proves the claim. For the second claim, from (26) we have

$$D_0 = F_0 p\left(\frac{Y_0}{F_0}\right) = F_0 \frac{c\left(1-\bar{\pi}\right)+\xi}{r+\xi}\left(1-\left(\frac{y_0}{y_b}\right)^{-\gamma}\right)$$

By setting $\xi \to \infty$ the term in the parentheses vanishes, while $F_0 \uparrow \frac{Y_0}{y_b}$ delivers the upper bound debt value $D_0 \uparrow Y_0 \frac{1-\bar{\pi}}{r-\mu}$. ■

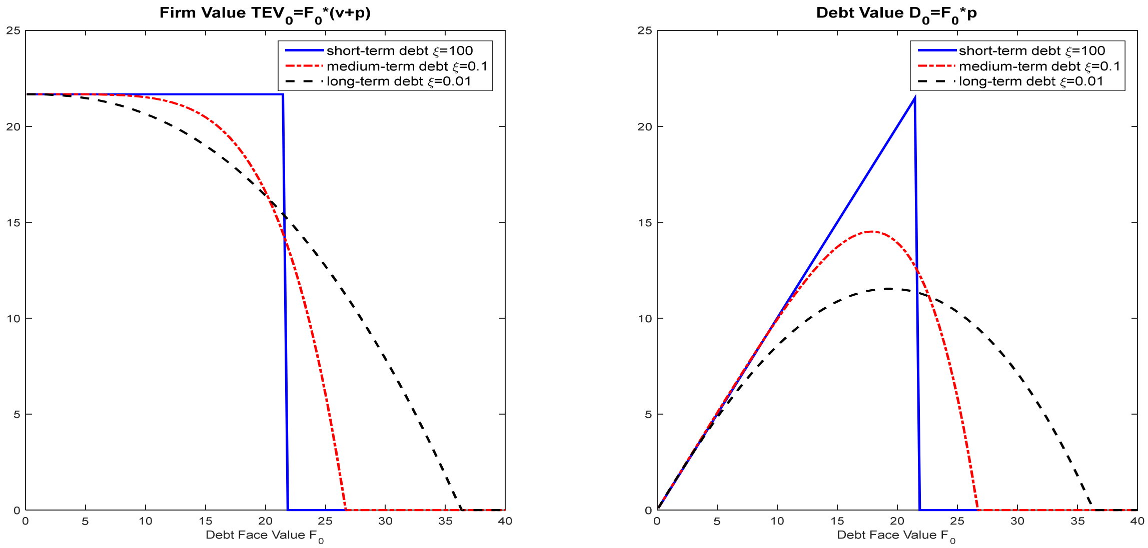

Figure 8. Firm value and debt value as a function of initial debt face value F_0 , for three levels of debt retiring rate ξ 's. Blue solid line is for shortest debt maturity $1/\xi = 0.01$, red dash-dotted line is for medium debt maturity $1/\xi = 10$, and black dashed line is for longest debt maturity $1/\xi = 100$ Both firm and debt values hit zero when $F_0 = Y_0 / y_b$. Parameters are $r = 5\%$, $c = 8\%$, $\xi = 0.1$, $\mu = 2\%$, $\sigma = 25\%$, $\bar{\pi} = 35\%$.

Figure 8 illustrates **PROPOSITION 10** by plotting the firm value TEV_0 (left panel) and debt value D_0 (right panel), both as a function of face value F_0 . We consider three debt maturities: long-term debt with a 100-year maturity ($\xi = 0.01$, dash-dotted), medium-term debt with a 10-year maturity ($\xi = 0.1$, dashed), and short-term debt with 3-day maturity ($\xi = 100$, solid). The left panel shows that the firm value is maximized by using 3-day maturity debt. This is simply because $\xi \to \infty$ implies that $\gamma \to \infty$, and hence for any $y > y_b$ the firm value achieves its upper bound $Y_0 \dfrac{1-\bar{\pi}}{r-\mu}$ in (45). Of course, a too high F_0 pushes the scaled cash-flow y_0 below y_b , triggering default---and the firm value drops to zero. Although firms with the shortest-term debt---with the highest default threshold y_b ---face the tightest constraint in setting a high F_0 ,[21] the right panel of Figure 8 shows that they achieve the highest market value of debt, thanks to the "Laffer" curve effect: Because long-term debt makes it difficult for the firm to reduce leverage in response to shocks, at some point an increase in the face value is more than offset by the increase in credit risk.

[21] Under a mild sufficient condition $r > c(1-\bar{\pi})$, one can show that the firm defaults earlier with shorter-term debt, i.e., default threshold y_b in (24) is increasing in debt maturity ξ . This is a general result in Leland-style models.

As shown, the upper bound of D_0, $Y_0 \frac{1-\overline{\pi}}{r-\mu}$, is achieved by setting F_0 at (slightly below) the default boundary for firms using "overnight" debt.

Our model differs from Leland type models in the firm's future leverage policy. In both settings, the debt maturity structure ξ captures the speed of debt retiring. In Leland, committing to refinance those retiring bonds (to maintain the aggregate debt face at a constant) leads equity to bear rollover gains/losses $\xi\left(p^{\xi}\left(y\right)-1\right)$ -- see Equation (40). When the debt price is low (because leverage is already high), rollover losses lower the option value to equity holders of keeping the firm alive and thus they will default earlier, creating rollover risk. The higher the ξ, the stronger the rollover risk.[22] To mitigate this rollover risk, a value-maximizing firm will set $\xi = 0$, corresponding to a consol bond that is free of rollover concerns.

When the firm can change its future debt burden freely, there emerges an important benefit of short-term debt. Given that short-term debt retires quickly, it allows the firm to swiftly adjust its leverage in response to profitability shocks. This result follows from the firm's mean-reverting leverage dynamics illustrated in Section 3.3, as (28) implies that the firm issues more (less) debt following positive (negative) performance.

It is worth noting that, typically, the flexibility brought on by short-term debt comes with a potential cost of lack of commitment;[23] in our particular setting the former benefit dominates the latter cost. We also caution that **PROPOSITION 10** heavily relies on a strong, and perhaps counterfactual, assumption that underlies all of these models: the firm is facing a frictionless equity market through which equity could inject liquidity at any time in a costless way. Following a sequence of negative shocks, to repay the mounting debts that are maturing instantaneously, firms issue equity as needed. This assumption allows a firm with very short-term debt to hold high leverage (as in the right panel of Figure 7) with out risking a liquidity induced default. We expect that modelling an equity market with realistic frictions will change many of these qualitative results, an interesting question left for future research.

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[22] See He and Xiong (2012) and Diamond and He (2014).

[23] For instance, long-term debt, because of its slow retiring speed, could potentially serve as a commitment device of leverage policy. He and Milbradt (2016) study a firm who cannot commit to debt maturity structure, and gives an example where the cost of lacking commitment dominates the benefit of flexibility.

Ultra-Short-term Debt and Commitment

The previous section shows that in our model it is optimal to set the debt maturity to be ultra-short-term, so that the debt matures instantaneously every dt (much like demand deposits).

In the literature, some papers (e.g. Tserlukevich, 2008) suggest that when the firm can adjust its leverage freely in response to the cash-flows shocks, then it is optimal to set $F_t \approx Y_t / c$ always so that the firm avoids default while capturing the entire debt tax shield. In particular, it seems that ultra-short-term debt, which at any moment matures entirely and allows the firm to reset its leverage to its favorable level in response to cash-flow shocks, can achieve this goal. For instance, setting issuance policy $d\Gamma_t$ as

$$d\Gamma_t = (\mu + \xi) F_t dt + \sigma F_t dZ_t \tag{46}$$

could potentially prevent the scaled cash flow y_t from fluctuating over time. Essentially, this captures the flexibility advantage offered by short-term debt.

Although the flexibility benefit does apply in our model, the above argument implicitly assumes that equity holders can commit to the first best leverage policy. In our model, the inability to commit to certain future leverage policy matters in a significant way – equity holders continue to raise debt until the likelihood of default impacts its price. This point is highlighted in (46) that the firm repurchases debt following negative cash-flow shocks $dZ_t < 0$, while in our model shareholders never find debt repurchases optimal. In the limit, even with instantaneously maturing debt, there is always a risk of bankruptcy in our model, so that the implied bankruptcy cost offsets the tax benefit.

4. Endogenous Investment and Debt Overhang

In this setting we extend our model by adding an endogenous investment decision also under the control of shareholders. Including investment allows us to explore the interaction of shareholder-creditor conflicts with regard to both investment and leverage choices. As expected from Myers (1977), debt overhang leads firms to underinvest. We show that when shareholders are unable to commit to future leverage decisions, debt overhang is more severe when profitability is high, but less severe when profitability is low, than when the debt level is fixed as in the Leland

(1998) case. In addition, we show that when shareholders cannot commit to future investment decisions, the leverage ratchet effect becomes more severe, and the firm will issue debt more rapidly, than the benchmark case in which investment is fixed.

4.1. General Analysis

Now we allow equity holders to choose the firm's endogenous investment policy i_t, which affects the drift of cash-flow process by $\mu\left(Y_t, i_t\right)$ at a cost of $K\left(Y_t, i_t\right)$. Both functions are smooth with $\mu_i\left(Y_t, i_t\right) > 0$, $K_i\left(Y_t, i_t\right) > 0$ and $K_{ii}\left(Y_t, i_t\right) > 0$.

For illustration purposes, we carry out the general analysis with investment under the general cash-flow diffusion process without jumps; the analysis for a cash-flow process as in (1) with jumps is similar.

The HJB equation for equity holders, who are choosing the firm's debt issuance G and investment i, can be written as

$$\underbrace{rV\left(Y,F\right)}_{\text{required return}} = \underbrace{\left(1-\pi\right)Y - \left(\left(1-\pi\right)c+\xi\right)F}_{\text{after-tax cash-flows net coupon-principal payment}} +$$

$$\max_{G,i}\left[\underbrace{p\left(Y,F\right)G + (G-\xi F)V_F\left(Y,F\right)}_{\text{evolution of } dF} + \underbrace{\mu(Y,i)V_Y\left(Y,F\right) + \frac{\sigma(Y)^2}{2}V_{YY}\left(Y,F\right)}_{\text{evolution of } dY} - \underbrace{K\left(Y,i\right)}_{\text{investment cost}}\right]$$

As before we focus on the equilibrium where G takes interior solutions, which implies that $p = -V_F$, and we can solve for the equity value function as if there is no issuance, i.e., $G = 0$. Suppressing the arguments for $V\left(Y,F\right)$, the HJB equation with $G = 0$ becomes

$$rV = \max_i\left[\left(1-\pi\right)Y - \left(\left(1-\pi\right)c+\xi\right)F - \xi F V_F - K\left(Y,i\right) + \mu(Y,i)V_Y + \frac{1}{2}\sigma(Y)^2 V_{YY}\right] \tag{47}$$

The first-order condition for the optimal investment policy i^* is

$$K_i\left(Y,i^*\right) = \mu_i\left(Y,i^*\right)V_Y \tag{47}$$

This condition characterizes the optimal investment policy i^* under the assumption that $\mu(Y,i)V_Y - K\left(Y,i\right)$ is strictly concave in i.

The equilibrium issuance policy can be derived as before. Plugging the optimal investment policy i^* into (47), and taking derivative with respect to F further, we have

$$rV_F = -\left(\left(1-\bar{\pi}\right)c+\xi\right)-\xi V_F - \xi F V_{FF} + \mu(Y,i^*)V_{YF} + \frac{1}{2}\sigma(Y)^2 V_{YYF}$$

As before, we have used the Envelope theorem that we can ignore the dependence of the optimal policy i^* on F.[24] Using $p=-V_F$, we have

$$-rp\left(Y,F\right) = -((1-\bar{\pi})c+\xi)-\mu(Y,i^*)p_Y\left(Y,F\right)+\xi p\left(Y,F\right)+\xi F p_F\left(Y,F\right)-\tfrac{\sigma(Y)^2}{2}p_{YY}\left(Y,F\right) \qquad (47)$$

The valuation equation for debt price $p\left(Y,F\right)$ in (5), with the equilibrium evolution of state variables $\left(Y,F\right)$, is

$$\underbrace{rp\left(Y,F\right)}_{\text{required return}} = \underbrace{c+\xi\left(1-p\left(Y,F\right)\right)}_{\text{coupon and principal payment}} + \underbrace{(G-\xi F)p_F\left(Y,F\right)}_{\text{evolution of } dF} + \underbrace{\mu(Y,i^*)p_Y\left(Y,F\right)+\tfrac{\sigma(Y)^2}{2}p_{YY}\left(Y,F\right)}_{\text{evolution of } dY} \qquad (47)$$

Combining (47) and (47) gives rise to the exact same equilibrium debt issuance policy as in PROPOSITION 3**:**

$$G\left(Y,F\right) = \frac{\bar{\pi}c}{-p_F\left(Y,F\right)} = \frac{\bar{\pi}c}{V_{FF}\left(Y,F\right)} > 0.$$

Again, the equilibrium requires the condition in PROPOSITION 1, i.e. $-p_F\left(Y,F\right)=V_{FF}\left(Y,F\right)>0$ holds always.

4.2. Log-normal Cash Flows and Quadratic Adjustment Costs

Consider the setting with a log-normal cash-flow process studied in Section 3, with $\mu(Y,i)=\left(\mu+i\right)Y$ and $K\left(Y,i\right)=0.5\kappa i^2 Y$, where $\kappa>0$ is a positive constant. Here, the investment i increases the cash-flow growth rate linearly, and the cost is proportional to cash-flow size Y but quadratic in investment i. Without debt, our model is similar to Hayashi (1982), and the optimal investment policy for unlevered firm, denoted by $i_{unlever}$, is given by:

[24] The envelope theorem readily applies if i^* takes interior solutions always. However, even if the optimal investment policy takes a binding solution, our logic goes through as long as the constraint does not depend on F (hence there is no first-order gain by changing the firm's debt).

$$i_{unlever} = r - \mu - \sqrt{(\mu - r)^2 - 2(1-\pi)/\kappa}\,. \tag{48}$$

Now we derive the solution to our model with leverage without commitment. Denote the optimal investment rate by i_t^* so that the evolution of scaled cash-flow $y_t = Y_t/F_t$ is

$$\frac{dy_t}{y_t} = \left(\mu + i_t^* + \xi - g_t\right)dt + \sigma dZ_t\,.$$

Equity holders default when y_t hits the endogenous default boundary y_b. The scaled equity value $v(y)$ without debt issuance satisfies

$$(r+\xi)v(y) = \max_i (1-\overline{\pi})(y-c) - \xi - \frac{\kappa i^2}{2}y + (\mu + i + \xi)yv'(y) + \frac{1}{2}\sigma^2 y^2 v''(y)\,. \tag{48}$$

Given the optimal investment $i^*(y) = v'(y)/\kappa$, the above equation becomes

$$(r+\xi)v(y) = (1-\overline{\pi})(y-c) - \xi + \frac{y\left[v'(y)\right]^2}{2\kappa} + (\mu+\xi)yv'(y) + \frac{1}{2}\sigma^2 y^2 v''(y), \tag{49}$$

with two boundary conditions: $v(y) = \kappa i_{unlever} y - \frac{(1-\overline{\pi})c + \xi}{r+\xi}$ for $y \to \infty$, and $v(y_b) = 0$. The default boundary y_b is determined by the smooth-pasting condition $v'(y_b) = 0$.

The equity value function $v(y)$ and default boundary y_b are readily solvable by standard Matlab built-in ODE solver. The debt price is then given by $p(y) = yv'(y) - v(y)$, and the debt issuance policy is $g(y) = \frac{\overline{\pi}c}{yp'(y)} > 0$.

Finally, we need to verify the key condition $v''(y) > 0$ (or equivalently $p'(y) > 0$) in Proposition 1. Although we no longer have closed-form solution in the model with investment, in the next proposition we show that $v''(y) > 0$ holds by analyzing the ODE (49) satisfied by the equity value.

PROPOSITION 11. In the log-normal cash-flow model with quadratic investment costs, equity value is strictly convex, i.e., $v''(y) > 0$, so that the debt price is decreasing in debt face value. This guarantees the optimality of smooth issuance policy and hence the investment policy $i^*(y) = v'(y)/\kappa$ and issuance policy $g(y) = \frac{\bar{\pi}c}{yp'(y)} > 0$, together with the debt price $p(y) = yv'(y) - v(y)$, constitute an equilibrium.

PROOF: Define a constant $B \equiv \kappa i_{unlever}$ and $w(y) \equiv v(y) - By + \frac{(1-\bar{\pi})c + \xi}{r+\xi}$; then $w(\cdot)$ is concave if and only if $v(\cdot)$ is concave. Using (49) and $v'(y) = w'(y) + B$, we have:

$$
\begin{aligned}
(r+\xi)w(y) &= (r+\xi)v(y) - (r+\xi)By + (1-\bar{\pi})c + \xi \\
&= (1-\bar{\pi})y + \frac{y(w'(y)+B)^2}{2\kappa} + (\mu+\xi)y(w'(y)+B) + \frac{\sigma^2 y^2}{2}w''(y) - (r+\xi)By \\
&= \underbrace{y\left[1-\bar{\pi} + \frac{B^2}{2\kappa} + (\mu+\xi)B - (r+\xi)B\right]}_{0\text{, because } B=\kappa(r-\mu)-\sqrt{\kappa^2(r-\mu)^2-2\kappa(1-\bar{\pi})}} \\
&\quad + y\frac{2Bw'(y) + (w'(y))^2}{2\kappa} + (\mu+\xi)yw'(y) + \frac{\sigma^2 y^2}{2}w''(y) \\
&= yw'(y)\left[\frac{B}{\kappa} + \mu + \xi\right] + y\frac{(w'(y))^2}{2\kappa} + \frac{\sigma^2 y^2}{2}w''(y).
\end{aligned}
$$

Hence $w(y)$ satisfies the following ODE:

$$(r+\xi)w(y) = yw'(y)\left[\frac{B}{\kappa} + \mu + \xi\right] + y\frac{(w'(y))^2}{2\kappa} + \frac{\sigma^2 y^2}{2}w''(y) \tag{50}$$

We need two steps to show that $w''(y) > 0$ for all $y > y_b$.

Step 1. $w(y) > 0$ for all $y \ge y_b$. We know that at default $w'(y_b) = v'(y_b) - B = -B < 0$, and $w(\infty) = 0$. This implies that if $w(y) \le 0$ ever occurs, then the global minimum must be nonpositive and interior. Pick that global minimum point y_1; we must have $w'(y_1) = 0$ and $w''(y_1) > 0$. Suppose that $w(y_1) < 0$; evaluating (50) at y_1, we find that the LHS is strictly negative while the RHS is positive, contradiction. Suppose that $w(y_1) = 0$; then there must exist some local maximum point $y_2 > y_1$, so that $w(y_2) > 0$, $w'(y_2) = 0$ and $w''(y_2) < 0$. But the same argument of evaluating (50) at y_2 leads to a contradiction.

Step 2. Because $w(y)$ approaches 0 from above when $y \to \infty$, we know that for y to be sufficiently large $w(y)$ is convex. Suppose counterfactually that $w(y)$ is not convex globally; we can take the largest inflection point y_2 with $w''(y_3) = 0$. We must have $w'(y_3) < 0$ and $w'''(y_3) > 0$ (it is because for $y > y_3$ the function $w(y)$ is convex and decreasing to zero from above). At this point, differentiate (50) and ignore the term with $w''(y_3) = 0$, and we have

$$\left(r - \mu - \frac{B}{\kappa}\right) w'(y_3) - \frac{\left(w'(y_3)\right)^2}{2\kappa} = \frac{\sigma^2 y_3^2}{2} w'''(y). \tag{50}$$

Recall $B = \kappa(r-\mu) - \sqrt{\kappa^2 (r-\mu)^2 - 2\kappa}$ which implies

$$\left(r - \mu - \frac{B}{\kappa}\right) w'(y_3) = \sqrt{\kappa^2 (r-\mu)^2 - 2\kappa} \cdot w'(y_3) < 0$$

As a result, the LHS of (50) is negative while the RHS of (50) is positive, contradiction. This implies that $w(y)$ is convex globally.

Combining all the results above, we have shown that $v''(y) > 0$ for $y > y_b$. ■

4.3. Optimal Investment and Leverage

The extension considered above with endogenous investment $i^*(y)$ and debt issuance $g^*(y)$ allows us to study implications on the endogenous firm growth and its interaction with the firm's leverage policies. To facilitate discussion, we consider the following three benchmark cases which features either a constant investment policy i (i.e., independent of the state y_t), or a constant debt issuance policy g. In each case, we isolate either endogenous investment or endogenous debt issuance separately, allowing us to study the interactions between these two policies.

In the first case, the firm commits to the socially optimal investment policy which is a constant:

$$i_{social} = r - \mu - \sqrt{(\mu - r)^2 - 2/\kappa} \ . \tag{51}$$

In the second case, the committed constant investment policy is optimal to an unlevered firm who pays taxes, as in equation (48). Both cases are relevant because equity holders in our model are maximizing the levered firm value, which includes debt tax shields.

Importantly, in each benchmark case, equity holders---given the respective constant investment policy---are making endogenous debt issuance decisions as in our base model. In contrast, in the third Leland (1998) benchmark, the firm commits to fully rollover always its debt so that $g(\cdot) = \xi$, but the investment decision is taken by equity holders endogenously. Appendix XX gives the details of solving these three benchmark cases.

Figure 9 plots the investment policies (left panel) and debt issuance polices (right panel) for our extension with both endogenous investment and debt issuance polices (which is indicated by "i^*, no commitment g^*," solid lines), together with three benchmark cases. Start with the left panel. The general take-away there is that the well-known debt overhang effect (Myers 1977; Hennessey, 2004) causes the firm to underinvest; to the extreme, equity holders stop investing when the firm is close to default. Because all debt tax shields get dissipated in our no-commitment model, the relevant benchmark is "$i_{unlever}$, no commitment g^*" plotted in dashed line; and as expected, the firm in our model underinvests.

Similarly, the Leland (1998) firm, red dash-dotted line "i^*, Leland '98 $g = \xi$", underinvests compared to the socially optimal level (black solid line with circle). Interestingly, relative to the Leland benchmark, the solid line of our model sits below (above) the dash-dotted line of Leland (1998) for low (high) y. In words, no commitment leverage policy leads to more severe debt overhang when profitability is high, but the overhang effect is less severe when profitability is low. This is a result of history dependent leverage policy discussed in Section 3.3---equity holders allow leverage to decline following negative shocks.

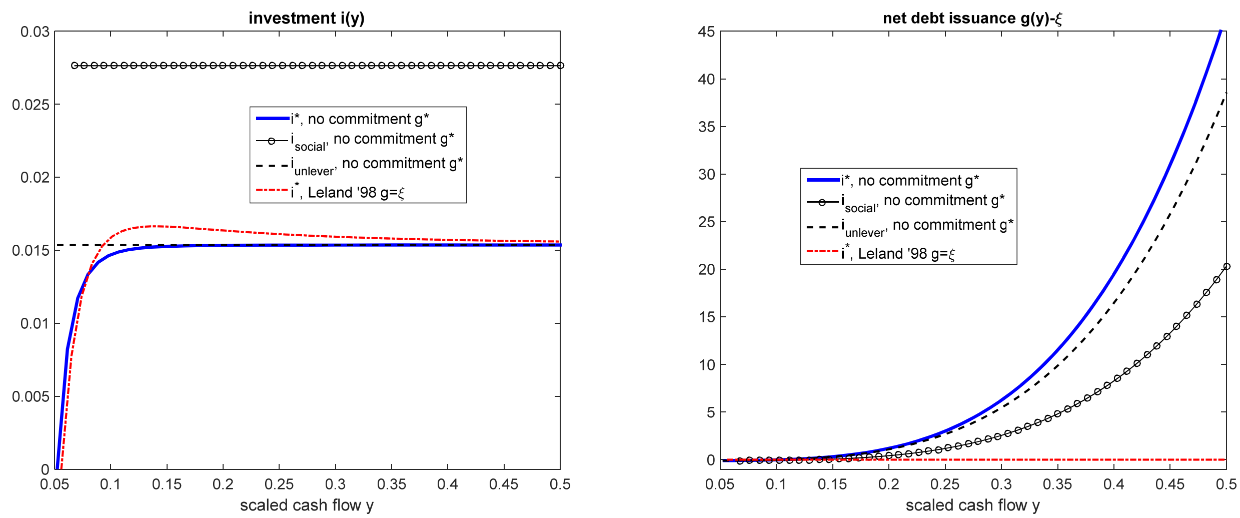

Figure 9: Endogenous investment and debt issuance policies, and comparison to three benchmarks.

Our model extension with endogenous investment is denoted by "i*, no commitment g*," with solid lines. "i_{social}, no commitment g*" with solid-circle lines and "$i_{unlever}$, no commitment g*" with dashed lines refer to the cases of constant investment $i(y)=i_{social}$ and $i(y)=i_{unlever}$, but shareholders has no commitment on the debt issuance policy. The third benchmark is Leland (1998) with constant debt issuance policy $g(y)=\xi$ but equity chooses investment endogenously. Parameters are $r=8\%, c=8\%, \sigma=25\%, \mu=0, \bar{\pi}=35\%, \xi=0.1, \kappa=10^3$.

The differences in net debt issuance polices $g(y)-\xi$ in the right panel of Figure 9 reveal an additional economic insight. By assumption, the net debt issuance is identically zero in the benchmark Leland (1998) case (dash-dotted line). For other cases, we observe that the solid line of "i^*, no commitment g^*" sits above both constant investment benchmarks (solid-circle line and dashed line). In words, equity holders are more aggressive in issuing new debt when they are also in charge of the endogenous investment policies of the firm. For intuition, recall that in deciding how much new debt to issue, equity holders are trading off the tax benefit against the losses caused by higher leverage that are borne by themselves. Compared to the case in which equity holders are forced to implement an (ex ante) optimal investment policy, the losses are mitigated when equity holders are also in charge of future investment policies that maximize equity value ex post, which explains the more aggressive leverage policy in this case.

5. Conclusions

To be added.

6. References

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7. Appendix.

To be completed.

7.1. Appendix for Leland (1998)

Following Leland (1998), we can first solve for the scaled levered firm value TEV as

$$\frac{TEV}{F} = \frac{(1-\bar{\pi})y}{r-\mu} + \frac{\bar{\pi}c}{r} - \left(\frac{\bar{\pi}c}{r} + \frac{(1-\bar{\pi})y_b^{\xi}}{r-\mu}\right)\left(\frac{y}{y_b^{\xi}}\right)^{-\gamma_v^{\xi}} \tag{52}$$

where the constant $\gamma_v^{\xi} \equiv \dfrac{\mu - 0.5\sigma^2 + \sqrt{(0.5\sigma^2-\mu)^2 + 2\sigma^2 r}}{\sigma^2}$. Then, the equity value $v^{\xi}(y) = \dfrac{TEV}{F} - p^{\xi}(y)$ equals:

$$v^{\xi}(y) = \frac{(1-\bar{\pi})y}{r-\mu} + \frac{\bar{\pi}c}{r} - \left(\frac{\bar{\pi}c}{r} + \frac{(1-\bar{\pi})y_b^{\xi}}{r-\mu}\right)\left(\frac{y}{y_b^{\xi}}\right)^{-\gamma_v^{\xi}} - \frac{c+\xi}{r+\xi}\left(1 - \left(\frac{y}{y_b^{\xi}}\right)^{-\gamma_p^{\xi}}\right) \tag{53}$$

7.2. Appendix for Section 4.2

With slight abuse of notation, consider a constant investment policy i, which could take either the constant value of $i_{unlever}$ in equation (48) or i_{social} in equation (51). The flow payoff to equity holders is $(1-\bar{\pi})(y-c) - \frac{\kappa i^2}{2} y$, and the method in our base model allows us to derive the equity holders' value to be

$$v(y)=\frac{y(1-\overline{\pi}-\frac{\kappa i^2}{2})}{r-\mu}-\frac{c\left(1-\overline{\pi}\right)+\xi}{r+\xi}\left(1-\frac{1}{1+\gamma}\left(\frac{y}{y_b}\right)^{-\gamma}\right),$$

with endogenous default boundary

$$y_b=\frac{\gamma}{1+\gamma}\frac{r-\mu}{r+\xi}\left[\frac{c\left(1-\overline{\pi}\right)+\xi}{1-\overline{\pi}-0.5\kappa i^2}\right].$$

Then we can solve for the endogenous debt issuance policy g^* as in (28).

The solution to the Leland (1998) model with endogenous investment is characterized by a pair of ODE, one for the equity value $v^{\xi}\left(y\right)$ and the other for the debt price $p^{\xi}\left(y\right)$. For equity value, we have

$$rv^{\xi}\left(y\right)=\max_i\left(1-\overline{\pi}\right)\left(y-c\right)+\xi\left(p^{\xi}\left(y\right)-1\right)+\left(\mu+i\right)yv^{\xi\prime}\left(y\right)-\frac{\kappa i^2}{2}y+\tfrac{1}{2}\sigma^2y^2v^{\xi\prime\prime}\left(y\right)$$

With optimal investment $i^*\left(y\right)=\frac{v^{\xi}{}'\left(y\right)}{\kappa}$, the above ODE becomes

$$rv^{\xi}\left(y\right)=\left(1-\overline{\pi}\right)\left(y-c\right)+\xi\left(p^{\xi}\left(y\right)-1\right)+\mu yv^{\xi\prime}\left(y\right)+\frac{v^{\xi\prime}\left(y\right)^2}{2\kappa}y+\tfrac{1}{2}\sigma^2y^2v^{\xi\prime\prime}\left(y\right)\quad(53)$$

With boundary conditions

$$v^{\xi}\left(y_b\right)=0, v^{\xi}{}'\left(y_b\right)=0, v^{\xi}{}'\left(y\right)=\kappa i_{FB}=\kappa\left(r-\mu-\sqrt{\left(\mu-r\right)^2-2\left(1-\pi\right)/\kappa}\right)\text{ for sufficient large } y,$$

For debt price $p^{\xi}\left(y\right)$, we have

$$\left(r+\xi\right)p^{\xi}\left(y\right)=c+\xi+\left(\mu+\frac{v^{\xi\prime}\left(y\right)}{\kappa}\right)yp^{\xi\prime}\left(y\right)+\tfrac{1}{2}\sigma^2y^2p^{\xi\prime\prime}\left(y\right)\quad(53)$$

with boundary conditions

$$p^{\xi}\left(y_b\right)=0, p^{\xi}{}'\left(y\right)=0\text{ for sufficient large } y.$$

One can easily solve for $\left\{v^{\xi}\left(y\right),p^{\xi}\left(y\right)\right\}$ by solving the ODE system (53)-(53), with respective boundary conditions.