

# Dynamic Agency and Real Options\*

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## Abstract

We present a model integrating dynamic moral hazard and real options. A risk-averse manager can exert costly hidden effort to increase the productivity growth of a firm. The risk-neutral owners of the firm can irreversibly increase the firm's capital stock. In contrast to the literature, we show that moral hazard can accelerate or delay investment relative to the first-best. When the agency problem is more severe, the firm will invest *earlier* than in the first best case because investment acts as substitute for effort. This mechanism provides an explanation for over-investment that does not rely on “empire-building” preferences. When the growth option is large, the firm will invest *later* than in the first-best. We also discuss the implications of the presence of real options for the manager's pay-performance sensitivity.

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# 1 Introduction

How firms make real investment decisions is a central topic in the study of corporate finance. As the investments of individual firms are typically lumpy and (partially) irreversible, they are well described as real options. In the standard real options model, cash flows of a firm are generated without any agency conflicts. In reality, cash flow growth often requires managerial effort and when this effort is costly and unobservable, a moral hazard problem arises. We investigate how this moral hazard problem affects investment timing decisions.

The optimal time to invest equates the benefit of investment with the direct cost plus the opportunity cost of investment. On the one hand, moral hazard will decrease the benefit causing a delay in investment. On the other hand, moral hazard will decrease the opportunity cost of investing and will thus accelerate investment. We show that when the moral hazard problem is severe or the size of the investment option is small, the agency conflict decreases the opportunity cost more than the benefit and hence accelerates investment. In contrast, when the moral hazard problem is moderate or the size of the investment option is large, the opposite is true and the agency problem delays investment.

Empirical evidence indicates that firms may either over- or under-invest relative to some first-best benchmark. For example, Bertrand and Mullainathan (2003) find that when external governance becomes weaker due to the passage of anti-takeover legislation, firms invest less in new plants. In contrast, Morck et al. (1990) document that negative price reactions to acquisitions are greater when the managers of bidding firms perform poorly before acquisitions, indicating that these managers are pursuing their own objectives. In theoretical literature on firm investment under agency conflicts, two main themes have developed. Grenadier and Wang (2005), DeMarzo and Fishman (2007), and DeMarzo et al. (2012) argue that moral hazard in effort induces firms to curtail investment and not to over-invest. While Jensen (1986), Stulz (1990), Harris and Raviv (1990), Hart and Moore (1995), and Zwiebel (1996) posit firms over-invest because managers have a preference for “empire-building.” Along similar lines, Roll (1986) and Bernardo and Welch (2001)) show that over-investment

may occur because managers are over-confident. We show that moral hazard in effort can also cause a form of over-investment; a firm plagued with a severe problem of incentivizing a manager to work will exercise investment options sooner than identical firms without a moral hazard problem.

To arrive at this result, we construct a continuous-time dynamic moral hazard model in which an investor contracts a manager to run a firm. The firm produces cash flow according to a linear production technology. The manager of the firm can exert effort to increase the expected growth rate of productivity. For example, the manager might need to work to increase market share or improve operational efficiency to increase productivity. This effort is costly to the manager and hidden to the investors so that the manager can potentially gain utility by exerting less effort than would be optimal from the perspective of the investor. In order to incentivize the manager to exert effort, the optimal contract will expose her to firm performance. This exposure is costly because it reduces risk sharing between the manager and the investor. In addition to increasing productivity by contracting with the manager, the investor has an option to irreversibly increase the firm's capital.

An important feature of our model is that capital and managerial effort are complements in the firm's production function. When the manager exerts more effort, productivity increases at a faster rate, and capital becomes more productive. Similarly, when the firm has more capital, managerial effort leads to more growth in cash flow. However, the complementarity of managerial effort and capital in the firm's production function does not necessarily mean the firm will invest less when the price of managerial effort rises. Indeed, if the price of managerial effort rises due to an increase in the cost of incentives, the firm may accelerate investment and decrease incentives. In this sense, investment serves as a substitute for managerial effort as a means of increasing cash flow. It is important to note that this substitution effect is not driven by the manager's preferences for investment. In fact, the manager in our model is indifferent between any particular investment policy.

In addition to implications for the investment behavior of firms, our model also generates

results for the manager's compensation and incentives. The power of incentives and pay-performance sensitivity are closely related to the size of the growth opportunity. All else equal, the productivity of managerial effort is increasing in the size of the growth option, and hence so is the power of incentives as measured by the sensitivity of the manager's wealth to firm output. However, increasing the size of the growth option also changes the sensitivity of firm value to output. Consequently, there is a wedge between pay-performance sensitivity and incentives. If the manager's cost of effort is increasingly convex, pay-performance sensitivity may actually decrease with the size of the growth option. This result is a caveat for empirical work on the power of incentives. In the presence of growth opportunities, there could be a negative relationship between actual incentives and the sensitivity of the manager's wealth to firm value (rather than output).

Our model further predicts that the manager's pay-performance sensitivity may increase or decrease at investment depending on the agency conflicts. When the moral hazard problem is less severe, the optimal contract will call for the manager to exert maximal effort before and after investment. As a result, pay-performance sensitivity, as measured by the sensitivity of the manager's wealth to firm value, will actually increase after investment. If, however, the moral hazard problem is more severe, the optimal contract will call for the manager to significantly decrease effort after investment, which causes a decrease in pay-performance sensitivity. An interesting feature of the results of our model is that the effect of investment on pay-performance sensitivity and investment timing are closely linked. On the one hand, when investment leads to an increase in pay-performance sensitivity, it must also be the case that investment is delayed relative to the first-best case. On the other hand, when investment leads to a decrease in pay-performance sensitivity, as is often empirically the case (e.g., Murphy (1999)), investment must be accelerated relative to the first-best case.

It is useful to illustrate the model in terms of some real world examples. First, consider a startup firm choosing the optimal time to raise its first round of venture capital. In this case, the initial capital stock of the firm is small and, as a consequence, managerial effort

is relatively cheap. In other words, the start-up manager's moral hazard problem prior to raising the first round of capital is relatively mild or perhaps even non-existent. After raising venture capital, the manager's moral hazard becomes more pronounced. As a result, increasing the cost of incentives decreases the present value of the added cash flow from the additional capital more than the value of the small firm, resulting in a delay in investment. Thus, our model predicts that startup firms with more severe moral hazard problems receive venture financing later than others.<sup>1</sup> A similar prediction applies to later stages of venture funding and initial public offerings.

In the preceding example, the size of the investment option is large relative to the initial capital stock of the firm, thus moral hazard delays investment timing. In the next example, the investment opportunity is small relative to the capital stock of the firm. For instance, consider a large mature firm choosing the optimal time to make an acquisition of a small target. In this setting, the acquisition allows the large firm to grow cash flows without providing costly incentives for additional managerial effort. This in turn implies that increasing the cost of incentives has a larger negative effect on the acquiring firm prior to the acquisition than on the merged firm and thus accelerates the acquisition. A prediction of the model is then that acquiring firms with more severe agency problems undertake acquisitions sooner and at lower levels of productivity than otherwise.

To gain a greater understanding of the forces at work in generating both accelerated and delayed investment, we generalize the model to allow for many different types of investment. The existing models of moral hazard and investment largely consider contracts that implement effort at the first-best level and show that moral hazard decreases or delays investment (e.g., Grenadier and Wang (2005); DeMarzo and Fishman (2007); Biais et al. (2010); DeMarzo et al. (2012)). So as a first step, we consider a model in the spirit of DeMarzo et al. (2012), i.e., a neoclassical model of investment, in which we allow optimal effort to deviate

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<sup>1</sup>Of course, other aspects of the agency problem may change as a result of the venture financing. Indeed, Gompers (1995) finds evidence that venture capitalists act as monitors since firms with more agency problems receive more frequent rounds of venture financing. However, this finding does not preclude our prediction.

from the first best. In this case, the marginal value of capital is a sufficient statistic for investment and always decreases with agency problems. Thus, investment decreases with the severity of the moral hazard problem even when effort is flexible. A similar argument applies to a setting with partially irreversible but perfectly divisible investment. However, when the investment technology implies lumpiness, as is often inherently the case with firm level investment as argued by Doms and Dunne (1998), Caballero and Engel (1999), and Cooper et al. (1999) among others, optimal investment is determined by the average value of new capital. Unlike the marginal value of capital, the average value of capital, and hence the effect of moral hazard on investment, increases or decreases with the severity of the agency problem depending on parameters.

This paper contributes to the growing literature on the intersection of dynamic agency conflicts and investment under uncertainty. On the dynamic contracting side, Holmstrom and Milgrom (1987) and Spear and Srivastava (1987) introduced the notion that providing agents with incentives may take place over many periods. More recently, a number of papers have built on the continuous time approach of Sannikov (2008) to characterize optimal dynamic contracts in a variety of settings. For example, DeMarzo and Sannikov (2006) consider the design of corporate securities when the manager may divert cash. Piskorski and Tchisty (2010, 2011) consider the optimal design of mortgages when lenders face stochastic interest rates or house prices are stochastic. He (2009) considers optimal executive compensation when firm size follows a geometric Brownian motion. Most closely related to our model of the dynamic agency problem is the capital structure model of He (2011), which allows for a risk-averse agent.

On the investment side, DeMarzo and Fishman (2007), Biais et al. (2010), and DeMarzo et al. (2012) consider dynamic moral hazard with investment. One important distinction between our paper and both Biais et al. (2010) and DeMarzo et al. (2012) is that their setups yield first-best effort even under moral hazard, and as such the substitutability of effort and investment is not present in their models. The investment technology we consider is based

on the classic real options models of Brennan and Schwartz (1985) and McDonald and Siegel (1986). Dixit and Pindyck (1994) offer a comprehensive guide to the real options literature. Two papers that use a similar model to ours to evaluate the effects of agency problems on real options investment are Grenadier and Wang (2005) and Philippon and Sannikov (2007). Grenadier and Wang consider a real option exercise problem in the presence of a static moral hazard and find that when there is an additional adverse selection over managerial ability, real option exercise is delayed. We consider a dynamic moral hazard problem and find that real option exercise may be either delayed or accelerated. Philippon and Sannikov consider real options in a dynamic moral hazard setting similar to ours. In their model, the first-best effort level is always optimal because cash flows follow an i.i.d process. As a result, they find that moral hazard can only delay investment. In contrast, we model cash flows that grow in expectation and as a consequence optimal managerial effort depends on the level of cash flows and may be lower than first best. This difference means that in our model, unlike in that of Philippon and Sannikov, moral hazard can accelerate investment.

The rest of the paper proceeds as follows. Section 2 introduces our model of moral hazard and real options. Section 3 provides the optimal contract and investment policy. Section 4 discusses the implications of the moral hazard problem for investment, compensation, and incentives. Section 5 considers a generalization of our basic model to build a greater understanding of the source of the effect of moral hazard on investment. Section 6 concludes.

## 2 The Model

In this section we present our model of dynamic moral hazard and real options. It resembles that of He (2011) in that we consider an agent (the firm's manager) with constant absolute risk-averse (CARA) preferences who can affect the productivity growth of the firm by exerting costly hidden effort. In addition, we endow the firm with an irreversible investment opportunity.

## 2.1 Technology and Preferences

Time is continuous, infinite, and indexed by  $t$ . The risk-free rate is  $r$ . A risk-neutral investor employs a risk-averse manager to operate a firm. Firm cash flows are  $X_t K_t dt$ , where  $K_t$  is the level of capital at time  $t$  and  $X_t$  is a productivity shock with dynamics given by:

$$dX_t = a_t \mu X_t dt + \sigma X_t dZ_t,$$

where  $a_t \in [0, 1]$  is the manager's effort and  $Z_t$  is a standard Brownian motion. Constants  $\mu$  and  $\sigma$  represent the (net of effort) drift and volatility of the productivity process. Managerial effort here corresponds to any action that increases the growth rate — not the current level — of productivity. For example, the manager may have to exert effort to increase market share or the operational efficiency of the firm. The firm starts with capital  $K_0 = k > 0$  and has a one time expansion option to increase capital to  $\hat{k}$  at cost  $p$ . In the notation that follows, a hat indicates a post investment quantity.

The manager has CARA preferences over consumption. She values a stream of consumption  $\{c_t\}$  and effort  $\{a_t\}$  as:

$$E \left[ \int_0^\infty e^{-rt} u(c_t, a_t) dt \mid \{a\} \right],$$

where  $u(c, a) = -e^{-\gamma(c - XKg(a))} / \gamma$  is the manager's instantaneous utility for consumption and effort and  $XKg(a)$  is the manager's cost of effort in units of consumption. This specification captures the notion that the manager's effort costs are increasing in both capital employed by the firm, as well as in productivity. In other words, it is more costly for the manager to increase productivity when the firm is larger or more productive. We assume the manager's normalized cost of effort  $g(a)$  is continuously differentiable, increasing, and convex,  $g(a) \in \mathcal{C}^1([0, 1])$ ,  $g'(a) \geq 0$ ,  $g''(a) > 0$ , and  $g'(0) = 0$ . When we consider specific parameterizations of the model, we assume a simple quadratic functional for  $g(a)$ . In addition to facing a cost

of effort, the manager may save at the risk-free rate  $r$ . We assume that the manager begins with zero savings. The manager's savings and effort are unobservable to the investor.

## 2.2 Contracts

A contract consists of a compensation rule, a recommended effort level, and an investment policy denoted  $\Pi = (\{c_t, a_t\}_{t \geq 0}, \tau)$ . The compensation rule  $\{c_t\}$  and recommended effort  $\{a_t\}$  are stochastic processes adapted to the filtration of public information,  $\mathcal{F}_t$ . For simplicity, we drop the subscript  $t$  whenever we are referring to the entire process of either consumption or effort. The investment policy  $\tau$  is  $\mathcal{F}_t$ -stopping time, which dictates when the firm exercises the option to increase capital. We assume that the investors can directly control investment and will pay the cost of investment. Note that the time  $t$  cash flow to the investor under a contract  $\Pi$  is given by:

$$dD_t = X_t K_t dt - c_t dt - \mathbb{I}(t = \tau)p,$$

where  $D_t$  denotes cumulative cash flow to the investor.

Since the agent can privately save, the compensation,  $c_t$ , specified by the contract need not be equal to the manager's time  $t$  consumption. Denote the manager's accumulated savings by  $S_t$  and her actual time  $t$  consumption and effort by  $\tilde{c}_t$  and  $\tilde{a}_t$ , respectively. Given a contract  $\Pi$ , the manager chooses a consumption and effort plan to maximize her utility from the contract:

$$W(\Pi) = \max_{\{\tilde{c}, \tilde{a}\}} E \left[ \int_0^\infty -\frac{1}{\gamma} e^{-\gamma(\tilde{c}_t - X_t K_t g(\tilde{a}_t)) - rt} dt \right] \quad (1)$$

$$\text{such that } dS_t = rS_t dt + (c_t - \tilde{c}_t) dt, S_0 = 0$$

$$dX_t = \tilde{a}_t \mu X_t dt + \sigma X_t dZ_t$$

$$K_t = k + (\hat{k} - k) \mathbb{I}(t \geq \tau).$$

The dynamics of savings  $S_t$  reflect that the difference between compensation  $c_t$  and con-

sumption  $\tilde{c}_t$  goes to increase (or decrease) savings while the balance grows at the risk-free rate  $r$ . In addition to the dynamics for  $S_t$  given above, we impose the standard transversality condition on the consumption process. The dynamics of productivity,  $X_t$ , reflect that the expected growth rate of productivity depends on the actual effort,  $\tilde{a}_t$ , of the manager. Finally, the time  $t$  capital stock of the firm depends on the investment policy set forth in the contract.

Given an initial outside option of the manager  $w_0$ , the investor then solves the problem:

$$B(X_0, w_0) = \max_{\{c, a\}, \tau} E \left[ \int_0^\infty e^{-rt} dD_t \right] \quad (2)$$

$$\text{such that } dX_t = \tilde{a}_t \mu X_t dt + \sigma X_t dZ_t$$

$$K_t = k + (\hat{k} - k) \mathbb{I}(t \geq \tau)$$

$$w_0 \leq E^{\{(\tilde{c}, \tilde{a}), \tau\}} \left[ \int_0^\infty -\frac{1}{\gamma} e^{-\gamma(\tilde{c}_t - X_t K_t g(\tilde{a}_t) - rt)} dt \right],$$

where  $\{\tilde{c}, \tilde{a}\}$  solves problem (1).

We call a contract  $\Pi$  *incentive compatible* and *zero savings* if the solutions  $\{\tilde{c}_t\}$  and  $\{\tilde{a}_t\}$  to Problem (1) are equal to the payment rule and recommended effort plan given in the contract. As is standard in the literature, we focus on contracts in which the solution to problem (1) is to follow the recommended action level and maintain zero savings by virtue of the following revelation-principle result.

**Lemma 1.** *For an arbitrary contract  $\tilde{\Pi}$ , there is an incentive compatible and zero savings contract  $\Pi$  that delivers at least as much value to the investor.*

### 3 Solution

The solution follows the now standard martingale representation approach developed by Sannikov (2008). The first step is to give a necessary and sufficient condition for a contract to implement zero savings. We then represent the dynamics of the manager's continuation

utility (the expected present value of her entire path of consumption) as the sum of a deterministic drift component and some exposure to the unexpected part of productivity growth shocks via the martingale representation theorem. With these dynamics in hand, we characterize the incentive compatibility condition as a restriction on the dynamics of continuation utility. Given the dynamics of continuation utility and productivity implied by incentive compatibility, we can represent the investor's optimal contracting problem as a dynamic program resulting in a system of ordinary differential equations (ODEs) for investor value together with boundary conditions that determine the investment policy. In the Appendix, we provide verification that the solution to this system of ODEs indeed achieves the optimum investor value.

### 3.1 The No-savings Condition

In this subsection, we follow He (2011) to characterize a necessary and sufficient condition for the manager to choose consumption equal to her compensation and thus maintain zero savings. In words, the condition states that the manager's marginal utility for consumption is equal to her marginal utility for savings. To determine the manager's marginal utility for an additional unit of savings, we first consider the impact of an increase in savings on her optimal consumption and effort plan going forward. Suppose  $\{\tilde{c}, \tilde{a}\}$  solves problem (1) for a given contract that implements zero savings. Now suppose we simply endow the manager with savings  $\mathcal{S} > 0$  at some time  $t > 0$ . How would her consumption and effort plan respond? Due to the absence of wealth effects implied by the manager's CARA preferences, the optimal consumption plan for  $s \geq t$  would be just  $\tilde{c}_s + r\mathcal{S}$ , while the effort plan would remain unchanged. Thus, an increase in savings from zero to  $\mathcal{S}$  increases the manager's utility flow by a factor of  $e^{-\gamma r \mathcal{S}}$  forever.<sup>2</sup> To make this intuition formal, it is useful to define the manager's continuation utility for a given contract when following the recommended

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<sup>2</sup>Since utility is always negative, the factor  $e^{-\gamma r \mathcal{S}} < 1$  represents an increase in utility.

effort policy and accumulating savings  $\mathcal{S}$  up to time  $t$ ,

$$W_t(\Pi, \{X_s, K_s\}_{s \leq t}; \mathcal{S}) = \max_{\{\tilde{c}_t, \tilde{a}_t\}} E \left[ \int_t^\infty -\frac{1}{\gamma} e^{-\gamma(\tilde{c}_s - X_s K_s g(\tilde{a}_t)) - r(s-t)} ds \mid \{X_s, K_s\} \right] \quad (3)$$

$$\text{such that } dS_s = rS_s ds + (\tilde{c}_s - c_s) ds \quad S_t = \mathcal{S}$$

$$dX_s = \tilde{a}_s \mu X_s ds + \sigma X_s dZ_s$$

$$K_s = k + (\hat{k} - k) \mathbb{I}(s \geq \tau).$$

The definition of continuation utility and the intuition given above lead to Lemma 2.

**Lemma 2** (He (2011)). *Let  $W_t(\Pi, \{X_s, K_s\}_{s \leq t}; \mathcal{S})$  be the solution to problem (3), then:*

$$W_t(\Pi, \{X_s, K_s\}_{s \leq t}; \mathcal{S}) = e^{-\gamma r \mathcal{S}} W_t(\Pi, \{X_s, K_s\}_{s \leq t}; 0). \quad (4)$$

Equation (4) allows us to determine the manager's marginal utility for savings under a contract that implements zero savings:

$$\frac{\partial}{\partial \mathcal{S}} W_t(\Pi, \{X_s, K_s\}_{s \leq t}; \mathcal{S}) \Big|_{\mathcal{S}=0} = -\gamma r W_t(\Pi, \{X_s, K_s\}_{s \leq t}; 0). \quad (5)$$

Since we are focused on zero savings contracts, we now drop the arguments and refer simply to continuation utility  $W_t$ . For the manager to maintain zero savings, her marginal utility of consumption must be equal to her marginal utility of savings:

$$u_c(c_t, a_t) = -\gamma r W_t$$

which, together with the CARA form of the utility function, implies the convenient no-savings condition:

$$u(c_t, a_t) = r W_t. \quad (6)$$

Thus, for a contract to implement zero savings, the manager's flow of utility from the contract

must be equal to the risk-free rate  $r$  times her continuation utility. This is intuitive; in order for the manager to have no incentive to save, the contract must deliver the risk-free yield of her continuation utility in units of utility flow. For the remainder of the paper, we only consider contracts that satisfy the no-savings condition given by Equation (6).

### 3.2 Incentive Compatibility

Now that we have characterized a necessary and sufficient condition for a contract to implement zero savings, we turn our attention to the incentive compatibility condition. For an arbitrary incentive compatible and zero savings contract, consider the following process:

$$F_t = E_t \left[ \int_0^\infty e^{-rs} u(c_s, a_s) ds \right].$$

This process is clearly a martingale with respect to the filtration of public information  $\mathcal{F}_t$ , thus the martingale representation theorem implies that there exists a progressively measurable process  $\beta_t$  such that:

$$dF_t = \beta_t(-\gamma r W_t) e^{-rt} (dX_t - a_t \mu X_t dt). \quad (7)$$

Now note that that  $F_t$  is related to the manager's continuation utility  $W_t$  (under the recommended consumption and effort plan) by:

$$dW_t = (rW_t - u(c_t, a_t))dt + e^{rt} dF_t. \quad (8)$$

Combining the no-savings condition (6) with Equations (7) and (8) gives the following dynamics for the manager's continuation utility:

$$dW_t = \beta_t(-\gamma r W_t) (dX_t - a_t \mu X_t dt). \quad (9)$$

The process  $\beta_t$  is the sensitivity of the manager's continuation utility to unexpected shocks to the firm's productivity. Since a deviation from the recommended effort policy results in an unexpected (from the investor's perspective) shock to productivity,  $\beta_t$  measures the manager's incentives to deviate from the contract's recommended effort policy.

For a given contract, Problem (1) implies that the manager chooses her current effort to maximize the sum of her instantaneous utility,  $u(c_t, a_t)dt$ , and the expected change in her continuation utility,  $W_t$ . The manager's expected change in continuation utility from deviating from the recommended effort policy  $a_t$  to  $\tilde{a}_t$  is:

$$E[dW_t|\tilde{a}] = \beta_t(-\gamma r W_t)(\tilde{a} - a_t)\mu X_t dt.$$

Thus, incentive compatibility requires that:

$$a_t = \arg \max_{\tilde{a}} \{u(c_t, \tilde{a}) + \beta_t(-\gamma r W_t)(\tilde{a} - a_t)\mu X_t\}. \quad (10)$$

Taking a first order condition for Problem (10) yields:

$$u_a(c_t, a_t) + \beta_t(-\gamma r W_t)\mu X_t = 0.$$

It is straightforward to show that this is a necessary and sufficient condition for the manager's optimal effort plan. Note that  $u_a(c_t, a_t) = -u_c(c_t, a_t)X_t K_t g'(a)$  and recall that the no-savings condition is  $u_c(c_t, a_t) = (-\gamma r W_t)$ , so that we can solve the first order condition above to find:

$$\beta_t = \frac{1}{\mu} K_t g'(a_t). \quad (11)$$

Intuitively, the sensitivity,  $\beta_t$ , that is required for incentive compatibility is the agent's marginal cost of effort,  $X_t K_t g'(a_t)$ , scaled by the marginal impact of effort on output,  $\mu X_t$ . Lemma 3 characterizes incentive-compatible no-savings contracts.

**Lemma 3.** *A contract is incentive compatible and has no savings if and only if the solution*

$W_t$  to Problem (3) has dynamics given by Equation (9), where  $\beta_t$  is defined by Equation (11).

It is useful to represent the agent's continuation utility,  $W_t$ , in terms of its certainty equivalent,  $V_t = -1/(\gamma r) \ln(-\gamma r W_t)$ . Applying Ito's lemma to (9) and combining it with Lemma 3 yields that the dynamics of  $V_t$  under an incentive-compatible no-savings contract are given by:

$$dV_t = \frac{1}{2} \gamma r \left( \frac{\sigma}{\mu} X_t K_t g'(a_t) \right)^2 dt + \frac{\sigma}{\mu} X_t K_t g'(a_t) dZ_t. \quad (12)$$

The drift term in Equation (12) comes from the difference in risk aversion between the investor and the manager. Since the manager is risk averse, the certainty equivalent of  $W$  must have additional drift for each additional unit of volatility. Since  $W$  is a martingale, the drift term in  $V$  is entirely due to this effect. This positive drift will show up in the investor's Hamilton-Jacobi-Bellman (HJB) equation as the cost of providing incentives.

### 3.3 First Best

As a benchmark, we first solve the model assuming effort is observable so that there are no agency conflicts. In this case, the investor simply pays the manager her cost of effort. Additionally, if the agent's promised utility is  $W$ , its certainty equivalent,  $V$ , can be paid out immediately. Thus, the investor's first-best value function,  $B_{FB}$ , depends linearly on the certainty equivalent of  $W$ , or  $B_{FB}(X, V) = b_{FB}(X) - V$ , for some function  $b_{FB}(X)$ . We refer to this function as the investor's gross value to indicate that it is equal to the investor's value gross of the certainty equivalent owed to the manager. To solve for the investor's first-best value function, we simply maximize the value of cash flows from the firm less the (direct) cost of effort. The investor's post-investment gross value function,  $\hat{b}_{FB}$ , then solves the following HJB equation:

$$r \hat{b}_{FB} = \max_{a \in [0,1]} \left\{ X \hat{k}(1 - g(a)) + a \mu X \hat{b}'_{FB} + \frac{1}{2} \sigma^2 X^2 \hat{b}''_{FB} \right\}. \quad (13)$$

Recall that a hat refers to a post-investment quantity. The first two terms in the brackets in Equation (13) are instantaneous cash flows and the cost of effort, respectively, and the other two terms reflect the impact of the dynamics of  $X$  on the value function. As all flows are proportional to  $X$ , the solution is also expected to be linear in  $X$  and as a result the optimal effort level given will be constant in  $X$ . We can solve Equation (13) to find the investor's first-best gross value function:

$$\hat{b}_{FB}(X) = \frac{1 - g(\hat{a}^{FB})}{r - \hat{a}^{FB}\mu} X \hat{k}.$$

Before investment, the firm's cash flows and the cost of effort are proportional to the lower level of capital,  $k$ . The HJB equation for the pre-investment gross firm value,  $b_{FB}(X)$ , is thus:

$$rb_{FB} = \max_{a \in [0,1]} \left\{ Xk(1 - g(a)) + a\mu X b'_{FB} + \frac{1}{2}\sigma^2 X^2 b''_{FB} \right\}. \quad (14)$$

Note that  $b'_{FB}$  is not constant due to the curvature implied by the option to invest. Consequently, optimal effort prior to investment,  $a^{FB}$ , will not necessarily be constant in  $X$ . To solve the first-best gross firm value prior to investment, we must identify a set of boundary conditions in addition to the HJB equation. At a sufficiently high level of  $X$ , denoted by  $\bar{X}_{FB}$ , the firm pays the investment cost  $p$  to increase the capital to  $\hat{k}$ . The firm value at  $\bar{X}_{FB}$  must satisfy the usual value-matching and smooth-pasting conditions:

$$\begin{aligned} b_{FB}(\bar{X}_{FB}) &= \hat{b}_{FB}(\bar{X}_{FB}) - p, \\ b'_{FB}(\bar{X}_{FB}) &= \hat{b}'_{FB}(\bar{X}_{FB}). \end{aligned}$$

Additionally, the firm value is equal to zero as  $X$  reaches its absorbing state of zero:

$$b_{FB}(0) = 0.$$

### 3.4 Optimal Contracting and Investment

We now present a heuristic derivation of the optimal contract in the full moral hazard case. First we characterize the payment rule to the manager. Recall that the no-savings condition in Equation (6) provides a link between instantaneous utility and continuation utility. This allows us to express the manager's compensation as a function of the current state of the firm  $(X_t, K_t)$ , the recommended effort level  $a_t$ , and the certainty equivalent of her continuation utility  $W_t$  as follows:

$$c_t = X_t K_t g(a_t) + rV_t. \quad (15)$$

The first term in Equation (15) is the manager's cost of effort in consumption units, while the second is the risk-free rate times the certainty equivalent of her continuation utility. In other words, the contract pays the manager her cost of effort plus the yield on her continuation utility.

The next task is to calculate the value of the firm to the investor before and after the investment. This amounts to expressing the investor's optimization problem given in (2) as a system of HJB equations. First, we consider the investor's problem post investment. An application of Ito's formula plus the dynamics of  $X_t$  and  $V_t$  yields the following HJB equation for the value function  $\hat{B}$  post investment:

$$r\hat{B} = \max_{a \in [0,1]} \left\{ X\hat{k}(1 - g(a)) - rV + a\mu X\hat{B}_X + \frac{1}{2}\sigma^2 X^2 \hat{B}_{XX} + \frac{1}{2}\gamma r \left( \frac{\sigma}{\mu} X_t \hat{k} g'(a) \right)^2 \hat{B}_V + \frac{\sigma^2}{\mu} X^2 \hat{k} g'(a) \hat{B}_{XV} + \frac{1}{2} \left( \frac{\sigma}{\mu} X \hat{k} g'(a) \right)^2 \hat{B}_{VV} \right\}. \quad (16)$$

We guess that  $\hat{B}(X, V) = \hat{b}(X) - V$ . Again, we refer to  $\hat{b}(X)$  as the investor's gross firm value as it measures the investor's valuation of the firm gross of the certainty equivalent promised to the manager. Then  $\hat{B}_V = -1$ ,  $\hat{B}_{XV} = 0$ , and  $\hat{B}_{VV} = 0$ . This leaves the following HJB

equation for  $\hat{b}(X)$ :

$$r\hat{b} = \max_{a \in [0,1]} \left\{ \hat{h}(X, a) + a\mu X\hat{b}' + \frac{1}{2}\sigma^2 X^2 \hat{b}'' \right\}, \quad (17)$$

where:

$$\hat{h}(X, a) = X\hat{k}(1 - g(a)) - \frac{1}{2}\gamma r \left( \frac{\sigma}{\mu} X\hat{k}g'(a) \right)^2 \quad (18)$$

is the total cash flow to the firm net of effort and incentive costs. It is instructive to note the difference between Equations (13) and (17). In the first-best case, the investor only needs to compensate the manager for her cost of effort, while in the moral hazard case, the investor must also bear incentive costs given by the second term in  $\hat{h}$ . These are costs for the risk-neutral investor of providing incentives to a risk-averse agent. The incentive cost of effort is proportional to the square of the level of cash flows,  $X\hat{k}$ , and thus the value function  $\hat{b}(X)$  is no longer linear in  $X$  as in the first-best case. It is left to specify the boundary conditions that determine a solution to ODE (17). The first boundary condition is that the firm, gross of the consumption claim to the agent, must be valueless when productivity is zero as this is an absorbing state:

$$\hat{b}(0) = 0. \quad (19)$$

The second boundary condition obtains by noting that the cost of positive effort goes to infinity as  $X$  goes to infinity, and as a result the optimal effort goes to zero. Thus, the value function must approach a linear function consistent with zero effort as  $X$  goes to infinity:

$$\lim_{X \rightarrow \infty} \left| \hat{b}'(X) - \frac{\hat{k}}{r} \right| = 0. \quad (20)$$

We now turn to the pre-investment firm value  $B$ . We again guess that  $B(X, V) = b(X) - V$ , where  $b$  is the investor's firm value gross of the certainty equivalent promised to

the manager. A similar argument to the above leads to the HJB equation for  $b$ :

$$rb = \max_{a \in [0,1]} \left\{ h(X, a) + a\mu Xb' + \frac{1}{2}\sigma^2 X^2 b'' \right\}, \quad (21)$$

where  $h(X, a)$  is as  $\hat{h}(X, a)$  in Equation (18) with  $\hat{k}$  replaced by  $k$ . Equation (21) is similar to the post-investment ODE given by Equation (17) but for the level of employed capital. A solution to ODE (21) is determined by investment-specific boundary conditions. As in the first-best case, the optimal investment policy will be a threshold  $\bar{X}$  in productivity at which the investor will increase the capital of the firm. Again the value-matching and smooth-pasting conditions apply<sup>3</sup>:

$$b(\bar{X}) = \hat{b}(\bar{X}) - p \quad (22)$$

$$b'(\bar{X}) = \hat{b}'(\bar{X}). \quad (23)$$

Additionally, as  $X$  reaches zero, the gross firm value is zero:

$$b(0) = 0. \quad (24)$$

We collect our results on the optimal contract in Proposition refoptimal.

**Proposition 1.** *The optimal contract is given by the payment rule (15) and investment time  $\tau = \min\{t : X_t \geq \bar{X}\}$  such that the investor's gross firm value before and after investment,  $b$  and  $\hat{b}$ , solve (21)-(24) and (17)-(20).*

Note that our choice to endow the manager with CARA preferences and the ability to privately save allows us to additively separate the dependence of the investor's value on productivity  $X_t$  and the certainty equivalent of the manager's continuation utility  $V_t$ . As a result, the investment problem reduces to the ODE in (21)-(24). If we had considered a risk-

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<sup>3</sup>These conditions derive from the usual value-matching and smooth-pasting conditions on the investor's value function  $B$ :  $B(\bar{X}, V) = \hat{B}(\bar{X}, V) - p$  and  $B_X(\bar{X}, V) = \hat{B}_X(\bar{X}, V)$ .

neutral manager, the resulting investment problem would be substantially more complex, with two state variables and the optimal investment threshold as a curve in  $(X_t, W_t)$  space.

## 4 Implications for Investment, Compensation, and Incentives

In this section, we discuss the implications of the optimal contract characterized in Proposition 1 for investment, compensation, and incentives. In numerical illustrations of these implications, we use particular parameterizations of our model and the following function form for the normalized cost of effort:

$$g(a) = \frac{1}{2}\theta a^2. \tag{25}$$

Following He (2011), we use a risk-free rate of  $r = 5\%$  and a standard deviation of productivity growth of  $\sigma = 0.25$ . We choose a slightly lower upper bound on the growth rate of productivity of  $\mu = 4\%$ , which reflects the that in our model the growth rate of productivity is bounded below by 0 due to the non-negativity of effort and the multiplicative specification for the effect of effort on productivity, while some calibrations (e.g., Goldstein et al. (2001)) find negative average growth rates. The parameter of risk aversion  $\gamma$  is set equal to 1. Investment increases capital from  $k = 0.5$  to  $\hat{k} = 1$  at cost 1 per unit of new capital. The cost of effort parameter is  $\theta = 1$ . We choose parameters for the cost of effort and investment so that the two are close substitutes.

### 4.1 Investment and Moral Hazard

Previous models of investment with moral hazard have largely agreed upon a central result: agency conflicts delay or curb firm investment. For example, DeMarzo and Fishman (2007) and DeMarzo et al. (2012) find that dynamic moral hazard problems decrease the rate of firm

investment, while Grenadier and Wang (2005) find that moral hazard combined with adverse selection delays the exercise of real options. In contrast to the extant literature, we find that agency conflicts can also accelerate investment relative to first best. This result is driven by the fact that although moral hazard decreases the value of the firm after investment, it also decreases the value of not investing. Moreover, the investment decision is driven by the difference between firm value with and without increased capital. Under conditions discussed below, increasing moral hazard increases this difference, and as a result, decreases the investment threshold.

The key intuition is that effort and investment are (imperfect) substitutes.<sup>4</sup> One period of high effort leads to one period of high expected cash flows growth. In a similar way, an investment in additional capital increases cash flows. A key difference between these methods of increasing cash flow growth is that effort is unobservable while investment is contractable. Thus, the relative cost of these two technologies depends on the severity of the moral hazard problem. Intuitively, when the moral hazard problem is severe, investment is a relatively cheap way of growing cash flows. Figures 1-3 show the investment threshold for the moral hazard and first-best cases over a range of parameter values. When the cost of effort  $\theta$ , the manager's risk aversion  $\gamma$ , or the volatility of growth  $\sigma$  are low, the moral hazard problem is less severe. In this case, higher effort is not too costly to implement and the investment threshold is higher for the moral hazard case than for the first best. In contrast, when any of these parameters are high, implementing high effort is costly relative to investment and the investment threshold for the moral hazard case is below that of the first-best case.

In order to make this intuition precise, we examine the comparative static properties of firm value before and after investment. Specifically, we consider the following comparative static:

$$\frac{\partial}{\partial \gamma} [\hat{b}(X) - b(X)]$$

for  $X$  close to  $\bar{X}$ . If this comparative static is negative, then an increase in  $\gamma$  decreases the

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<sup>4</sup>The substitutability of effort and investment was first emphasized in Holmstrom and Weiss (1985).

difference between the firm's value before and after investment. In other words, investment is less attractive and will be delayed. However, when this comparative static is positive, an increase in  $\gamma$  increases the profitability of investment and investment accelerates. To compute the derivative above, we apply the method of comparative statics developed by DeMarzo and Sannikov (2006). The details of this derivation are given in the Appendix. The main intuition is that for  $X$  very close to the investment boundary, the difference between the pre- and post-investment firm net of the cost of new capital is essentially just the difference between cash flows over the final instant before investment. We can then differentiate cash flow with respect to  $\gamma$  to get:

$$\frac{\partial}{\partial \gamma} [\hat{b}(X) - b(X)] \approx \frac{1}{2} r \left( \frac{\sigma}{\mu} \theta X \right)^2 ((kg'(a^*))^2 - (\hat{k}g'(\hat{a}^*))^2). \quad (26)$$

When the right-hand side of Equation (26) is positive, a small increase in the manager's risk aversion  $\gamma$  leads to an increase in the difference between  $\hat{b}(\bar{X})$  and  $b(\bar{X})$ . By the value-matching condition, this means that the investment threshold must decrease. We formally state this result in Proposition 2.

**Proposition 2.** *An increase in  $\gamma$  decreases the investment threshold  $\bar{X}$  if and only if the marginal cost of effort at the optimum drops by a sufficiently large amount at investment, i.e., if and only if:*

$$\frac{g'(\hat{a}^*(\bar{X}))}{g'(a^*(\bar{X}))} \leq \frac{k}{\hat{k}}. \quad (27)$$

Proposition 2 highlights one of our main findings: increased moral hazed problems do not necessarily lead to delayed or decreased investment. In fact, in our model, an increase in managerial risk aversion can lead to a decrease in the investment threshold. Much of the literature on agency conflicts and investment following Jensen (1986) has focused on problems of free cash flow, in which managers may invest funds in pet projects that are not beneficial to shareholders. This view posits that a central conflict between managers and shareholders is that a manager may want to invest even when it does not benefit shareholders to do so, i.e.,

manager’s wish to “empire-build.” At the same time, another strand of the literature (e.g., DeMarzo and Fishman (2007) and DeMarzo et al. (2012)) has focused on the assumption that motivating managers to apply effort is more costly for larger firms. This view implies that managers either have no preferences over investment or prefer less investment. Consequently, moral hazard in effort models typically predict that investment is curtailed or delayed. As such, it seems hard to reconcile this type of moral hazard with empirical evidence that firms sometimes over-invest. Proposition 2 demonstrates that over-investment can be perfectly natural in a standard moral hazard setting without empire-building preferences if we allow for flexible effort.

In Proposition 2, we posit that it is possible for moral hazard to accelerate real option exercise; we also provide some guidance for when such acceleration may take place. Specifically, an increase in the manager’s risk aversion  $\gamma$ , which in turn makes incentive provision more costly, accelerates investment when the marginal cost of effort at the optimum is greater before investment than after it. Thus, the effect of  $\gamma$  on investment timing depends on how the optimal effort changes when the firm invests. To investigate this effect further, it useful to consider the quadratic effort cost given by Equation (25). For this special case, we characterize the optimal effort policy by a simple first-order condition:

$$\hat{a}^*(X) = \min \left\{ \frac{\mu^3 \hat{b}'(x)}{\theta \hat{k}(\mu^2 + \gamma r \sigma^2 X \hat{k})}, 1 \right\},$$

$$a^*(X) = \min \left\{ \frac{\mu^3 b'(x)}{\theta k(\mu^2 + \gamma r \sigma^2 X k)}, 1 \right\}.$$

When optimal effort is interior both before and after investment, i.e., when  $a^*(\bar{X}), \hat{a}^*(\bar{X}) < 1$ , Inequality (27) simplifies to:

$$\frac{\mu^2 + \gamma r \sigma^2 \theta \bar{X} k}{\mu^2 + \gamma r \sigma^2 \theta \bar{X} \hat{k}} \leq 1,$$

which is always satisfied. When  $a^*(\bar{X}) = 1$ , i.e., when the optimal contract calls for the

manager to exert full effort before investment, inequality (27) simplifies to:

$$\hat{a}^*(\bar{X}) = \frac{\mu^3 \hat{b}'(\bar{X})}{\hat{k} \theta (\mu^2 + \gamma r \sigma^2 \theta \bar{X} \hat{k})} \leq \frac{k}{\hat{k}}.$$

This condition states that if optimal managerial effort drops immediately after investment by a sufficiently large (small) amount, then increasing (decreasing) agency costs decreases (increases) the investment threshold.

We now return to the examples we discuss in the introduction. Consider a startup firm with no (or a very small amount of) initial capital choosing the optimal time to increase its capital stock to start producing. In this case, the capital stock after investment is much larger than before investment,  $\hat{k} \gg k$ , so that the right-hand side of Inequality (27) is essentially zero. Note that in the left-hand side of the inequality, the ratio of the manager's marginal cost of effort before and after investment is always strictly positive. Thus the inequality is violated and an increase in the severity of the moral hazard problem delays investment. Intuitively, if the startup firm is not subject to a moral hazard problem prior to investment, then a relatively large post-investment moral hazard problem will delay investment.

Now consider the example of a large firm considering the acquisition of a relatively small target. In this case, the capital stock after investment is not much larger than before investment,  $\hat{k} - k \ll k$ , so that the right-hand side of inequality (27) is close to one. The HJB equations together with the smooth-pasting condition imply that optimal effort always decreases at investment, so that the left-hand side of Inequality (27) is always strictly below one. Thus the inequality is satisfied and an increase in the severity of the moral hazard problem accelerates the acquisition. The intuition here is that the acquisition allows the firm to grow its cash flows without requiring its manager to work more. This in turn allows the firm to save on incentive costs, so that when incentive costs are larger, the acquisition is accelerated.

Although Proposition 2 gives a condition to determine the sign of the effect of the agency

problem on investment, it does so in terms of the endogenously chosen effort levels before and after investment. This evokes the following question: Under what circumstances would optimal effort decrease after investment by a sufficiently large amount such that increasing the agency problem accelerates investment? Consider the optimal effort choice under first best ( $\gamma = 0$ ). Intuitively, when the cost of effort  $\theta$  is high, optimal effort will be interior both before and after investment. Using a similar technique to compute comparative statics to the one employed above, it is possible to show that effort weakly decreases with the severity of the agency problem:

$$\frac{\partial a^*}{\partial \gamma}, \frac{\partial \hat{a}^*}{\partial \gamma} \leq 0,$$

which implies that if optimal effort is interior in the first best, it will be interior in the presence of agency conflicts as well. Thus, when the cost of effort is high, an increase in the severity of the agency problem decreases the investment threshold and accelerates investment. When the cost of effort  $\theta$  is small, first-best effort prior to investment will be high, i.e., full effort will be employed. In this case, an increase in the severity of the agency problem from  $\gamma = 0$  will accelerate investment if and only if:

$$\mu \hat{b}'_{FB}(\bar{X}) \leq \theta k. \tag{28}$$

Moreover, we have seen that in the first-best case,  $\hat{b}(X)$  takes a simple linear form. We can then state Proposition 3:

**Proposition 3.** *Suppose the cost of effort is quadratic and given by Equation (25). If the cost of effort  $\theta$  is small, specifically when:*

$$\theta < \frac{2\mu \hat{k}^2}{k(2r\hat{k} - \mu k)},$$

*an increase in  $\gamma$  will delay investment when  $\gamma$  is small and accelerate investment when  $\gamma$  is large. Moreover, the investment threshold  $\bar{X}$  will be above the first best threshold  $\bar{X}_{FB}$  when*

$\gamma$  is small and below when  $\gamma$  is large. When the cost of effort is  $\theta$  is large, an increase in  $\gamma$  will always accelerate investment. Moreover, the investment threshold  $\bar{X}$  will always be below the first-best threshold,  $\bar{X}_{FB}$

Proposition 3 shows how, for the special case of quadratic effort costs, exogenous parameters determine accelerated and delayed investment. In Figure 2, we can see that when  $\gamma$  is small, the investment threshold increases in  $\gamma$ . Since at  $\gamma = 0$  the investment threshold under moral hazard and first best are equivalent, this means that for small  $\gamma$ , the investment threshold is higher under moral hazard than under first best. Investment is then delayed relative to first best. For large enough  $\gamma$ , the sign of the comparative static of the investment threshold with respect to  $\gamma$  is negative, and the investment threshold under moral hazard is lower than under first best. Investment is then accelerated relative to first best.

## 4.2 Incentives and Pay-Performance Sensitivity

The manager's compensation and incentives depend on the level of effort stipulated by the optimal contract. Therefore, we begin this section with a discussion of managerial effort. For interior solutions of effort  $a$ , we use the HJB equations (17) and (21) to characterize the optimal effort policies  $a^*(X)$  and  $\hat{a}^*(X)$  by the first order conditions:

$$g'(a^*(X)) = \frac{\mu^3 b'(X)}{k(\mu^2 + \gamma r \sigma^2 X k g''(a^*(X)))}, \quad (29)$$

$$g'(\hat{a}^*(X)) = \frac{\mu^3 \hat{b}'(X)}{\hat{k}(\mu^2 + \gamma r \sigma^2 X \hat{k} g''(\hat{a}^*(X)))}. \quad (30)$$

In the following analysis, we restrict our attention to parameter values such that the maxima  $a^*(X)$  and  $\hat{a}^*(X)$  satisfy the second order conditions.<sup>5</sup> Optimal effort is time-varying with productivity  $X_t$ , depends on the primitive parameters of the model, and on the presence of growth opportunities. Figure 4 illustrates some of the key properties of the optimal effort for

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<sup>5</sup>If the second-order derivative of the objective function is zero (a knife-edge case given its dependence of  $X$ ), then the implicit function theorem is not applicable.

our baseline parameter values. Efforts in young (pre-investment), mature (post-investment), and small no-growth (permanently small) firms are plotted at two levels of the cost of effort,  $\theta$ . Effort implemented in the mature and no-growth firms decreases and goes to zero as  $X$  approaches infinity. This is because the cost of providing incentives grows more in  $X$  than does the benefit of effort. A related effect makes effort decrease in response to exogenous changes in capital (that is, abstracting from growth options; to see this, compare the efforts of the no-growth and mature firms).

Effort implemented in the young firm is above that of the mature firm due to two reasons. First, the young firm employs a low level of capital. This property also manifests itself in the fact that effort (weakly) decreases at the moment of investment. The second reason for high effort in young firms is due to the presence of growth options. As is standard in real options models, growth options increase the sensitivity of firm value to productivity shocks as the firm approaches the investment threshold. As the optimal effort increases in  $b'(X)$  (see Equation (29)), this indicates that effort may increase in  $X$  in the young firm. Intuitively, the prospect of capital investment makes contracting high effort additionally attractive from the investor's point of view.

To implement any of the optimal effort levels under moral hazard, the manager needs to be appropriately incentivized. Our main interest is how investment and investment opportunities affect the power of incentives. We look at two alternative measures of incentives: one implied by our model and another commonly used in practice. A direct measure of a manager's incentives in our model is  $\beta_t$ . To see this, note that the certainty equivalent of a manager's promised continuation utility,  $V_t$  with the dynamics given in (12), can be interpreted as her financial wealth. Its sensitivity to output surprises (unexpected changes in  $X_t$ ) divided by the volatility of output,  $\sigma X_t$ , is exactly equal to  $\beta_t$ . In other words,  $\beta_t$  is an output-based pay-performance sensitivity (PPS) that measures the sensitivity of the manager's wealth to changes in output affected by the manager.

A standard approach to the measurement of PPS is to compute the sensitivity of the

manager's wealth to changes in firm value. This approach is particularly convenient from an empirical point of view as it is based on firm value changes, which are easy to measure. In contrast, an output-based PPS measure must isolate that output process, which is most directly attributable to the manager. In cases in which firm value is linear in output  $X_t$ , this simplification would be inconsequential as value-based PPS would be equivalent to direct, output-based PPS, such as  $\beta$ . However, growth options are known to generate a non-linear relationship between firm value and output. Thus firms with growth opportunities may have a wedge between output-based and value-based PPS.

In our model, the manager's dollar value-based PPS is measured by the sensitivity of the manager's dollar value,  $V_t$ , to changes in firm value,  $b(X_t)$ . Looking at the nondeterministic components of the two values (that is, the volatility terms), pre-investment value-based PPS, denoted by  $\phi$ , is given by:

$$\phi_t = \frac{\beta_t}{b'(X_t)} = \frac{g'(a^*(X_t))k}{\mu b'(X_t)}. \quad (31)$$

Post-investment  $\hat{\phi}_t$  is characterized by a similar equation substituting appropriately  $\hat{k}$ ,  $\hat{a}^*$ , and  $\hat{b}$ . As expected,  $\phi$  is closely related to  $\beta$ . However, it is scaled by the slope of the value function in output  $X$ . The presence of investment opportunities affects  $\phi$  by changing  $\beta$  and by changing the slope of  $b$  in output  $X$ .

In the next proposition, we posit that growth options can impact the two pay-performance sensitivity measures,  $\beta$  and  $\phi$ , differently.

**Proposition 4.** *By increasing investment opportunities, the manager's value-based PPS,  $\phi$ , changes in the opposite direction than the output-based PPS,  $\beta$ , if the convexity of the cost of effort increases in effort,  $g'''(a) > 0$ , and effort is interior.*

To provide the intuition for this result, suppose for concreteness that optimal effort increases in the size of investment opportunities. Incentives measured by  $\beta$  increase in the size of investment opportunities because the marginal value of effort for the investor increases and thus a higher optimal effort is contracted. However, in the case of  $\phi$ , scaling by  $b'(X)$

may change the relation. Indeed,  $\phi$  decreases in the size of the investment opportunity when the growth option creates a large sensitivity of the pre-investment firm value to productivity shocks for which the manager does not have to be compensated. In such risky firms, it is then optimal to set output-based PPS to a relatively low level. Thus, a low  $\phi$  does not necessarily mean that the manager's incentives are low-powered but that a strong response of firm value to output allows the principal to set a low value-based PPS. The condition  $g'''(a) > 0$  means that the cost function of the manager's effort is strongly convex (increasingly convex) and additional effort is costly.

Another important observation is that the power of incentives may either increase or decrease at investment. This is despite the fact that optimal effort never increases at investment. Proposition 5 give the precise conditions under which incentives increase or decrease at investment.

**Proposition 5.** *The manager's power of incentives, measured by either  $\beta$  or  $\phi$ , increases at investment if:*

$$\frac{g'(\hat{a}^*(\bar{X}))}{g'(a^*(\bar{X}))} > \frac{k}{\hat{k}}$$

*and decreases otherwise.*

Proposition 5 states that incentives increase at investment when the drop of the implemented effort at investment is sufficiently small relative to the inverse of the size of the growth option. This is the case for firms with low costs of effort (low agency conflicts) and large growth opportunities.

It is interesting to note that the condition in Proposition 5 is closely related to the one given in Proposition 2 for the negative sign of the effect of risk aversion on the investment threshold. This means that the response of the manager's incentives to investment can be linked to the distortion in investment timing due to agency conflicts. Specifically, our model predicts that the power of incentives decreases at investment if moral hazard conflicts accelerate investment and increases at investment if moral hazard conflicts delay investment.

This does not depend on whether or not the manager's incentives are measured by output- or value-based metrics.

## 5 A Generalized Model of Investment

In this section we extend our basic real options model to consider other specifications of the investment problem. The main goal of this exercise is to determine under what conditions an increase in the severity of the moral hazard problem as measured by the manager's risk aversion,  $\gamma$ , leads to increases in investment. To that end, we make the following modifications to the model of the previous sections. First, productivity now follows a general diffusion of the form:

$$dX_t = a_t \mu(X_t) dt + \sigma(X_t) dZ_t,$$

where the drift and volatility terms,  $\mu$  and  $\sigma$ , are continuously differentiable functions of productivity  $X$ . We maintain our restriction that effort must fall in the interval  $a_t \in [0, 1]$ ; however, the cost of effort is now given by the general function  $G(X_t, K_t, a_t)$  such that  $G$  is twice continuously differentiable in its arguments and convex in effort  $a_t$ . Next, the firm's cash flows are given by a general function  $\pi(X_t, K_t)$ , which may exhibit increasing or decreasing returns to scale, and may depend on either the increment or the level of productivity as well. Finally, capital accumulates according to:

$$dK_t = (I_t - \delta K_t) dt,$$

where  $\delta$  is capital depreciation and investment  $I_t$  is at the cost  $C(X_t, K_t, I_t) dt$  that may feature convex adjustment cost, partial reversibility, and stock fixed costs. A contract in this more general setting is then a triple  $(a_t, c_t, I_t)$  consisting of a recommended effort level  $a_t$ , a compensation plan  $c_t$ , and an investment rule  $I_t$ . In the following subsections we give a heuristic analysis of the generalized investment model, with formal proofs provided in the

Appendix.

Note that the arguments leading to a characterization of the no-savings condition and incentive-compatibility conditions did not depend on a specification of the investment technology. Consequently, continuation utility arising from a contract without savings under this more general model must be a martingale and satisfy  $u_c(c_t, a_t) = -\gamma r W_t$ . The incentive-compatibility condition is then:

$$a_t = \arg \max_{\tilde{a}} \{u(c_t, \tilde{a}) + \beta_t(-\gamma r W_t)(\tilde{a} - a_t)\mu(X)\}, \quad (32)$$

which implies that:

$$\beta_t = \frac{1}{\mu(X)} G_a(X_t, K_t, a_t)$$

and

$$dW_t = \frac{\sigma(X_t)}{\mu(X_t)} (-\gamma r W_t) G_a(X_t, K_t, a_t) dZ_t.$$

The characterization of incentive compatibility given above allows us to proceed to analyze the effect moral hazard on investment in this more general setting.

## 5.1 Tobin's $q$

For many of the models subsumed by our general setup, the optimal investment policy is an increasing function of the investor's marginal value of capital, commonly referred to as Tobin's  $q$ . For this class of models, including the neoclassical and capacity choice models, the effect of the agency problem on optimal investment operates entirely through  $q$ . Thus, to determine the effect of the moral hazard problem on optimal investment in these models, it is sufficient to determine its effect on the marginal value of capital. We now show that increasing the severity of the moral hazard problem decreases the marginal value of capital and hence curtails investment.

To determine the effect of the moral hazard problem on the investor's marginal value

of capital, we again apply the method of comparative statics developed in DeMarzo and Sannikov (2006). Since the dynamics of continuation utility remain essentially unchanged from the previous sections, the investor's value function is still additively separable as  $B(X, K, V) = b(X, K) - V$ . Taking as given the optimal investment and effort policies  $I^*$  and  $a^*$ , an application of Ito's formula, the envelope theorem, and the Feynman-Kac formula detailed in the Appendix yields the following expression for the derivative of the marginal value of capital,  $b_k$ , with respect to the manager's risk aversion  $\gamma$ :

$$b_{K\gamma} = E \left[ \int_0^\infty e^{-(r+\delta)t} h_{K\gamma}(X_t, K_t, a^*(X_t, K_t)) dt \mid X_0, K_0 \right], \quad (33)$$

where  $h(X, K, a)$  is defined as in Equation (18) and represents the total cash flow to the firm net of effort and incentive costs. Equation (33) states that the derivative of the marginal value of capital with respect to  $\gamma$  is just the expected present value of all future derivatives of the marginal products of capital with respect to  $\gamma$ . For any given point  $(X, K, a)$ , it is straightforward to compute the derivative of the marginal product of capital with respect to  $\gamma$  to find:

$$h_{K\gamma}(X, K, a) = -r \left( \frac{\sigma(X)}{\mu(X)} \right)^2 G_a(X, K, a) G_{aK}(X, K, a) \leq 0. \quad (34)$$

Equations (33) and (34) together imply the following proposition.

**Proposition 6.** *The investor's marginal value of capital  $b_K$  is decreasing in the manager's risk aversion  $\gamma$ .*

Proposition 6 confirms the intuition of the previous literature (e.g., DeMarzo et al. (2012) and DeMarzo and Fishman (2007)) that the moral hazard problem decreases the marginal value capital. This in turn implies that if the optimal investment policy can be expressed as an increasing function of the marginal value of capital otherwise independent of the severity of the moral hazard problem, then optimal investment decreases the severity of the moral hazard problem. For example, in the neoclassical model with convex adjustment costs ( $C_{II}(X, K, I) > 0$ ), the optimal investment rate equates the marginal value of capital with

the marginal cost of capital:

$$b_K(X, K, a^*) = C_I(X, K, I^*). \quad (35)$$

Since investment costs are independent of the manager's risk aversion, we can differentiate both sides of Equation (35) to find:

$$I_\gamma^* = \frac{b_{K\gamma}(X, K, a^*)}{C_{II}(X, K, I^*)} \leq 0,$$

so that investment decreases with the severity of the moral hazard problem. Similarly, in a capacity choice model with partial reversibility, as in Abel and Eberly (1996), the optimal investment policy is to invest (divest) only when the marginal value of capital is greater (less) than the marginal purchase price of capital. For any given level of productivity  $X$ , increasing the severity of the moral hazard problem through an increase in  $\gamma$  decreases the marginal value of capital. Thus both the thresholds in productivity at which the firm invests and divests increase with the severity of the agency problem; thus, investment will be delayed while divestment will be accelerated.

## 5.2 Lumpy Investment

We have found that so long as investment is an increasing function of the marginal value capital, moral hazard serves to decrease or delay investment. Thus, our main results concerning investment timing must follow from the fact that in our real options setting, the effect of the agency problem on investment does not operate solely through  $q$ . We now show that in a wide class of models, that is those in which optimal investment is lumpy and infrequent, investment will be accelerated by moral hazard when the agency problem is more severe. The acceleration in investment occurs because the location of the threshold in productivity at which the firm invests is now given by the average value of the additional capital. In contrast to the marginal value of capital, the average value of additional capital can increase

with the severity of the agency problem because of the substitutability of effort and capital we have demonstrated in the previous sections. If the average value of additional capital at the moment of investment increases with the severity of the moral hazard problem, then the moral hazard problem will accelerate investment.

To make this argument more precise, consider a production function with convex regions in capital and partial reversibility. For example, the firm may be endowed with a sequence of real options (rather than a single option in the simple model we consider above). Whenever the current level of capital falls within the convex region of  $\pi$ , the optimal investment will always be in a “lump.” That is, suppose  $\pi_{KK}(X_t, K_t) > 0$ . If it is optimal to invest at the point  $(X_t, K_t)$ , then it must be that the path of  $K_t$  has a discontinuous jump up at time  $t$ . Indeed any smooth investment process would not take advantage of the increasing returns to scale in  $\pi$ . Moreover, partial reversibility implies that it cannot be optimal to invest at all moments in time by the standard real option intuition. In this setup, let the current level of capital be  $k$  and the optimal amount of new capital at investment be  $I^*$ , then the optimal investment threshold is given by the familiar value-matching and smooth-pasting conditions:

$$\begin{aligned} b(\bar{X}, k) &= b(\bar{X}, k + I^*) - p_+ I^*, \\ b_X(\bar{X}, k) &= b_X(\bar{X}, k + I), \end{aligned}$$

where  $p_+$  is the purchase price of new capital. We can rearrange the value-matching condition to get:

$$\frac{b(\bar{X}, k + I^*) - b(\bar{X}, k)}{I^*} = p_+,$$

which reads: invest when the average value of new capital  $I$  is equal to the average price of new capital  $p_+$ . Thus, the effect of the moral hazard problem on the optimal investment time is given by its effect on the average value of capital.

To see why the average value of additional capital can increase with the severity of the moral hazard problem, we again compute comparative statics with respect to the manager’s

risk aversion. Using a similar argument as in the previous sections, we can calculate:

$$\text{sign} \left[ \frac{\partial}{\partial \gamma} \left[ \frac{b(\bar{X}, k + I^*) - b(\bar{X}, k)}{I^*} \right] \right] = \text{sign} \left[ (G_a(\bar{X}, k, a^*(X, k)))^2 - (G_a(X, k + I^*, a^*(X, k + I^*)))^2 \right]. \quad (36)$$

Equation (36) states that the average value of new capital is increasing in  $\gamma$  if the marginal cost of effort to the manager is reduced after investment. This condition is equivalent to the one given in Proposition 2, however it is stated in more general terms. Again, the intuition is that the moral hazard problem may have a more negative effect on the firm after investment than before.

## 6 Conclusion

We presented a model of real options and dynamic moral hazard. We find that the effect of agency conflicts on investment timing depends on the severity of the conflict. When the moral hazard problem is less severe, the optimal contract will implement high effort, but delay investment. When the moral hazard problem is more severe, the optimal contract will implement lower effort but will call for accelerated investment. The finding that moral hazard may accelerate investment is new and provides an alternative to empire-building or managerial hubris-based explanations of over-investment.

The effect of investment on pay-performance sensitivity also depends on the severity of the moral hazard problem. When the moral hazard problem is less severe, pay-performance sensitivity increases after investment. When the moral hazard problem is more severe, pay-performance sensitivity decreases with investment. These results link pay-performance sensitivity, which is easily measurable, with the nature the distortion on investment timing imposed by moral hazard, which is more difficult to measure. This link could be exploited to empirically evaluate the effect of agency problems on the timing of investment.

Our results also provide guidance to empirical work on pay-performance sensitivity on its own. We show that in the presence of growth options, there is a wedge between output-based and value-based measures of incentives. In fact, if the manager's cost of effort is increasingly convex, value-based pay-performance sensitivity may decrease even though true incentives (i.e., output-based pay-performance sensitivity) increase. Thus, it is important to control for the presence of growth options when using value-based measures of pay-performance sensitivity as a proxy for the level of incentives.

Finally, although the primary real options model we consider is fairly simple, we show that the main intuition carries over into more realistic settings, so long as the optimal investment path is lumpy. A promising direction for future work would be to enrich the current model so that it is suitable for a structural estimation.

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# Appendix

## A Proofs

*Proof of Lemma 1.* Consider an arbitrary contract  $\Pi = (\{c_t, a_t\}, \tau)$  and suppose the solution to the manager's optimization problem (1) for this contract is given by  $\{\tilde{c}_t, \tilde{a}_t\}$  and the manager's associated value for this contract is  $\tilde{W}_0$ .

Now consider the alternative contract  $\tilde{\Pi} = (\{\tilde{c}_t, \tilde{a}_t\}, \tau)$ . Note that under this contract the manager again gets utility  $\tilde{W}_0$  from the consumption effort pair  $\{\tilde{c}_t, \tilde{a}_t\}$ . We claim that the solution to manager's optimization problem (1) is again  $\{\tilde{c}_t, \tilde{a}_t\}$ . Indeed suppose it is not and that there is a alternative feasible consumption effort pair  $\{\check{c}_t, \check{a}_t\}$  such that this policy yields utility  $\check{W}_0 > \tilde{W}_0$  to the manager. The consumption effort pair  $\{\check{c}_t, \check{a}_t\}$  is also feasible under the original contract  $\Pi$  since:

$$\begin{aligned} \lim_{t \rightarrow \infty} E \left[ e^{-rt} \int_0^t (c_s - \check{c}_s) ds \right] &= \lim_{t \rightarrow \infty} \left( E \left[ e^{-rt} \int_0^t (c_t - \tilde{c}_t) dt \right] + E \left[ e^{-rt} \int_0^t (\tilde{c}_s - \check{c}_s) ds \right] \right) \\ &= \lim_{t \rightarrow \infty} E \left[ e^{-rt} \int_0^t (c_s - \tilde{c}_s) ds \right] + \lim_{t \rightarrow \infty} E \left[ e^{-rt} \int_0^s (\tilde{c}_s - \check{c}_s) ds \right] \\ &= 0. \end{aligned}$$

Thus, the manager could achieve utility  $\check{W}_t > \tilde{W}_t$  under the original contract  $\Pi$ , a contradiction.

Finally note that the investor achieves the same value under the new contract  $\tilde{\Pi}$  as under the original contract  $\Pi$ , since effort and investment are unchanged, and the transversality condition implies that the two consumption streams have the same present value. □

*Proof of Lemma 2.* Suppose  $S_t = \mathcal{S}$  and recall the definition of  $W_t(\Pi, \{X_s, K_s\}_{s \leq t}; \mathcal{S})$  and let  $\{\tilde{c}, \tilde{a}\}$  solve problem (3). We claim that  $\{\tilde{c} - rS, \tilde{a}\}$  solves problem (3) for  $S_t = 0$ . This plan gives the manager a utility of  $W_t(\Pi, \{X_s, K_s\}_{s \leq t}; 0) = e^{\gamma r S} W_t(\Pi, \{X_s, K_s\}_{s \leq t}; \mathcal{S})$ . Suppose

there is some alternative  $\{\check{c}, \check{a}\}$  that yields a higher utility to the agent  $\check{W}_t(\Pi, \{X_s, K_s\}_{s \leq t}; 0) > W_t(\Pi, \{X_s, K_s\}_{s \leq t}; 0)$ . Now consider the plan  $\{\check{c} + rS, \check{a}\}$  and note that this plan is feasible under  $S_t = \mathcal{S}$  but under this plan the manager can achieve the following utility:

$$\begin{aligned} \check{W}_t(\Pi, \{X_s, K_s\}_{s \leq t}; 0) &= e^{-\gamma r S} \check{W}_t(\Pi, \{X_s, K_s\}_{s \leq t}; 0) \\ &\geq e^{-\gamma r S} W_t(\Pi, \{X_s, K_s\}_{s \leq t}; 0) \\ &= W_t(\Pi, \{X_s, K_s\}_{s \leq t}; 0), \end{aligned}$$

which contradicts the optimality of the  $\{\check{c}, \check{a}\}$ . □

*Proof of Proposition 1.* We show that the candidate policies are indeed optimal for the investor. Note that the compensation policy  $c_t$  is pinned down by the no-savings condition. Define the stopped gain process by:

$$G_t = \int_0^t e^{-rs} (X_t K_t - c_t) dt + e^{-rt} B(X_t, V_t) + \mathbb{I}(t \geq \tau) (e^{-rt} (\hat{B}(X_t, V_t) - B(X_t, V_t)) - e^{-r\tau} p).$$

When  $V_t$  evolves according to (12), Ito's formula gives the following dynamics:

$$\begin{aligned} e^{rt} dG_t &= \left[ X_t K_t - \frac{1}{2} \theta X_t K_t a^2 + -rV_t + a_t \mu X_t \mathcal{B}_X + \frac{1}{2} \sigma^2 X_t^2 \mathcal{B}_{XX} \right. \\ &\quad \left. + \frac{1}{2} \gamma r \left( \frac{\sigma}{\mu} X_t \hat{k} g'(a) \right)^2 \mathcal{B}_V + \frac{\sigma^2}{\mu} X_t^2 \hat{k} g'(a) \mathcal{B}_{XV} + \frac{1}{2} \left( \frac{\sigma}{\mu} X_t \hat{k} g'(a) \right)^2 \mathcal{B}_{VV} - r \mathcal{B} \right] dt \\ &\quad + \left( \mathcal{B}_X + \frac{\sigma}{\mu} X_t K_t g'(a_t) \mathcal{B}_V \right) \sigma X_t dZ_t + \mathbb{I}(t = \tau) (\hat{B}(X_t, V_t) - B(X_t, V_t) - p), \end{aligned}$$

where  $\mathcal{B}(X_t, V_t) = B(X_t, V_t) + \mathbb{I}(t \geq \tau) (\hat{B}(X_t, V_t) - B(X_t, V_t))$ . Note that the drift term (the term in the square brackets) is clearly negative for any alternative policy, while it is zero for the candidate policy. Now examine the last term. Under the candidate policy, this term is zero due to the value-matching condition. Under any alternative investment time  $\tau$ , this term is negative due to the smooth-pasting condition and the concavity of  $\hat{B}(X_t, V_t) - B(X_t, V_t)$ .

As a result, the process  $G_t$  is a martingale under the proposed contract and a supermartingale otherwise. The rest of the argument proceeds along the standard lines.  $\square$

*Proof of Proposition 2.* The following lemmas aid in the proof of the proposition. For ease of exposition, we leave their proofs to the end of the main argument.

**Lemma 4.** *Suppose  $X_t$  evolves according to  $dX_t = \mu(X_t)dt + \sigma(X_t)dZ_t$ . Then for some bounded functions  $f: (0, Y] \rightarrow \mathbb{R}$ ,  $r: (0, Y] \rightarrow \mathbb{R}^+$ , and  $\Omega: \mathbb{R} \rightarrow \mathbb{R}$ , a function  $F: (0, Y] \rightarrow \mathbb{R}$  solves both:*

$$r(X)F(X) = f(X) + \mu(X)F_X(X) + \frac{1}{2}\sigma(X)^2F_{XX}(X), \quad (\text{A.1})$$

with a boundary condition  $F(Y) = \Omega(Y)$  and

$$F(X) = E \left[ \int_0^\tau e^{-\int_0^t r(X_s)ds} f(X_t)dt + e^{-\int_0^\tau r(X_s)ds} \Omega(Y) | X_0 = X \right], \quad (\text{A.2})$$

where  $\tau = \inf\{t | X_t \geq Y\}$ .

**Lemma 5.**  $\hat{b}_{XX}(\bar{X}) - b_{XX}(\bar{X}) \neq 0$ .

**Lemma 6.** *There exists  $\epsilon > 0$  such that  $\hat{b}_X(X) - b_X(X) > 0$  for all  $X \in (\bar{X} - \epsilon, \bar{X})$ .*

**Lemma 7.** *There exists  $\epsilon > 0$  such that when  $X \in (\bar{X} - \epsilon, \bar{X})$  we have  $\text{sign}(\hat{b}_\gamma(X) - b_\gamma(X)) = \text{sign}(kg'(a^*(\bar{X})) - \hat{k}g'(\hat{a}^*(\bar{X})))$ .*

We now proceed to the main argument. The first step in determining the sign of  $\bar{X}_\gamma$  is to differentiate the smooth-pasting condition with respect to  $\gamma$  to get:

$$\bar{X}_\gamma(\hat{b}_{XX}(\bar{X}) - b_{XX}(\bar{X})) = \hat{b}_{\gamma X}(\bar{X}) - b_{\gamma X}(\bar{X}). \quad (\text{A.3})$$

Lemma 6 together with an application of the one-sided version of l'Hopital's rule yields the following expression for  $\bar{X}_\gamma$ :

$$\bar{X}_\gamma = -\frac{\hat{b}_{X\gamma}(\bar{X}) - b_{X\gamma}(\bar{X})}{\hat{b}_{XX}(\bar{X}) - b_{XX}(\bar{X})} = \lim_{X \uparrow \bar{X}} -\frac{\hat{b}_{X\gamma}(X) - b_{X\gamma}(X)}{\hat{b}_{XX}(X) - b_{XX}(X)} = \lim_{X \uparrow \bar{X}} -\frac{\hat{b}_\gamma(X) - b_\gamma(X)}{\hat{b}_X(X) - b_X(X)}, \quad (\text{A.4})$$

so that determining the sign of  $\overline{X}_\gamma$  is equivalent to determining the sign of the last limit above.

If  $\frac{g'(\hat{a}^*(\overline{X}))}{g'(a^*(\overline{X}))} < \frac{k}{\hat{k}}$ , Lemmas 6 and 7 imply there exists  $\epsilon > 0$  such that:

$$-\frac{\hat{b}_\gamma(X) - b_\gamma(X)}{\hat{b}_X(X) - b_X(X)} < 0$$

for all  $X \in (\overline{X} - \epsilon, \overline{X})$ , which in turn implies:

$$\lim_{X \uparrow \overline{X}} -\frac{\hat{b}_\gamma(X) - b_\gamma(X)}{\hat{b}_X(X) - b_X(X)} \geq 0,$$

since  $\hat{b}_\gamma(X) - b_\gamma(X)$  and  $\hat{b}_X(X) - b_X(X)$  are nonzero and continuous. Thus,  $\overline{X}_\gamma \leq 0$ . If  $\frac{g'(\hat{a}^*(\overline{X}))}{g'(a^*(\overline{X}))} < \frac{k}{\hat{k}}$ , a similar argument shows  $\overline{X}_\gamma \geq 0$ .

*Proof of Lemma 4.* The proof essentially follows the proof of DeMarzo and Sannikov (2006) for Lemma 4. Suppose  $V$  solves Equation (A.1) and define the process  $H_t$  by:

$$H_t = \int_0^t e^{-\int_0^s r(X_u) du} f(X_s) ds + e^{-\int_0^t r(X_s) ds} V(X_s).$$

An application of Ito's formula gives the dynamics for  $H_t$ :

$$e^{\int_0^t r(X_s) ds} dH_t = (f(X_t) + \mu(X_t)V_X(X_t) + \frac{1}{2}\sigma(X_t)^2 V_{XX}(X_t) - r(X_t)V(X_t))dt + \sigma(X_t)V(X_t)dZ_t.$$

By Equation (A.1), the drift in the above dynamics is zero, so that  $H_t$  is a martingale. Since  $V(X)$  is bounded on  $[0, \overline{X}]$ ,  $H_\tau$  is a martingale and:

$$\begin{aligned} V(X_0) = H_0 &= E[H_\tau | X_0] = E \left[ \int_0^\tau e^{-\int_0^t r(X_s) ds} f(X_t) dt + e^{-\int_0^\tau r(X_s) ds} V(X_\tau) | X_0 \right] \\ &= E \left[ \int_0^\tau e^{-\int_0^t r(X_s) ds} f(X_t) dt + e^{-\int_0^\tau r(X_s) ds} W(Y) | X_0 \right]. \end{aligned}$$

where the last equality follows from the definition of the stopping time  $\tau$  and the boundary

condition  $V(Y) = W(Y)$ . □

*Proof of Lemma 5.* Assume that  $\hat{b}_{XX}(\bar{X}) - b_{XX}(\bar{X}) = 0$ . We show that this leads to a contradiction. First, differentiate the smooth-pasting condition (23) with respect to  $p$  to obtain:

$$b_{Xp}(\bar{X}) = -(\hat{b}_{XX}(\bar{X}) - b_{XX}(\bar{X}))\bar{X}_p.$$

$\hat{b}_{XX}(\bar{X}) - b_{XX}(\bar{X}) = 0$  implies that  $b_{Xp}(\bar{X}) = 0$ . Next, differentiate the ODE (21) with respect to  $X$  and  $p$  to get an ODE for  $b_{Xp}$ :

$$(r - a^*(X)\mu)b_{Xp} = (a^*(X)\mu + \sigma^2)Xb_{XXp} + \frac{1}{2}\sigma^2X^2b_{XXXp}.$$

Lemma 4 then implies:

$$b_{Xp}(X) = E \left[ e^{-\int_0^\tau (r - a^*(\tilde{X}_s)\mu) ds} b_{Xp}(\bar{X}) | \tilde{X}_0 = X \right] = 0,$$

where  $\tilde{X}$  is a process with the following dynamics:

$$d\tilde{X} = (a^*(\tilde{X})\mu + \sigma^2)\tilde{X} dt + \sigma\tilde{X} dZ.$$

Since  $b_{Xp}(X) = 0$ , then  $b_p(X)$  must be a constant in  $X$ . By differentiating the value-matching condition (22) with respect to  $p$ , we get:

$$b_p(\bar{X}) = (\hat{b}_X(\bar{X}) - b_X(\bar{X}))\bar{X}_p - 1 = -1,$$

where the second equality follows from the smooth-pasting condition (23). Differentiating the ODE (21) with respect to  $p$  to gives an ODE for  $b_p$ :

$$rb_p(X) = a(X)\mu Xb_{Xp}(X) + \frac{1}{2}\sigma^2X^2b_{XXXp}(X).$$

Lemma 4 then implies:

$$b_p(X) = E [e^{-r\tau} b_p(\bar{X}) | X_0 = X].$$

In particular,

$$\lim_{X \rightarrow 0} b_p(X) = \lim_{X \rightarrow 0} E [e^{-r\tau} b_p(\bar{X}) | X_0 = X] = 0,$$

as  $X = 0$  is an absorbing point for the process  $X_t$ . This means that  $b_p(X)$  cannot be a constant, which is a contradiction.  $\square$

*Proof of Lemma 6.* Suppose there does not exist  $\epsilon > 0$  such that  $b_X(X) < \hat{b}_X(X)$  for all  $X \in (\bar{X} - \epsilon, \bar{X})$ , then for all  $\epsilon > 0$  there exists  $X \in (\bar{X} - \epsilon, \bar{X})$  such that  $b_X(X) \geq \hat{b}_X(X)$ . Now since  $b_X(\bar{X}) = \hat{b}_X(\bar{X})$  and  $b_X$  and  $\hat{b}_X$  are continuous, this implies that there exists  $\epsilon > 0$  such that  $b_X(X) \geq \hat{b}_X(X)$  for all  $X \in (\bar{X} - \epsilon, \bar{X})$ . This implies that for  $X \in (\bar{X} - \epsilon, \bar{X})$ , we have:

$$\hat{b}(X) - b(X) = \hat{b}(\bar{X}) - b(\bar{X}) - \int_X^{\bar{X}} (\hat{b}_X(x) - b_X(x)) dx \geq p,$$

which is a contradiction to the definition of  $\bar{X}$ .  $\square$

*Proof of Lemma 7.* First, we differentiate the HJB equations (17) and (21) with respect to  $\gamma$  to get:

$$r\hat{b}_\gamma = \hat{h}_\gamma(X, \hat{a}^*(X)) + \hat{a}^*(X)\mu X\hat{b}_{X\gamma} + \frac{1}{2}\sigma^2 X^2 \hat{b}_{XX\gamma},$$

and:

$$rb_\gamma = h_\gamma(X, a^*(X)) + a^*(X)\mu Xb_{X\gamma} + \frac{1}{2}\sigma^2 X^2 b_{XX\gamma},$$

subject to  $b_\gamma(\bar{X}) = \hat{b}_\gamma(\bar{X})$ . For the remainder of the proof of this lemma, we will suppress the dependence of  $h_\gamma$  and  $\hat{h}_\gamma$  on  $a^*$  and  $\hat{a}^*$  to ease notation. Note that  $\hat{h}_\gamma(\bar{X}) \geq h_\gamma(\bar{X})$  is equivalent to  $\frac{g'(\hat{a}^*(\bar{X}))}{g'(a^*(\bar{X}))} < \frac{k}{\bar{k}}$ .

Let  $X^*$  and  $\hat{X}^*$  be given by the following dynamics:

$$\begin{aligned} dX^* &= a^*(X^*)\mu X^* dt + \sigma X^* dZ \\ d\hat{X}^* &= \hat{a}^*(\hat{X}^*)\mu \hat{X}^* dt + \sigma \hat{X}^* dZ. \end{aligned}$$

Applying Proposition 4 we have:

$$b_\gamma(X) = E \left[ \int_0^\tau e^{-rt} h_\gamma(X_t^*) dt + e^{-r\tau} \hat{b}_\gamma(\bar{X}) | X_0^* = X \right]$$

and

$$\hat{b}_\gamma(X) = E \left[ \int_0^{\hat{\tau}} e^{-rt} \hat{h}_\gamma(\hat{X}_t^*) dt + e^{-r\hat{\tau}} \hat{b}_\gamma(\bar{X}) | \hat{X}_0^* = X \right],$$

where  $\tau = \inf\{t | X_t^* \geq \bar{X}\}$  and  $\hat{\tau} = \inf\{t | \hat{X}_t^* \geq \bar{X}\}$ . Subtracting we get:

$$\hat{b}_\gamma(X) - b_\gamma(X) = E \left[ \int_0^{\hat{\tau}} e^{-rt} \hat{h}_\gamma(\hat{X}_t^*) dt - \int_0^\tau e^{-rt} h_\gamma(X_t^*) dt + (e^{-r\hat{\tau}} - e^{-r\tau}) \hat{b}_\gamma(\bar{X}) | X_0^* = \hat{X}_0^* = X \right].$$

Now by continuity and the fact that  $\hat{\tau}, \tau \xrightarrow{\text{a.s.}} 0$  as  $\hat{X}_0^*, X_0^* \rightarrow \bar{X}$ , there exists  $\epsilon > 0$  such that when  $\hat{X}_0^*, X_0^* \in (\bar{X} - \epsilon, \bar{X})$  we have:

$$\text{sign}(\hat{b}_\gamma(X) - b_\gamma(X)) = \text{sign}(\hat{h}_\gamma(\bar{X}) - h_\gamma(\bar{X})) \tag{A.5}$$

$$= \text{sign} \left( kg'(a^*(\bar{X})) - \hat{k}g'(\hat{a}^*(\bar{X})) \right), \tag{A.6}$$

which is the desired result. □

□

□

*Proof of Proposition 4.* To consider the effect of the size of investment opportunities on incentives, we analyze the effect of increasing post-investment capital  $\hat{k}$ .

First, consider the effect on pre-investment  $\beta_t$ :

$$\frac{d\beta_t}{d\hat{k}} = \frac{1}{\mu} k g''(a^*(X_t)) \frac{da^*(X_t)}{d\hat{k}}.$$

Consider next the effect of  $\hat{k}$  on  $\phi_t$ :

$$\frac{d\phi_t}{d\hat{k}} = -\frac{\mu^2}{(\mu^2 + \gamma r \sigma^2 X_t k g''(a^*(X_t)))^2} \gamma r \sigma^2 X_t k g'''(a^*(X_t)) \frac{da^*(X_t)}{d\hat{k}}.$$

So:

$$\text{sign}\left(\frac{d\phi_t}{d\hat{k}}\right) = \text{sign}\left(-g'''(a^*(X_t)) \frac{da^*(X_t)}{d\hat{k}}\right).$$

This shows that the sign of the effect of  $\hat{k}$  on  $\phi_t$  is the same as on  $\beta_t$  if  $g'''(a^*) < 0$  and the opposite (negative) if  $g'''(a^*) > 0$ . □

*Proof of Proposition 5.* The result for  $\beta$  follows directly by comparing pre- and post-investment  $\beta_t$  at the moment of investment. The relation for  $\phi$  is found similarly using the smooth-pasting condition (23). □

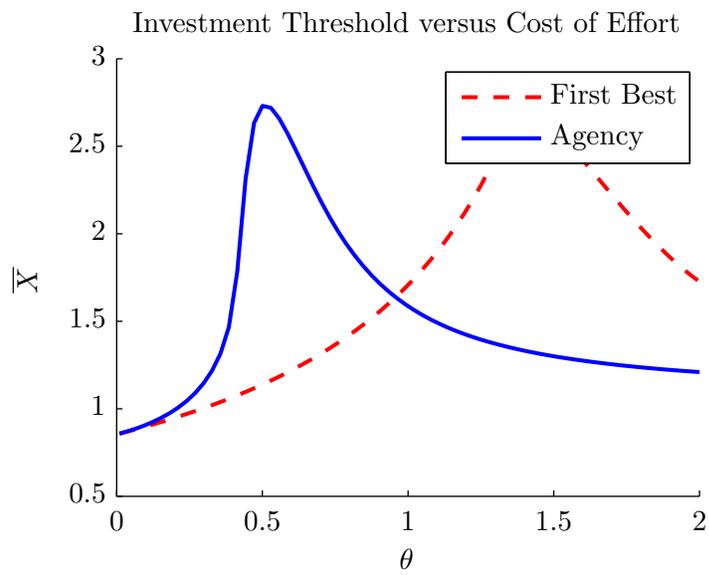


Figure 1: The investment threshold  $\bar{X}$  as a function of the cost of effort. For low costs of effort, the investment threshold in the moral hazard setting is above that of the first best. For high costs of effort, the relationship is reversed.

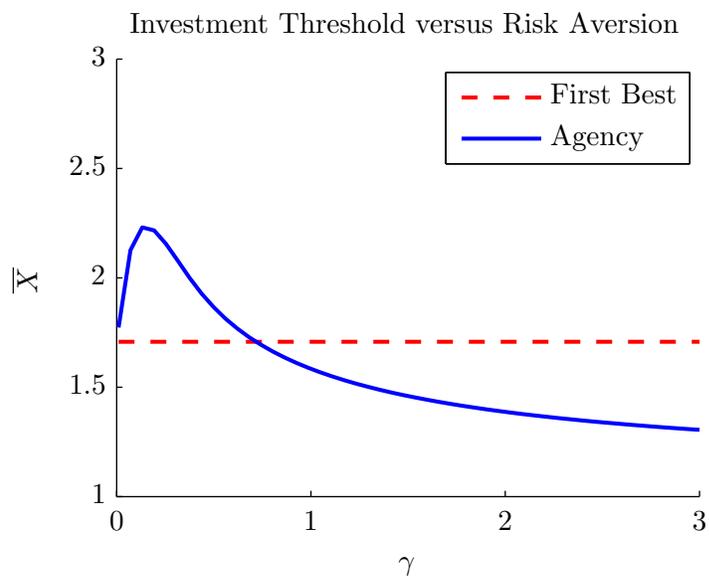


Figure 2: The investment threshold  $\bar{X}$  as a function of the risk aversion.

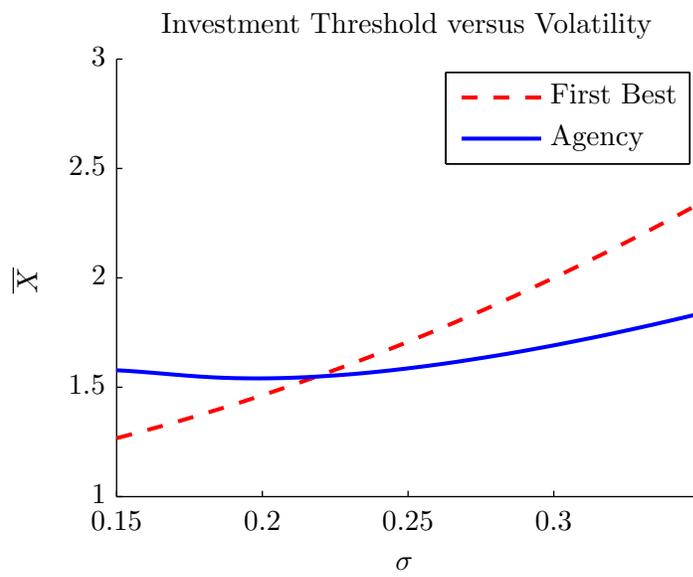


Figure 3: The investment threshold  $\bar{X}$  as a function of the volatility.

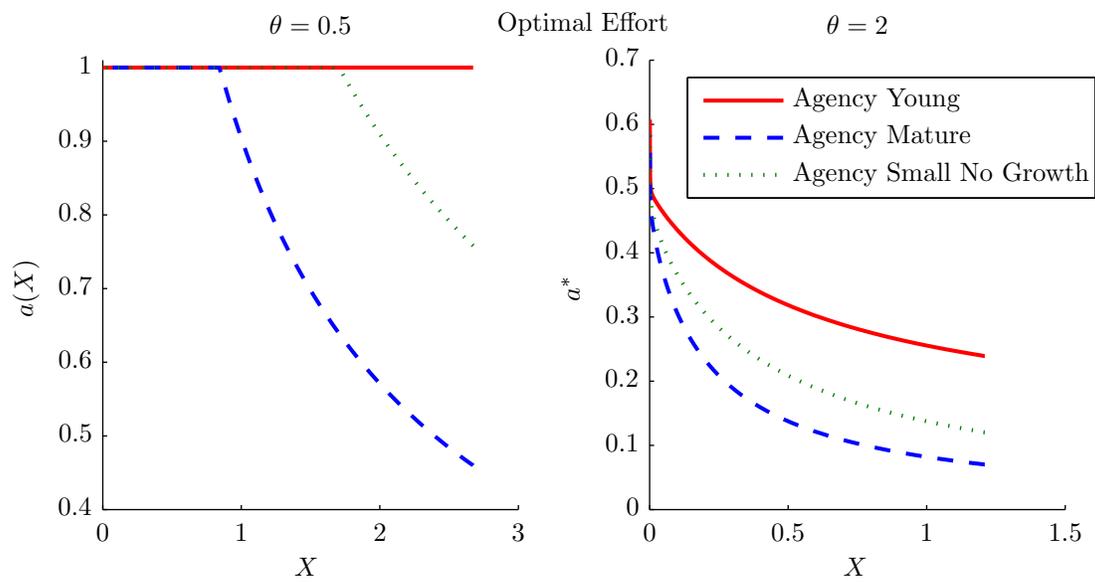


Figure 4: The optimal effort at two levels of the cost of effort:  $\theta = 1$  (left) and  $\theta = 2$  (right).