

PRELIMINARY

The Analytics of Investment, q , and Cash Flow*

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Abstract

This paper analyzes the relationships among investment, q , and cash flow in a tractable stochastic model in which marginal q and average q are equal. Using a Markov regime switching process to generate the marginal operating profit of capital, the paper analyzes the impact of a mean-preserving spread on the unconditional distribution and a change in the persistence of the marginal operating profit of capital. Extending the analysis to include measurement error provides an analytic explanation for positive cash flow coefficients in regressions of investment on q and cash flow, and the cash flow coefficient is larger for faster-growing firms.

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Empirical investment equations typically find that Tobin's q has a positive effect on capital investment by firms, and that even after taking account of the effect of Tobin's q on investment, cash flow has a positive effect on investment. Of course, the interpretations of these results rely on some theoretical model of investment. Typically, the theoretical model that underlies the relationship between Tobin's q and investment is based on convex capital adjustment costs.¹ In this framework, marginal q is a sufficient statistic for investment. No other variables, in particular, cash flow, should have any explanatory power for investment, once account is taken of marginal q . The fact that cash flow has a positive impact on investment, even after taking account of q , is interpreted by many researchers as evidence of financing constraints facing firms. That interpretation is bolstered by the finding that the cash flow coefficient is larger for firms that are classified as likely to be financially constrained.

In this paper, I develop and analyze a tractable stochastic model of investment, q , and cash flow and use it to interpret the empirical results described above. In modeling adjustment costs, the first choice is whether to specify these costs as a function of investment only (usually either gross investment or net investment) or to specify these costs as a function of the capital stock as well as of investment. The former specification is more tractable and easier to analyze in some ways, especially in the context of perfect competition and constant returns to scale in production. In that context, the marginal contribution of capital to operating profits is a function only of exogenous factors such as the price of output, the wage rate, and the level of productivity. Marginal q equals the expected present value of the stream of marginal contributions to operating profit accruing to the undepreciated portion of a unit of capital installed today. When this stream of marginal operating profits depends only on exogenous factors, the value of marginal q is exogenous to the firm, and in particular, does not depend on current or future investment decisions of the firm. In Abel (1983), I exploit this exogeneity of the stream of marginal operating profits to derive closed-form expressions for marginal q and for the value of the firm.

One unfortunate implication of specifying adjustment costs to depend only on investment, and not on the capital stock also, is that the optimal level of investment is independent of the size of the firm. Two firms with the same value of marginal q would undertake the same level of investment even if one firm's capital stock is a thousand times the size of the other firm's capital stock. Alternatively, as shown by Lucas (1967), if the cash flow of the firm, after deducting all costs associated with investment, is linearly homogeneous in capital, labor, and investment, the *growth rate* of the firm is independent of its size. Later, Hayashi (1982) showed that this linear homogeneity implies that Tobin's q , often called average q , is equal to marginal q . This equality of marginal q and average q is particularly powerful, because average q , which is in principle observable, can be used to measure marginal q , which is the appropriate shadow value of capital that determines the optimal

¹Lucas and Prescott (1971) and Mussa (1977) demonstrate the link between securities prices, which are related to Tobin's q , and investment in an adjustment cost framework.

rate of investment. In addition, this linearly homogeneous framework relates the investment-capital ratio to q and most empirical analyses, in fact, use the investment-capital ratio as the dependent variable in regressions.

Although the linearly homogeneous framework has some convenient properties, it can be less tractable because the adjustment cost function depends on the firm's capital stock as well as on its rate of investment. In general, an additional unit of capital stock reduces the adjustment cost, and this marginal benefit of capital must be added to the marginal operating profit of capital in computing marginal q . Even though the marginal operating profit of capital is exogenous, the marginal reduction in the adjustment cost depends on the firm's choice of investment, and this dependence on the firm's decisions complicates the calculation of marginal q . Nevertheless, I show that if the exogenous marginal operating profit of capital is known with certainty to be constant forever, it is simple to calculate a closed-form expression for the optimal rate of investment, and hence for q , if the adjustment cost function is quadratic.

Ideally, to shed light on the response of investment to q requires a framework with variation in the firm's marginal operating profit, which induces variation in q and in optimal investment. In this paper, I develop a model in which stochastic variation in the marginal operating profit of capital is generated by a Markov regime-switching process. With this stochastic specification, the model turns out to be very tractable. Although I cannot derive closed-form solutions for marginal q or investment, I present a closed-form solution for optimal investment and q , up to a constant, in the case of quadratic adjustment costs. More importantly, however, the framework is tractable enough to permit straightforward analysis of the effects on q and investment of changes in cash flow for a particular firm, even if the adjustment cost function is not quadratic. The model can also be used to compare q and investment across firms that face different interest rates, different depreciation rates, and different stochastic processes for exogenous marginal operating profit of capital. I apply this tractable framework to analyze the impact on marginal q and investment of a mean-preserving spread in the unconditional distribution of marginal operating profit of capital, as well as the impact of a change in the persistence of the Markov regime-switch process generating these marginal operating profits.

As mentioned earlier, a common feature of adjustment cost models of investment is that marginal q is a sufficient statistic for investment. In particular, cash flow should not add any explanatory power for investment after taking account of marginal q . This feature holds in the model I present here and might appear to be an obstacle to accounting for the empirical cash flow effect on investment described above. To circumvent that obstacle, I introduce classical measurement error. It has been argued in the literature that if q is measured with error, then since the true value of q is an increasing function of cash flow, cash flow will have some additional explanatory power for investment, and the coefficient on cash flow will be positive in a regression of investment on q and cash flow. I present

a simple expression to illustrate the impact of measurement error on estimated coefficients on q and cash flow. The model I present here allows the analysis to go beyond the existing argument for a positive cash flow coefficient by showing that the size of the cash flow coefficient will be larger for firms that grow more rapidly. Since rapidly growing firms are more likely to be classified as facing binding financial constraints, the model's implication that rapidly-growing firms have larger cash flow coefficients is consistent with empirical studies that find larger cash flow coefficients for firms classified as financially constrained. However, because the model has perfect capital markets, without any financial frictions, the results described here imply that the finding of positive cash flow coefficients that are larger for faster-growing firms cannot be taken as evidence of financing constraints.

Because the analysis of the model relies on the equality of marginal q and average q , I begin, in Section 1, by re-stating, and extending to a stochastic framework, the Hayashi condition under which average q and marginal q are equal. Section 2 introduces the model of the firm and analyzes the valuation of a unit of capital and the optimal investment decision in the case in which the marginal operating profit of capital is known to be constant forever. More than simply serving as a warm up to the stochastic model, Section 2 introduces a function that facilitates the analysis of the stochastic model that follows in later sections. I introduce a Markov regime-switching process for the marginal operating profit of capital in Section 3 to generate stochastic variation in q and optimal investment. In Section 4, I analyze the impact of various changes in the stochastic properties of the marginal operating profit of capital, including changes to the unconditional distribution and to the persistence of this exogenous random variable. In order to account for the positive impact of cash flow on investment, even after taking account of q , I introduce measurement error in Section 5. In addition, I show that this cash flow coefficient tends to be larger for firms that are growing more rapidly. Concluding remarks are in Section 6. The proofs of lemmas, propositions, and corollaries are in the Appendix.

1 The Hayashi Condition

Before describing the specific framework that I analyze in this paper, it is useful to begin with a simple, yet more general, description of the conditions under which average q and marginal q are equal. Consider a competitive firm with capital stock K_t at time t . The firm accumulates capital by undertaking gross investment I_t at time t , and capital depreciates at rate δ_t , so the capital stock evolves according to

$$\frac{dK_t}{dt} = I_t - \delta_t K_t. \tag{1}$$

The firm uses capital, K_t , and labor, L_t , to produce and sell output at time t . I assume that the price of capital goods is constant and normalize it to be one. Define $\pi_t(K_t, I_t) =$

$\max_{L_t} R(K_t, L_t, I_t) - I_t$, where $R(K_t, L_t, I_t)$ is revenue net of wage payments to labor and net of any investment adjustment costs. For now, I will simply assume that $\pi_t(K_t, I_t)$ is concave, increasing in K_t , and decreasing in I_t . Letting $M(t, s)$ be the stochastic discount factor used to discount cash flows at time $s \geq t$ back to time t , the value of the firm at time t is

$$V_t(K_t) = \max_{\{K_s, I_s\}} E_t \left\{ \int_t^\infty \pi_s(K_s, I_s) M(t, s) ds \right\}, \quad (2)$$

subject to equation (1). The following proposition presents conditions for the equality of average q and marginal q , which are essentially the same as in Hayashi (1982), though the method of proof is different from Hayashi's proof and the framework is generalized to include uncertainty and possible non-separability of costs of adjustment.

Proposition 1 (*Hayashi*) *If $\pi_s(K_s, I_s)$ is linearly homogeneous in K_s and I_s , then for any $\omega \geq 0$, $V_t(\omega K_t) = \omega V_t(K_t)$, i.e., the value function is linearly homogeneous in K_t , and hence average q , $\frac{V_t(K_t)}{K_t}$, equals marginal q , $V_t'(K_t)$.*

For the remainder of this paper, I will assume that $\pi_s(K_s, I_s)$ is linearly homogeneous in K_s and I_s so that average q and marginal q are equal.

2 Model of the Firm

Consider a competitive firm that faces convex costs of adjustment that are separable from the production function. The firm uses capital, K_t , and labor, L_t , to produce non-storable output, Y_t , at time t according to the production function $Y_t = A_t f(K_t, L_t)$, where $f(K_t, L_t)$ is linearly homogeneous in K_t and L_t , and A_t is the exogenous level of total factor productivity. If the amount of labor is costlessly and instantaneously adjustable, the firm chooses L_t at time t to maximize instantaneous revenue less wages $p_t A_t f(K_t, L_t) - w_t L_t$, where p_t is the price of the firm's output at time t and w_t is the wage rate per unit of labor at time t . The linear homogeneity of $f(K_t, L_t)$ and the assumption that the firm is a price-taker in the markets for its output and its labor together imply that the maximized value of revenue less wages is $\Phi_t K_t$, where $\Phi_t \equiv \max_l [p_t A_t f(1, l) - w_t l]$. The marginal (and average) operating profit of capital, Φ_t , is a deterministic function of A_t , p_t , and w_t , all of which are exogenous to the firm and possibly stochastic. Therefore, Φ_t is exogenous to the firm and, henceforth, I will treat Φ_t as the fundamental exogenous variable facing the firm, comprising the effects of productivity, output price and the wage rate.

I assume that the depreciation rate of capital is constant, so that net investment, $\frac{dK_t}{dt}$, is given by equation (1) with δ_t equal to the constant δ . Define $\gamma_t \equiv \frac{I_t}{K_t}$ to be the investment-capital ratio at time t . Therefore, the growth rate of the capital stock, g_t , is

$$g_t \equiv \frac{1}{K_t} \frac{dK_t}{dt} = \gamma_t - \delta, \quad (3)$$

so that for $s \geq t$

$$K_s = K_t \exp\left(\int_t^s g_u du\right). \quad (4)$$

Finally, I will specify the stochastic discount factor, $M(t, s)$, to be simply $\exp(-r(s - t))$, so that net cash flows are discounted at the constant rate r .

At time t , the firm chooses gross investment, I_t . The cost of this investment has two components. The first component is the cost of purchasing capital at a price per unit that I assume to be constant over time and normalize to be one. Thus, this component of the cost of gross investment at rate I_t is simply I_t , which, of course, would be negative if the firm sells capital so that $I_t < 0$. The second component is the cost of adjustment, $C(I_t, K_t)$, which I assume to be linearly homogeneous in I_t and K_t . It will be convenient to use the definition of the investment-capital ratio, $\gamma_t \equiv \frac{I_t}{K_t}$, to write the adjustment cost function as $c(\gamma_t) K_t$, where $c(\gamma_t) \geq 0$ is at least twice differentiable, $c'(r + \delta) > 0$, and for some $\zeta > 0$, $c''(\gamma_t) > \zeta$ for all γ_t so that the $c(\gamma_t)$ is strictly convex. Therefore, after choosing the optimal usage of labor, the amount of revenue less wages and less the cost of investment is

$$\pi_t(K_t, I_t) \equiv [\Phi_t - \gamma_t - c(\gamma_t)] K_t. \quad (5)$$

2.1 Constant Φ_t

Consider the case in which the marginal operating profit of capital, Φ_t , is constant forever. This case is simple enough that closed-form solutions for the value of the firm and optimal investment are readily available when the adjustment cost function $c(\gamma_t)$ is quadratic. More importantly, however, the analytic apparatus developed in this case will prove to be useful in later sections when Φ_t evolves according to a Markov regime-switching process.

I begin by defining an admissible set of values for the constant marginal operating profit of capital, Φ . First, a lower bound on Φ is provided by the assumption that Φ must be large enough so that revenue net of wages, adjustment costs, and expenditure on investment, $\pi_t(K_t, I_t)$, is positive for some value of γ_t . Note that $\pi_t(K_t, I_t)$ is a concave function of γ_t and let γ^m be the unique value of γ_t that maximizes $\pi_t(K_t, I_t) = \pi_t(1, \gamma_t) K_t$. Because $c(\gamma_t)$ is twice differentiable, $c'(\gamma^m) = -1$, which means that the marginal cost of purchasing and installing capital at rate γ^m , $1 + c'(\gamma^m)$, equals zero. To make sure that Φ is large enough for $\pi_t(K_t, I_t)$ to be positive for some γ_t , I confine attention to cases in which $\pi_t(1, \gamma^m) = \Phi - c(\gamma^m) - \gamma^m$, is positive. Therefore, I assume that $\Phi > c(\gamma^m) + \gamma^m$.

To motivate an upper bound on admissible values of Φ , I require that the value of the firm is finite. The requirement of finite value implies that $\Phi \leq c(r + \delta) + r + \delta$. To see why, suppose that $\Phi > c(r + \delta) + r + \delta$, which implies that there exists $\hat{\gamma} > r + \delta$ for which $\pi_t(1, \hat{\gamma}) = \Phi - c(\hat{\gamma}) - \hat{\gamma} > 0$. If a firm chooses to set $\gamma = \hat{\gamma}$ forever, then $\pi_t(K_t, I_t) = \pi(1, \hat{\gamma}) K_t > 0$ would grow at the rate $\hat{\gamma} - \delta > r$ forever, which would make the value of the firm infinite. Therefore, finiteness of firm

value requires $\Phi \leq c(r + \delta) + r + \delta$. To avoid a singularity in equation (9) below, I rule out $\Phi = c(r + \delta) + r + \delta$. Therefore, I confine attention to $\Phi < c(r + \delta) + r + \delta$.

Putting together the lower and upper bounds on Φ described above, define the set of admissible values of Φ to be²

$$G \equiv \{\Phi : c(\gamma^m) + \gamma^m < \Phi < c(r + \delta) + r + \delta\}. \quad (6)$$

2.1.1 The Value of a Unit of Capital

With a constant marginal operating profit of capital, Φ , constant depreciation rate, δ , and constant discount rate, r , the optimal investment-capital ratio, γ_t , is constant also. In this case, the value of the firm in equation (2) can be written as

$$V_t(K_t) = \max_{\gamma} \int_t^{\infty} [\Phi - \gamma - c(\gamma)] K_s e^{-r(s-t)} ds. \quad (7)$$

Dividing both sides of equations (7) by K_t and using equation (4) with $g_u = \gamma - \delta$, yields an expression for the average value of a unit of capital, v , which is

$$v = \max_{\gamma} \frac{\Phi - \gamma - c(\gamma)}{r + \delta - \gamma}. \quad (8)$$

Differentiating the maximand on the right-hand side of equation (8) with respect to γ and setting the derivative equal to zero yields

$$1 + c'(\gamma) = \frac{\Phi - \gamma - c(\gamma)}{r + \delta - \gamma}. \quad (9)$$

Solving equation (9) for γ yields

$$\Phi - (r + \delta) - c(\gamma) - (r + \delta - \gamma) c'(\gamma) = 0. \quad (10)$$

It will be useful to define a function $H(\gamma, \Phi, \rho)$ to characterize the optimal investment-capital ratio. Specifically,

$$H(\gamma, \Phi, \rho) \equiv \Phi - \rho - c(\gamma) - (\rho - \gamma) c'(\gamma), \quad (11)$$

and the optimal value of γ , characterized by equation (10), satisfies

$$H(\gamma, \Phi, r + \delta) = 0. \quad (12)$$

Note that $H(\gamma, \Phi, \rho)$ is an increasing, linear function of Φ and, for $\gamma > \gamma^m$, $H(\gamma, \Phi, \rho)$ is a decreasing, linear function of ρ (because $H_{\rho} = -(1 + c'(\gamma))$, which is negative for $\gamma > \gamma^m$). To describe the

²Since $c'(\gamma^m) = -1$ and $c'(r + \delta) > 0$, the strict convexity of $c(\gamma_t)$ implies that $\gamma^m < r + \delta$ and that $c(\gamma_t) + \gamma_t$ is strictly increasing in γ_t for all $\gamma_t > \gamma^m$. Therefore, $c(r + \delta) + r + \delta > c(\gamma^m) + \gamma^m$ so that G is non-vacuous.

dependence of $H(\gamma, \Phi, \rho)$ on γ , differentiate $H(\gamma, \Phi, \rho)$ with respect to γ to obtain

$$H_\gamma(\gamma, \Phi, \rho) = -(\rho - \gamma) c''(\gamma). \quad (13)$$

Since $c''(\gamma) > 0$, $H(\gamma, \Phi, \rho)$ is strictly decreasing in γ for $\gamma < \rho$, strictly increasing in γ for $\gamma > \rho$, and minimized with respect to γ at $\gamma = \rho$. The minimized value of $H(\gamma, \Phi, \rho)$ is $H(\rho, \Phi, \rho) = \Phi - \rho - c(\rho)$, which is negative for any $\Phi \in G$ if $\rho \geq r + \delta$.³

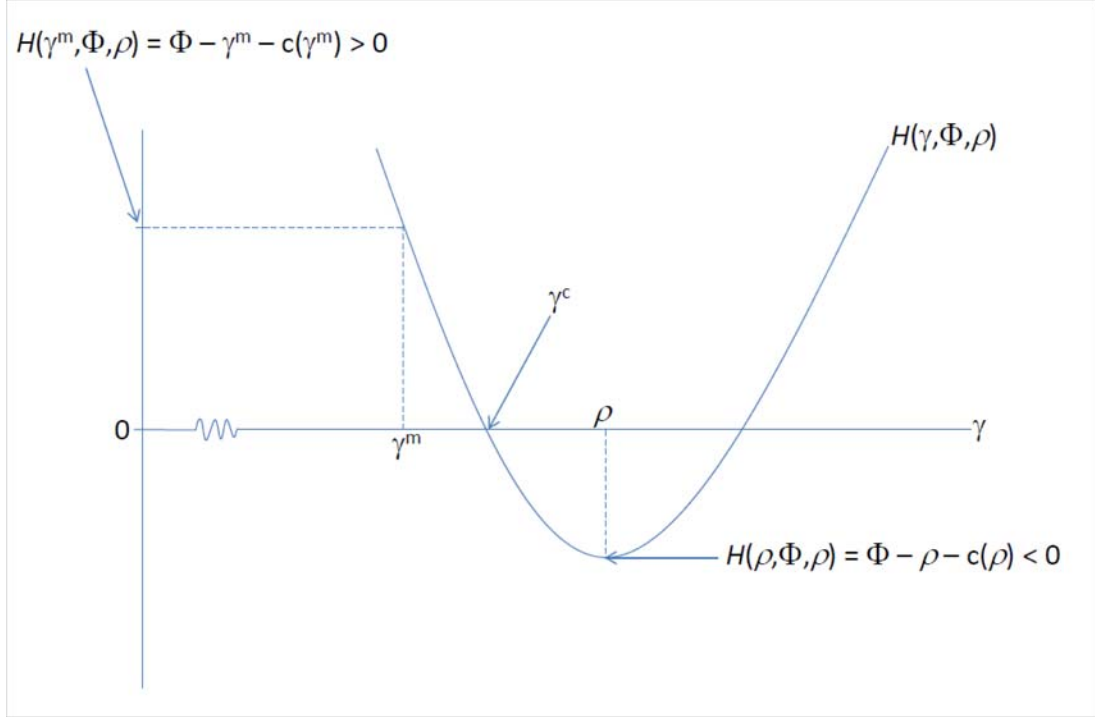


Figure 1 illustrates $H(\gamma, \Phi, \rho)$ as a function of γ for given values of Φ and ρ . As pointed out above, the function $H(\gamma, \Phi, \rho)$ is strictly decreasing in γ for $\gamma < \rho$, strictly increasing in γ for $\gamma > \rho$, and attains its minimum value of $H(\rho, \Phi, \rho) = \Phi - \rho - c(\rho) < 0$ at $\gamma = \rho$. Also shown in Figure 1 is $H(\gamma^m, \Phi, \rho) = \Phi - c(\gamma^m) - \gamma^m > 0$ for $\Phi \in G$. If the adjustment cost function $c(\gamma)$ is quadratic, then $H(\gamma, \Phi, \rho)$ is quadratic in γ and thus is a convex function of γ . In general, however, $H(\gamma, \Phi, \rho)$ need not be convex in γ . However, it must slope downward monotonically for $\gamma^m < \gamma < \rho$ and it must slope upward monotonically for $\gamma > \rho$, as shown in Figure 1. These properties lead to the following lemma, which is proved in the Appendix.

Lemma 1 : *If $\Phi \in G$ and if $\rho \geq r + \delta$, then $H(\gamma, \Phi, \rho) = 0$ has a one real root in (γ^m, ρ) and one real root in (ρ, ∞) .*

Because the optimal value of the investment-capital ratio, γ , is a root of $H(\gamma, \Phi, \rho) = 0$ when

³If $\rho \geq r + \delta$, then $H(\rho, \Phi, \rho) \leq H(\rho, \Phi, r + \delta) \leq H(r + \delta, \Phi, r + \delta) = \Phi - (r + \delta) - c(r + \delta) < 0$ for any $\Phi \in G$.

$\rho = r + \delta$, and Lemma 1 states that this equation has two roots, the next step is to determine which root is the optimal value. Denoting these roots as $\gamma_1 < \gamma_2$, and using the fact that $H(\gamma_i, \Phi, \rho) = 0$, for $i = 1, 2$, yields

$$\Phi - \gamma_i - c(\gamma_i) = (\rho - \gamma_i) [1 + c'(\gamma_i)], \quad i = 1, 2. \quad (14)$$

The left hand side of equation (14) is $\pi(1, \gamma_i)$, revenue less wages and investment costs, for a unit of capital, at γ_i . Since $1 + c'(\gamma) > 0$ for $\gamma > \gamma^m$, it follows that $\pi(1, \gamma_i) > 0$ for $\gamma_i = \gamma_1 \in (\gamma^m, \rho)$ and $\pi(1, \gamma_i) < 0$ for $\gamma_i = \gamma_2 \in (\rho, \infty)$. Since $\pi(1, \gamma_1) > 0 > \pi(1, \gamma_2)$, the optimal value of the investment-capital ratio is γ_1 , which proves the following proposition. I denote this root as γ^c , where the superscript "c" indicates the optimal value of γ under *certainty*.

Proposition 2 *If $\Phi \in G$ is known with certainty to be constant forever, then the optimal investment-capital ratio, $\gamma^c(\Phi, r + \delta)$, is the unique value of $\gamma \in (\gamma^m, r + \delta)$ that satisfies $H(\gamma, \Phi, r + \delta) = 0$.*

Corollary 1 *If $\rho \geq r + \delta$, then $\frac{\partial \gamma^c(\Phi, \rho)}{\partial \Phi} = \frac{1}{(\rho - \gamma^c)c''(\gamma^c)} > 0$ and $\frac{\partial \gamma^c(\Phi, \rho)}{\partial \rho} = -\frac{1 + c'(\gamma^c)}{(\rho - \gamma^c)c''(\gamma^c)} < 0$.*

Corollary 1 states that a firm with a higher deterministic value of marginal operating profit of capital, Φ , will have a higher optimal value of the investment-capital ratio. It also states that a firm with a higher user cost of capital, $r + \delta$, will have a lower optimal value of the investment-capital ratio.

Corollary 2 *If $\rho \geq r + \delta$, then for given ρ , $\frac{\partial v}{\partial \Phi} = \frac{1}{\rho - \gamma^c} > 0$, $\frac{\partial^2 v}{(\partial \Phi)^2} = \frac{1}{(\rho - \gamma^c)^3} \frac{1}{c''(\gamma^c)} > 0$, and $\frac{\partial v}{\partial \rho} = -\frac{1 + c'(\gamma^c)}{\rho - \gamma^c} < 0$.*

Corollary 2 states that v , which is the common value of marginal q and average q , is an increasing convex function of the marginal profit of capital, Φ , and a decreasing function of the user cost of capital.

2.2 Changes in the Adjustment Cost Function

Now consider the effects of a change in the adjustment cost function that reduces the adjustment cost at each value of γ . Specifically, let the adjustment cost function be $c(\gamma, \varepsilon)K$, where $c \geq 0$, $c_{\gamma\gamma} > 0$, and $c_\varepsilon > 0$, and consider the impact of a decrease in ε . In the face of a decrease in ε , the firm could maintain the originally-planned path of the investment-capital ratio, γ , and hence would maintain the originally-planned path of the capital stock. However, cash flow per unit of capital, $\Phi - \gamma - c(\gamma, \varepsilon)$, would be higher than its originally-planned level at each point of time. Hence, the value of the firm would increase, which for a given path of the capital stock, would increase average q , and hence increase marginal q . The following proposition states this result formally. The proof of this proposition, which is in the Appendix, takes a different line of reasoning.

Proposition 3 *If $c_\varepsilon(\gamma, \varepsilon) > 0$, then $\frac{\partial v}{\partial \varepsilon} < 0$.*

A change in the adjustment cost function can affect the optimal value of γ through two distinct channels: by changing the total adjustment cost and by changing the marginal adjustment cost. To illustrate the first channel, consider a change that increases the adjustment cost function by a constant amount at each level of γ , thereby leaving the marginal adjustment cost, $c_\gamma(\gamma, \varepsilon)$ unchanged. In this case, Proposition 3 indicates that v , the common value of average q and marginal q , would fall and hence the optimal value of γ would fall because optimal γ satisfies $1 + c_\gamma(\gamma, \varepsilon) = v$ and $c_{\gamma\gamma}(\gamma, \varepsilon) > 0$. The second channel, which also reduces optimal γ , operates through a change in the marginal adjustment cost function. Specifically, an increase in the marginal adjustment cost function would reduce the optimal value of γ associated with any given value of v . These two channels, which both reduce optimal γ in response to an increase in ε , are captured in the following corollary to Proposition 3.

Corollary 3 *If $c_\varepsilon(\gamma, \varepsilon) \geq 0$ and $c_{\gamma\varepsilon}(\gamma, \varepsilon) \geq 0$, and if $c_\varepsilon(\gamma, \varepsilon) + c_{\gamma\varepsilon}(\gamma, \varepsilon) > 0$, then $\frac{\partial\gamma}{\partial\varepsilon} < 0$.*

Corollary 3 implies that either an upward shift of the adjustment cost function or an upward shift of the marginal adjustment cost function is sufficient to reduce the optimal value of γ .

2.3 Quadratic Adjustment Cost Example

Now consider an example with a quadratic adjustment cost function, $c(\gamma)$, which permits easy derivation of closed-form solutions for the optimal value of γ and for the common value of average q and marginal q . Assume that $c(\gamma) = \theta\gamma^2$, which implies that G , the permissible set of marginal operating profit of capital, Φ , is $\left\{ \Phi : -\frac{1}{4\theta} < \Phi < \theta(r + \delta)^2 + r + \delta \right\}$. With this quadratic adjustment cost function, and setting $\rho = r + \delta$, the function $H(\gamma, \Phi, \rho)$ in equation (11) is $H(\gamma, \Phi, r + \delta) = \Phi - (r + \delta) - 2\theta\gamma(r + \delta) + \theta\gamma^2$. The root of $H(\gamma, \Phi, r + \delta) = 0$ that is smaller than $r + \delta$ is

$$\gamma^c = \left[1 - \sqrt{1 - \frac{\Phi - (r + \delta)}{\theta(r + \delta)^2}} \right] (r + \delta) < r + \delta, \quad (15)$$

where, as defined earlier, γ^c as the optimal value of γ under certainty.

Since $v = 1 + c'(\gamma) = 1 + 2\theta\gamma$ at the optimal value of γ , equation (15) implies

$$v = 1 + 2\theta \left[1 - \sqrt{1 - \frac{\Phi - (r + \delta)}{\theta(r + \delta)^2}} \right] (r + \delta). \quad (16)$$

The values of γ^c and v in equations (15) and (16) are real because $\Phi \in G$ implies that $\Phi < \theta(r + \delta)^2 + r + \delta$, which implies that $1 - \frac{\Phi - (r + \delta)}{\theta(r + \delta)^2} > 0$. If the marginal operating profit of capital, Φ , exceeds the user cost of capital, $r + \delta$, then v , the common value of average q and marginal q , is greater than one and the optimal investment-capital ratio, γ^c , is positive. However, if the marginal operating profit of capital, Φ , is smaller than the user cost of capital, $r + \delta$, then v , the common value of average q and marginal q , is smaller than one and the optimal investment-capital ratio, γ^c , is

negative. Finally, if the marginal operating profit of capital, Φ , equals the user cost of capital, $r + \delta$, then the common value of average q and marginal q equals one and the optimal investment-capital ratio is zero.

3 Markov Regime-Switching Process for Φ_t

In this section I develop and analyze a model of a firm facing stochastic variation in the marginal operating profit of capital, Φ_t , governed by a Markov regime-switching process. Specifically, a regime is defined by a constant value of Φ_t . If the marginal operating profit of capital at time t , Φ_t , equals ϕ , it remains equal to ϕ until a new regime arrives. The arrival of a new regime is a Poisson process with probability λdt of a new arrival during a time interval of length dt . When a new regime arrives, a new value of the marginal operating profit of capital, Φ , is drawn from a distribution with c.d.f $F(\Phi)$, where the support of $F(\Phi)$ is in G , defined in equation (6). $F(\Phi)$ can be continuous or not continuous, so the random variable Φ can be continuous or discrete.

The Markovian nature of Φ implies that the value of the firm at time t depends only on the capital stock at time t , K_t , and the value of the marginal operating profit at time t , ϕ . The value of the firm $V(K_t, \phi)$ is

$$\begin{aligned} V(K_t, \phi) = & \max_{\gamma_t} \int_t^{t+dt} [\phi - \gamma_t - c(\gamma_t)] K_t e^{-r(s-t)} ds \\ & + e^{-\lambda dt} e^{-r dt} V(K_{t+dt}, \phi) \\ & + (1 - e^{-\lambda dt}) e^{-r dt} \int_G V(K_{t+dt}, \Phi) dF(\Phi), \end{aligned} \quad (17)$$

which is the maximized sum of three terms. The first term is the present value of $\pi(K_s, I_s) = [\phi - \gamma_s - c(\gamma_s)] K_s$ over the infinitesimal interval of time from t to $t + dt$. The second term is the present value of the firm at time $t + dt$, conditional on Φ remaining equal to ϕ at time $t + dt$, weighted by the probability, $e^{-\lambda dt}$, that $\Phi_{t+dt} = \phi$. The third term is the present value of the expected value of the firm at time $t + dt$ conditional on a new regime for Φ at time $t + dt$, weighted by the probability of new regime at time $t + dt$.

The Hayashi conditions in Proposition 1 hold in this framework so that the value of the firm is proportional to the capital stock. Therefore, the average value of the capital stock, $\frac{V(K_t, \phi)}{K_t}$, is independent of the capital stock and depends only on ϕ . I will define $v(\phi) \equiv \frac{V(K_t, \phi)}{K_t}$ to be Tobin's q , or equivalently, the average value of the capital stock. Since average q and marginal q are identically equal in this framework, $v(\phi)$ is also marginal q .

Use the definition $v(\phi) \equiv \frac{V(K_t, \phi)}{K_t}$ and the fact that $\frac{K_{t+dt}}{K_t} = e^{g_t dt} = e^{(\gamma_t - \delta) dt}$ and perform the

first integration on the right-hand side of equation (17) to obtain

$$\begin{aligned}
v(\phi) &= \max_{\gamma} [\phi - \gamma - c(\gamma)] \frac{1 - e^{-rdt}}{r} \\
&\quad + e^{-\lambda dt} e^{-rdt} e^{(\gamma_t - \delta)dt} v(\phi) \\
&\quad + (1 - e^{-\lambda dt}) e^{-rdt} e^{(\gamma_t - \delta)dt} \int_G v(\Phi) dF(\Phi).
\end{aligned} \tag{18}$$

Take the limit of equation (18) as dt goes to zero to obtain

$$0 = \max_{\gamma} \phi - \gamma - c(\gamma) - (r + \delta + \lambda - \gamma) v(\phi) + \lambda \bar{v}, \tag{19}$$

where

$$\bar{v} \equiv \int_G v(\Phi) dF(\Phi) \tag{20}$$

is the unconditional expected value of a unit of capital, which is also the unconditional expected value of both average q and marginal q .

The maximization in equation (19) has the first-order condition

$$1 + c'(\gamma) = v(\phi). \tag{21}$$

The optimal value of γ equates the marginal cost of investment, including the purchase price of capital and the marginal adjustment cost, with marginal q and average q .

3.1 Alternative Derivations of Average q and Marginal q

In this section I present alternative derivations of average q and marginal q . Because the model presented here is a special case of Proposition 1, I have already proved that average q and marginal q are equal. Nevertheless, it is helpful to examine different expressions for average q and marginal q and to understand why these expressions, which at first glance look different, are equivalent.

Marginal q at time t is commonly expressed as the expected present value of the stream of contributions to revenue, less wages and investment costs, of the remaining undepreciated portion of a unit of capital installed at time t , which is

$$q_t = E_t \left\{ \int_t^{\infty} \frac{\partial \pi_s(K_s, I_s)}{\partial K_s} e^{-(r+\delta)(s-t)} ds \right\}. \tag{22}$$

Suppose that $\Phi_s = \phi$ for all $t \leq s < x$ and the regime switches at time x , with a new drawing of Φ from the unconditional distribution $F(\Phi)$. The expression for $\pi_s(K_s, I_s)$ in equation (5) can be written as $\pi_s(K_s, I_s) = \left[\phi - c\left(\frac{I_s}{K_s}\right) \right] K_s - I_s$. Therefore, $\frac{\partial \pi_s(K_s, I_s)}{\partial K_s} = \phi - c(\gamma_s) + \gamma_s c'(\gamma_s)$, which equals $\phi - c(\gamma_t) + \gamma_t c'(\gamma_t)$ for all $t \leq s < x$. As for the stream of marginal contributions of capital accruing from time x onward, their expected present value as of time x is $e^{-\delta(x-t)} q_x$; the expected present value of $e^{-\delta(x-t)} q_x$ as of time t is $e^{-(r+\delta)(x-t)} \bar{q}$, where \bar{q} is the unconditional expected value

of q_x . Therefore, $q_{t|x}$, the value of q_t , conditional on the next regime switch occurring at time $x > t$ is

$$q_{t|x} = \frac{1 - e^{-(r+\delta)(x-t)}}{r + \delta} (\phi - c(\gamma_t) + \gamma_t c'(\gamma_t)) + e^{-(r+\delta)(x-t)} \bar{q}. \quad (23)$$

The first term on the right-hand side of equation (23) is the present value of $\frac{\partial \pi_s(K_s, I_s)}{\partial K_s} e^{-\delta(s-t)} = \frac{\partial \pi_s(1, \gamma_s)}{\partial K_s} e^{-\delta(s-t)}$ from time t to time x .⁴ The second term is the expected present value, discounted to time t , of $\frac{\partial \pi_s(K_s, I_s)}{\partial K_s} e^{-\delta(s-t)}$ from time x onward.

The probability that the first switch in the regime after time t occurs at time $x > t$ is $\lambda e^{-\lambda(x-t)}$, so that

$$q(\phi) = \int_t^\infty \lambda e^{-\lambda(x-t)} q_{t|x} dx, \quad (24)$$

where $q(\phi)$ is the value of marginal q at time t conditional on $\Phi_t = \phi$. Substituting equation (23) into equation (24) and performing the integration yields

$$q(\phi) = \frac{\phi - c(\gamma) + \gamma c'(\gamma)}{r + \delta + \lambda} + \frac{\lambda}{r + \delta + \lambda} \bar{q}, \quad (25)$$

where γ in this equation is the optimal value of γ when $\Phi = \phi$.

Average q at time t is the value of the firm at time t divided by K_t . Dividing both sides of equation (2) by K_t , using the linear homogeneity of $\pi_s(K_s, I_s)$, and using equation (4) and $M(t, s) = \exp(-r(s-t))$ yields

$$v_t = E_t \left\{ \int_t^\infty \pi_s(1, \gamma_s) \exp\left(-\int_t^s (r - g_u) du\right) ds \right\}, \quad (26)$$

where γ_s and g_u are evaluated along the path of optimal behavior. Again, suppose that $\Phi_s = \phi$ for all $t \leq s < x$ and the regime switches at time x , with a new drawing of Φ from the unconditional distribution $F(\Phi)$. With $\pi_s(K_s, I_s)$ specified as in equation (5), $\pi_s(1, \gamma_s) = \phi - \gamma_t - c(\gamma_t)$ for all $t \leq s < x$ and $g_u = g_t = \gamma_t - \delta$ for all $t \leq u < x$. Therefore, $v_{t|x}$, the value of v_t , conditional on the next regime switch occurring at time $x > t$, is

$$v_{t|x} = \frac{1 - e^{-(r+\delta-\gamma)(x-t)}}{r + \delta - \gamma} (\phi - \gamma_t - c(\gamma_t)) + e^{-(r+\delta-\gamma)(x-t)} \bar{v}, \quad (27)$$

where \bar{v} is the unconditional expected value of $v(\phi)$ defined in equation (20). The first term on the right-hand side of equation (27) is the present value of $\pi_s(1, \gamma_s) e^{(\gamma-\delta)(s-t)}$ from time t to time x . The second term on the right-hand side of equation (27) is the expected present value, discounted to time t , of $\pi_s(1, \gamma_s) e^{(\gamma-\delta)(s-t)}$ from time x onward.

Since the probability that the first switch in the regime after time t occurs at time $x > t$ is

⁴The equality of $\frac{\partial \pi_s(K_s, I_s)}{\partial K_s} e^{-\delta(s-t)}$ and $\frac{\partial \pi_s(1, \gamma_s)}{\partial K_s} e^{-\delta(s-t)}$ is an implication of the linear homogeneity of $\pi_s(K_s, I_s)$.

$\lambda e^{-\lambda(x-t)}$, the average value of capital is

$$v(\phi) = \int_t^\infty \lambda e^{-\lambda(x-t)} v_t|_x dx, \quad (28)$$

where, as in equation (18), $v(\phi)$ is the value of average q at time t when $\Phi_t = \phi$.

Next substitute equation (27) into equation (28) and perform the integration to obtain

$$v(\phi) = \frac{\phi - \gamma - c(\gamma)}{r + \delta + \lambda - \gamma} + \frac{\lambda}{r + \delta + \lambda - \gamma} \bar{v}, \quad (29)$$

where γ in this equation is the optimal value of γ when $\Phi = \phi$.

Proposition 1 implies that marginal q in equation (25) and average q in equation (29) are equal to each other, and therefore that \bar{q} and \bar{v} are equal to each other, yet these equations look different. They have different effective discount rates that in appear in the denominator ($r + \delta + \lambda$ vs. $r + \delta + \lambda - \gamma$); and marginal q discounts the flow $\frac{\partial \pi_s(1_s, \gamma_s)}{\partial K_s} e^{-\delta(s-t)}$ while average q discounts the flow $\pi_s(1, \gamma_s) e^{(\gamma-\delta)(s-t)}$. To reconcile the expressions for marginal q in (25) and average q in (29), add and subtract γ to the numerator of the first of the two terms in equation (25) to obtain

$$q(\phi) = \frac{\phi - \gamma - c(\gamma) + \gamma(1 + c'(\gamma))}{r + \delta + \lambda} + \frac{\lambda}{r + \delta + \lambda} \bar{q}. \quad (30)$$

Next use Proposition 1 to re-write the first-order condition for optimal investment in equation (21) as $1 + c'(\gamma) = q(\phi)$, and then substitute $q(\phi)$ for $1 + c'(\gamma)$ in equation (30) to obtain

$$q(\phi) = \frac{\phi - \gamma - c(\gamma) + \gamma q(\phi)}{r + \delta + \lambda} + \frac{\lambda}{r + \delta + \lambda} \bar{q}, \quad (31)$$

which can be rearranged to yield

$$q(\phi) = \frac{\phi - \gamma - c(\gamma)}{r + \delta + \lambda - \gamma} + \frac{\lambda}{r + \delta + \lambda - \gamma} \bar{q}. \quad (32)$$

Notice that the expression for average q in (29) is equivalent to the expression for marginal q in (32), as must be the case from Proposition 1. The key step in reconciling the expression for marginal q in equation (25) and the expression for average q in equation (29) is to use the basic first-order condition that drives the q -theory of investment, namely, $1 + c'(\gamma) = q$.

3.2 Quadratic Adjustment Cost Example

Now continue with the quadratic adjustment cost example introduced in subsection 2.3, where $c(\gamma) = \theta\gamma^2$, which implies that

$$c'(\gamma) = 2\theta\gamma \quad (33)$$

and

$$-c(\gamma) + \gamma c'(\gamma) = \theta\gamma^2. \quad (34)$$

Substitute equation (33) into the first-order condition in equation (21), and use the fact that average q , $v(\phi)$, and marginal q , $q(\phi)$, are equal to obtain

$$1 + 2\theta\gamma = q(\phi). \quad (35)$$

Now replace the right hand side of equation (35) with the right hand side of equation (25), use equation (34), and rearrange terms to get the following quadratic equation in γ

$$\theta\gamma^2 - 2(r + \delta + \lambda)\theta\gamma + \phi + \lambda\bar{q} - (r + \delta + \lambda) = 0. \quad (36)$$

The quadratic equation in equation (36) has two real roots⁵. The optimal value of γ is the root smaller⁶ than $r + \delta$, which is

$$\gamma = \left[1 - \sqrt{1 - \frac{\phi - (r + \delta) + \lambda(\bar{q} - 1)}{\theta(r + \delta + \lambda)^2}} \right] (r + \delta + \lambda). \quad (37)$$

Therefore, equations (35) and (37) imply that

$$q = 1 + 2\theta \left[1 - \sqrt{1 - \frac{\phi - (r + \delta) + \lambda(\bar{q} - 1)}{\theta(r + \delta + \lambda)^2}} \right] (r + \delta + \lambda). \quad (38)$$

The expressions for optimal γ and q in equations (37) and (38) are closed-form functions of parameters plus one other variable, \bar{q} , which is constant over time for any given firm. I discuss the calculation of \bar{q} in subsection 3.4.

3.3 Optimal Investment

In this section I exploit the first-order condition for optimal investment in equation (21) to analyze several properties of optimal γ . The optimal value of γ depends on \bar{v} . For now, I will treat \bar{v} as a parameter and defer further analysis and computation of \bar{v} to subsection 3.4.

To analyze optimal γ , substitute the first-order condition for optimal γ from equation (21) into equation (19) to obtain⁷

$$0 = \phi - c(\gamma) - (r + \delta + \lambda) - (r + \delta + \lambda - \gamma) c'(\gamma) + \lambda\bar{v}. \quad (39)$$

⁵To prove that equation (36) has two real roots, it suffices to show that $L(\gamma) \equiv \theta\gamma^2 - 2(r + \delta + \lambda)\theta\gamma + \phi + \lambda\bar{q} - (r + \delta + \lambda) < 0$ for some γ . In particular, $L(r + \delta + \lambda) = -\theta(r + \delta + \lambda)^2 + \phi + \lambda\bar{q} - (r + \delta + \lambda)$. Corollary 8 implies that $\bar{q} < 1 + c'(r + \delta) = 1 + 2\theta(r + \delta)$, so $L(r + \delta + \lambda) < -\theta(r + \delta + \lambda)^2 + \phi + \lambda + 2\lambda\theta(r + \delta) - (r + \delta + \lambda) = -\theta\lambda^2 + [\phi - (r + \delta) - \theta(r + \delta)^2]$. Since $\phi \in G$, $\phi < r + \delta + c(r + \delta) = (r + \delta) + \theta(r + \delta)^2$. Therefore, $L(r + \delta + \lambda) < 0$.

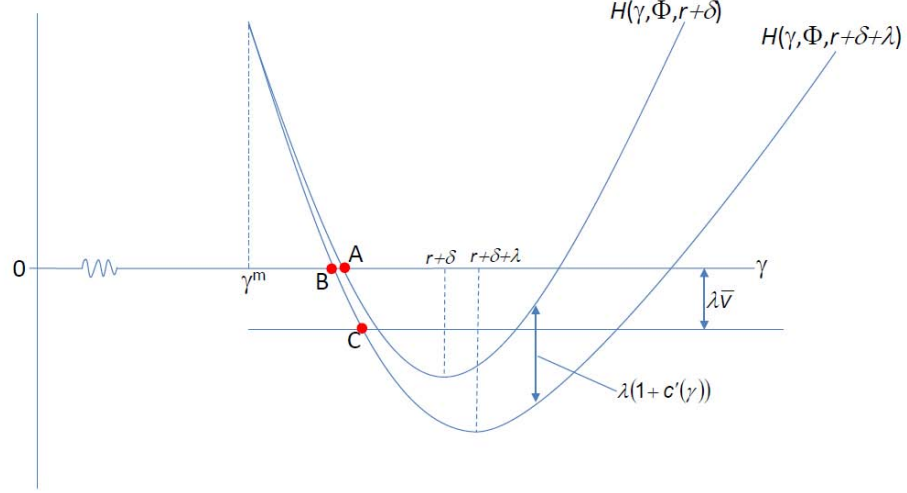
⁶The value of γ in equation (37) is less than $r + \delta$ if $\lambda < (r + \delta + \lambda) \sqrt{1 - \frac{\phi - (r + \delta) + \lambda(\bar{q} - 1)}{\theta(r + \delta + \lambda)^2}} = \sqrt{(r + \delta + \lambda)^2 - \frac{\phi - (r + \delta) + \lambda(\bar{q} - 1)}{\theta}}$, which will be the case if $J \equiv (r + \delta + \lambda)^2 - \frac{1}{\theta}(\phi - (r + \delta) + \lambda(\bar{q} - 1)) - \lambda^2 > 0$. Simplify the expression for J to obtain $J = \frac{1}{\theta} [\theta(r + \delta)^2 + 2\lambda\theta(r + \delta) - (\phi - (r + \delta) + \lambda(\bar{q} - 1))]$. Since $\phi \in G$, we have $\phi - (r + \delta) < c(r + \delta) = \theta(r + \delta)^2$, which implies $J > \frac{\lambda}{\theta} [1 + 2\theta(r + \delta) - \bar{q}]$. Using the fact that \bar{v} is the same as \bar{q} , Corollary 8 implies that $\bar{q} < 1 + c'(r + \delta) = 1 + 2\theta(r + \delta)$, which implies that $J > 0$ if $\lambda > 0$.

⁷Equation (39) can also be derived by substituting $1 + c'(\gamma)$ for $v(\phi)$ in equation (29) and rearranging terms.

Using the definition of $H(\gamma, \Phi, \rho)$ in equation (11), rewrite equation (39) as

$$H(\gamma, \phi, r + \delta + \lambda) = -\lambda\bar{v}. \quad (40)$$

- Point A : $\gamma = \gamma^c \equiv \gamma(\phi, 0, r + \delta, 0)$, certainty
- Point B : $\gamma = \gamma(\phi, 0, r + \delta, \lambda)$, firm is worthless in new regime
- Point C : $\gamma = \gamma(\phi, \bar{v}, r + \delta, \lambda)$, firm is worth \bar{v} in new regime



Equation (40) characterizes the optimal value of γ when there is a constant instantaneous probability, λ , of a regime switch. Of course, when $\lambda = 0$, this equation is equivalent to equation (12), which characterizes the optimal value of γ under certainty. The optimal value of γ when $\lambda = 0$ is shown in Figure 2 as point A where $H(\gamma, \phi, r + \delta) = 0$. The introduction of a positive value of λ , which introduces stochastic variation in the future values of $\pi_s(1, \gamma_s)$ and $\frac{\partial \pi_s(1, \gamma_s)}{\partial K_s}$, has two opposing effects on optimal γ in equation (40). First, the introduction of a positive value of λ increases the effective user cost of capital, ρ , from $r + \delta$ to $r + \delta + \lambda$. This increase in the user cost, ρ , reduces the value of $H(\gamma, \phi, \rho)$ by $\lambda(1 + c'(\gamma))$ at each value of γ , which induces the downward shift of the curve shown in Figure 2. This downward shift of the curve reduces the value of γ for which $H(\gamma, \phi, \rho) = 0$, as illustrated by the movement from point A to point B. The value of γ for which $H(\gamma, \phi, r + \delta + \lambda) = 0$ is the optimal value of γ that would arise if the firm were to disappear, with zero salvage value, when the regime switches. Thus, not surprisingly, the introduction of the possibility of a stochastic death of the firm reduces the optimal investment-capital ratio. However, if the new regime does not eliminate the firm, there is a second impact on optimal γ of the introduction of a positive value of λ . Specifically, if the firm receives a new draw of Φ from the unconditional distribution $F(\Phi)$ when the regime changes, then \bar{v} is the expected value of a unit of capital in the

new regime. With $\bar{v} > 0$, the term $-\lambda\bar{v}$ on the right-hand side of equation (40) is negative, so that $H(\gamma, \phi, r + \delta + \lambda) < 0$ at the optimal value of γ . Reducing the value of $H(\gamma, \phi, r + \delta + \lambda)$ from zero to a negative value requires an increase in γ , as shown in Figure 2 by the movement from point B to point C. To summarize, the introduction of stochastic variation in future Φ has two opposing effects on the optimal value of γ . As I will show in Section 4, for some values of ϕ the introduction of uncertainty will increase the optimal value of γ , and for other values of ϕ it will decrease the optimal value of γ .

Define $\gamma(\phi, \kappa, r + \delta, \lambda)$ to be the optimal value of γ for given values of $r + \delta$ and λ if $\Phi = \phi$ and $\bar{v} = \kappa$. Formally, $\gamma(\phi, \kappa, r + \delta, \lambda)$ is defined by

$$H(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, r + \delta + \lambda) = -\lambda\kappa. \quad (41)$$

Of course, this definition is meaningful only if $\min_{\gamma} H(\gamma, \phi, r + \delta + \lambda) \leq -\lambda\kappa$. Recall that $H(\gamma, \phi, \rho)$ is minimized at $\gamma = \rho$ and that $H(\rho, \Phi, \rho) = \Phi - \rho - c(\rho)$. Therefore, the definition of $\gamma(\phi, \kappa, r + \delta, \lambda)$ is meaningful only if $\phi - (r + \delta + \lambda) - c(r + \delta + \lambda) \leq -\lambda\kappa$. Observe that $H(\gamma^m, \phi, \rho) = \phi - \gamma^m - c(\gamma^m)$, which is positive for $\phi \in G$. Therefore, since $H_{\gamma}(\gamma^m, \Phi, \rho) < 0$ for $\gamma < \rho$

$$\gamma(\phi, \kappa, r + \delta, \lambda) > \gamma^m, \text{ for } 0 \leq \kappa \leq \frac{1}{\lambda} [(r + \delta + \lambda) + c(r + \delta + \lambda) - \phi]. \quad (42)$$

Note that $\gamma(\phi, 0, r + \delta, 0) = \gamma^c(\phi, r + \delta)$, which is the optimal value of the investment-capital ratio, γ , in the case in which $\Phi_t = \phi$ with certainty forever.

The following lemma and its corollary list several properties of $\gamma(\phi, \kappa, r + \delta, \lambda)$ and $c'(\gamma(\phi, \kappa, r + \delta, \lambda))$.

Lemma 2 Define $\rho \equiv r + \delta + \lambda$. If $\phi \in G$ and if $0 \leq \kappa \leq \frac{1}{\lambda}(\rho + c(\rho) - \phi)$, then

1. $\frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial \phi} = \frac{1}{(\rho - \gamma)c''(\gamma)} > 0$,
2. $\frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial \kappa} = \frac{\lambda}{(\rho - \gamma)c''(\gamma)} > 0$,
3. $\frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial (r + \delta)} = -\frac{1 + c'(\gamma)}{(\rho - \gamma)c''(\gamma)} < 0$,
4. $\frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial \lambda} = -\frac{1 + c'(\gamma) - \kappa}{(\rho - \gamma)c''(\gamma)}$.

Corollary 4 Define $\rho \equiv r + \delta + \lambda$. If $\phi \in G$ and if $0 \leq \kappa \leq \frac{1}{\lambda}(\rho + c(\rho) - \phi)$, then

1. $\frac{\partial c'(\gamma(\phi, \kappa, r + \delta, \lambda))}{\partial \phi} = \frac{1}{\rho - \gamma} > 0$,
2. $\frac{\partial c'(\gamma(\phi, \kappa, r + \delta, \lambda))}{\partial \kappa} = \frac{\lambda}{\rho - \gamma} > 0$,
3. $\frac{\partial c'(\gamma(\phi, \kappa, r + \delta, \lambda))}{\partial (r + \delta)} = -\frac{1 + c'(\gamma)}{\rho - \gamma} < 0$,
4. $\frac{\partial c'(\gamma(\phi, \kappa, r + \delta, \lambda))}{\partial \lambda} = -\frac{1 + c'(\gamma) - \kappa}{\rho - \gamma}$
5. $\frac{\partial^2 c'(\gamma(\phi, \kappa, r + \delta, \lambda))}{(\partial \phi)^2} = \frac{1}{(\rho - \gamma)^3 c''(\gamma)} > 0$.

Lemma 2 and its corollary show for any $\kappa \in [0, \frac{1}{\lambda}(\rho + c(\rho) - \phi)]$ and $\phi \in G$, both $\gamma(\phi, \kappa, r + \delta, \lambda)$ and $c'(\gamma(\phi, \kappa, r + \delta, \lambda))$ are increasing functions of ϕ and κ , and decreasing functions of $r + \delta$. The impact of a higher value of λ depends on the size of ϕ . This result is easiest to articulate for the case in which $\kappa = \bar{v}$, so that κ is the expected value of a unit of capital when ϕ is drawn from the unconditional distribution. In this case, an increase in λ hastens the arrival of a new regime in which the expected value of a unit of capital is κ . For values of ϕ that are small enough that $1 + c'(\gamma(\phi, \bar{v}, r + \delta, \lambda)) < \kappa = \bar{v}$, hastening the arrival of a new regime increases the value of a unit of capital, thereby increasing optimal γ and the optimal value of $c'(\gamma)$. Alternatively, for values of ϕ that are large enough that $1 + c'(\gamma(\phi, \bar{v}, r + \delta, \lambda)) > \kappa = \bar{v}$, hastening the arrival of a new regime means an earlier end to the current regime with a high ϕ . As a result, capital is less valuable and the optimal values of γ and $c'(\gamma)$ decline. Finally, the corollary shows that $c'(\gamma(\phi, \kappa, r + \delta, \lambda))$ is strictly convex in ϕ . This convexity will be helpful in subsection 4.2.1 when I analyze the impact on the value of a unit of capital of increased uncertainty in the unconditional distribution $F(\Phi)$.

3.4 Computation of Unconditional Optimal Value of a Unit of Capital

Equation (40) is a simple expression that characterizes the optimal value of γ . However, this expression depends on $\bar{v} \equiv \int_G v(\Phi) dF(\Phi)$, which is the unconditional expectation of the optimal value of a unit of installed capital. In this subsection, I describe how to compute \bar{v} as the unique fixed point of particular function. This discussion will serve four purposes. First, it will demonstrate that the fixed point exists; second, it will demonstrate that the fixed point is unique; third, it will provide a recursive procedure that converges to the fixed point; and fourth, it will help analyze the impact on optimal investment of changes in the distribution $F(\Phi)$ and changes in λ .

Define

$$\alpha(\kappa) \equiv 1 + \int_G c'(\gamma(\Phi, \kappa, r + \delta, \lambda)) dF(\Phi) \quad (43)$$

as the unconditional expectation of the marginal cost of investment, including the purchase cost of capital and the marginal adjustment cost, where $\gamma(\phi, \kappa, r + \delta, \lambda)$ is defined in equation (41) as the optimal value of the investment-capital ratio if $\Phi = \phi$ and $\bar{v} = \kappa$. Since the value of a unit of capital when $\Phi = \phi$ is $v(\phi) = 1 + c'(\gamma(\phi, \bar{v}, r + \delta, \lambda))$, optimal behavior by the firm implies that \bar{v} satisfies $\alpha(\bar{v}) = \bar{v}$.

Lemma 3 *Suppose that the support of the distribution $F(\Phi)$ is contained in $G \equiv \{\Phi : c(\gamma^m) + \gamma^m < \Phi < c(r + \delta) + r + \delta\}$. The function $\alpha(\kappa) \equiv 1 + \int_G c'(\gamma(\Phi, \kappa, r + \delta, \lambda)) dF(\Phi)$ has the following three properties: (1) $\alpha(0) > 0$; (2) $\alpha(1 + c'(r + \delta)) < 1 + c'(r + \delta)$; and (3) $\alpha'(\kappa) < 1$.*

Lemma 3 together with the continuity of $\alpha(\kappa)$ leads to the following proposition.

Proposition 4 *Suppose that the support of the distribution $F(\Phi)$ is contained in $G \equiv \{\Phi : c(\gamma^m) + \gamma^m < \Phi < c(r + \delta) + r + \delta\}$. Then \bar{v} is the unique positive value of $\kappa < 1 + c'(r + \delta)$ that satisfies $\alpha(\kappa) = \kappa$.*

Lemma 3 also leads to the following corollary, which describes a simple algorithm to compute \bar{v} .

Corollary 5 *Consider the sequence of κ_i , $i = 0, 1, 2, \dots$, defined by $\kappa_i = \alpha(\kappa_{i-1})$ where the initial value of κ_i is $\kappa_0 \in [0, 1 + c'(r + \delta)]$. Then $\lim_{i \rightarrow \infty} \kappa_i = \bar{v}$.*

Lemma 3 also leads to the following corollary, which will prove useful in analyzing the effects of changes in the distribution $F(\Phi)$ and changes in λ .

Corollary 6 *For any $\kappa^* \in [0, 1 + c'(r + \delta)]$, $\text{sign}[\alpha(\kappa^*) - \kappa^*] = \text{sign}[\bar{v} - \kappa^*]$.*

4 Effect of Changing the Stochastic Properties of Φ

In the section I consider the impact of the stochastic properties of the marginal operating profit of capital, Φ . As a first step, I consider the impact of introducing the Markov regime-switching process by comparing optimal behavior under Markov regime-switching in which new regimes are drawn from $F(\Phi)$ with optimal behavior under certainty. Then I consider changes in the unconditional distribution $F(\Phi)$ and the regime-switching probability, λ .

4.1 Markov Regime Switching vs. Certainty

The introduction of uncertainty in the form of a Markov regime-switching process for Φ can either increase or decrease the optimal investment-capital ratio, γ , associated with any particular value of Φ , compared to the optimal value of γ under certainty. For relatively low values of Φ , the possibility of switching regimes and drawing a new Φ from $F(\Phi)$ increases the value of an additional unit of capital, v , because the new regime is likely to have a higher value of Φ than the low value in the current regime. This increase in the value of an additional unit of capital will increase optimal γ . On the other hand, for relatively high values of Φ , the possibility of switching regimes reduces the value of an additional unit of capital, v , because the new regime is likely to have a lower value of Φ than the high value in the current regime. This reduction in v reduces optimal γ . These impacts of the introduction of Markov regime switching on the value of an additional unit of capital and on optimal investment are summarized in the following proposition and corollary.

Proposition 5 *Define $\hat{\phi}$ by $1 + c'(\gamma(\hat{\phi}, \bar{v}, r + \delta, \lambda)) = \bar{v}$ so that $\hat{\phi}$ is the value of ϕ for which the value of a unit of capital equals the unconditional average value of a unit of capital. Then $\text{sign}[\gamma(\phi, \bar{v}, r + \delta, \lambda) - \gamma^c(\phi, r + \delta)] = -\text{sign}[\phi - \hat{\phi}]$, where $\gamma^c(\phi, r + \delta) = \gamma(\phi, 0, r + \delta, 0)$ is the optimal value of γ under certainty.*

Proposition 5 implies that for $\phi < \widehat{\phi}$, the introduction of a Markov regime-switching process increases the optimal value of γ , and for $\phi > \widehat{\phi}$, the introduction of Markov regime-switching reduces the optimal value of γ .⁸

The first-order condition for optimal γ in equation (21), $1 + c'(\gamma) = v(\phi)$, implies that, for a given adjustment cost function $c(\gamma)$, a change in the optimal value of γ will be associated with a change in the same direction of the value of a unit of capital. This reasoning leads immediately to the following corollary to Proposition 5.

Corollary 7 *If $\phi < \widehat{\phi}$, then the introduction of the Markov regime-switching process increases the value of a unit of capital. If $\phi > \widehat{\phi}$, then the introduction of the Markov regime-switching process decreases the value of a unit of capital.*

The following corollary presents upper and lower bounds on the unconditional value of a unit of capital. Since the value of a unit of capital in the case of certainty is bounded between zero and $1 + c'(r + \delta)$, and since the Markov regime-switching process reverts toward its central tendency, it is not surprising that the unconditional value of a unit of capital under Markov regime switching lies within the same bounds.

Corollary 8 *The unconditional value of a unit of capital \bar{v} satisfies $0 < \bar{v} < 1 + c'(r + \delta)$.*

4.2 Effect of Changing the Distribution $F(\Phi)$

In this subsection, I analyze the impact of changing $F(\Phi)$, the unconditional distribution of the marginal operating profit of capital. I begin by considering a change in the distribution $F(\Phi)$ from $F_1(\Phi)$ to $F_2(\Phi)$, where $F_2(\Phi)$ first-order stochastically dominates $F_1(\Phi)$. Let \bar{v}_i be the unconditional expected value of a unit of capital when the distribution of Φ is $F_i(\Phi)$, $i = 1, 2$. Also, let $\gamma(\Phi, \bar{v}_i, r + \delta, \lambda)$ be the optimal value of γ for given Φ when the distribution of Φ is $F_i(\Phi)$, and let $\Gamma_i(\gamma)$ be the induced distribution of the optimal value of γ when the distribution of Φ is $F_i(\Phi)$, $i = 1, 2$.

Proposition 6 *If $F_2(\Phi)$ strictly first-order stochastically dominates $F_1(\Phi)$, then $\bar{v}_2 > \bar{v}_1$ and $\Gamma_2(\gamma)$ strictly first-order stochastically dominates $\Gamma_1(\gamma)$.*

Proposition 6 states that moving to a more favorable distribution of Φ that first-order stochastically dominates the original distribution will increase \bar{v} , the average value of a unit of capital. The increase in \bar{v} will increase the optimal value of γ at each value of Φ , and because the distribution of Φ becomes more favorable and optimal γ is increasing in Φ , the distribution of optimal γ also becomes more favorable in the sense of first-order stochastic dominance.

⁸Define $\bar{\phi} \equiv \int_G \Phi dF(\Phi)$ as the unconditional mean of the marginal operating profit of capital. Corollary 9 states that for non-degenerate $F(\Phi)$ and $\lambda > 0$, $\gamma(\bar{\phi}, \bar{v}, r + \delta, \lambda) > \gamma(\bar{\phi}, 0, r + \delta, 0) = \gamma^c(\bar{\phi}, r + \delta)$, so $\text{sign}[\gamma(\bar{\phi}, \bar{v}, r + \delta, \lambda) - \gamma^c(\bar{\phi}, r + \delta)]$ is positive. Therefore, since Proposition 5 implies that $\text{sign}[\gamma(\bar{\phi}, \bar{v}, r + \delta, \lambda) - \gamma^c(\bar{\phi}, r + \delta)] = -\text{sign}[\bar{\phi} - \widehat{\phi}]$, $\text{sign}[\bar{\phi} - \widehat{\phi}]$ is negative and $\widehat{\phi} > \bar{\phi}$.

4.2.1 Effect of a Mean-Preserving Spread on $F(\Phi)$

Now consider the effect on optimal investment of an increase in uncertainty that takes the form of a mean-preserving spread on the distribution $F(\Phi)$. This question was first addressed in a model with convex costs of adjustment by Hartman (1972) and then by Abel (1983). In both papers, the production function is linearly homogeneous in capital and labor and the firm is perfectly competitive, so that, as in the framework in this paper, the marginal operating profit of capital, Φ , is independent of the capital stock. Both of them found that an increase in the uncertainty of the price of output leads to an increase in the optimal rate of investment.⁹ The channel through which this effect operates is the convexity of $\Phi_t \equiv \max_l [p_t A_t f(1, l) - w_t l]$ in $p_t A_t$ and w_t . This convexity implies that a mean-preserving spread on $p_t A_t$ or w_t at some future time t increases the expected value of future Φ_t and thus increases the expected present value of the stream of future Φ_t , which increases (marginal) q and hence increases investment. In the current paper, I analyze a different channel for increased uncertainty to affect investment. To focus on that channel, I analyze mean-preserving spreads in the distribution of Φ_t directly. Since the expected value of Φ_t is constrained to remain unchanged, any effects on the optimal value of γ will operate through a different channel than in Hartman (1972) and Abel (1983).

Proposition 7 *A mean-preserving spread of $F(\Phi)$ that maintains the support within G increases \bar{v} .*

The proof of Proposition 7 is in the Appendix, but it is helpful to examine a key step to get a sense for what is driving the result. As shown in the Appendix, this result relies on the fact that $c'(\gamma(\Phi, \kappa, r + \delta, \lambda))$ is convex in Φ , even though $c'(\gamma)$ may not be convex in γ and $\gamma(\Phi, \kappa, r + \delta, \lambda)$ may not be convex in Φ . Notice that $c'(\gamma(\Phi, \kappa, r + \delta, \lambda))$ will be convex in Φ if $\frac{\partial c'(\gamma(\Phi, \kappa, r + \delta, \lambda))}{\partial \Phi} = c''(\gamma(\Phi, \kappa, r + \delta, \lambda)) \times \frac{\partial \gamma(\Phi, \kappa, r + \delta, \lambda)}{\partial \Phi}$ is increasing in Φ . However, neither $c''(\gamma(\Phi, \kappa, r + \delta, \lambda))$ nor $\frac{\partial \gamma(\Phi, \kappa, r + \delta, \lambda)}{\partial \Phi} = \frac{1}{(\rho - \gamma(\Phi, \kappa, r + \delta, \lambda))c''(\gamma(\Phi, \kappa, r + \delta, \lambda))}$ is necessarily increasing in Φ . But their product, $\frac{1}{\rho - \gamma(\Phi, \kappa, r + \delta, \lambda)}$, is increasing in Φ , so $c'(\gamma(\Phi, \kappa, r + \delta, \lambda))$ is convex in Φ . Therefore, a mean-preserving spread on Φ increases the unconditional expected value of $c'(\gamma(\Phi, \kappa, r + \delta, \lambda))$ and hence increases \bar{v} .

The following corollary uses the fact that the introduction of mean-preserving variation in Φ increases \bar{v} .

Corollary 9 *Let $\bar{\phi} \equiv \int_G \Phi dF(\Phi)$ be the unconditional mean of the marginal operating profit of capital. Then for non-degenerate $F(\Phi)$ and $\lambda > 0$, $\gamma(\bar{\phi}, \bar{v}, r + \delta, \lambda) > \gamma(\bar{\phi}, 0, r + \delta, 0) = \gamma^c(\bar{\phi}, r + \delta)$. That is, the introduction of the Markov regime-switching process increases optimal γ at $\bar{\phi}$.*

⁹Caballero (1991) showed that positive impact of uncertainty on optimal investment can be reversed by relaxing the assumption of perfect competition or by relaxing the linear homogeneity of the production function in capital and labor.

The following corollary uses the fact that with a quadratic adjustment cost function $c(\gamma)$, the marginal adjustment cost function is linear in γ , so that optimal γ is a linear function of q , or equivalently, v .

Corollary 10 *If the adjustment cost function $c(\gamma)$ is quadratic, then a mean-preserving spread of $F(\Phi)$ that maintains the support within G increases $\bar{\gamma} \equiv \int_G \gamma(\Phi, \bar{v}, r + \delta, \lambda) dF(\Phi)$, the unconditional expected value of γ .*

4.3 Effect of a Change in Persistence of Regimes

Next I analyze the impact of a change in the persistence of regimes governing Φ . With a constant probability λ of a switch in the regime, the expected life of a regime is $\frac{1}{\lambda}$, so an increase in λ reduces the persistence of the regime.

Proposition 8 *If $F(\Phi)$ is non-degenerate, then $\frac{\partial \bar{v}}{\partial \lambda} < 0$, so that an increase in the persistence of regimes (which is a reduction in λ) increases \bar{v} .*

An increase in λ has two, potentially opposing, effects on the value of capital for a given regime. By hastening the arrival of the next regime, an increase in λ reduces the value of regimes with high Φ because the expected value of the next regime is lower than the value of the current regime. In addition, Proposition 8 states that an increase in λ reduces the expected value of the next regime. Therefore, an increase in λ unambiguously reduces the value of a unit of capital for regimes with high Φ . On the other hand, for regimes with low Φ , hastening the arrival of the next regime, which has a higher expected value than the value of the current regime, increases the value of a unit of capital. However, working in the opposite direction is the fact that an increase in λ reduces the expected value of capital in the next regime. But as Proposition 8 states, the probability-weighted average value of a unit of capital is reduced by an increase in λ .

For each regime, the optimal value of the investment-capital ratio, γ , moves in the same direction as the value of a unit of capital moves when λ increases. Thus, an increase in λ increases γ for regimes with small Φ and decreases γ for regimes with large Φ . The following corollary shows that in the case of quadratic adjustment costs, the sign of the effect on $\bar{\gamma}$, the unconditional expected value of γ , can be determined.

Corollary 11 *If $F(\Phi)$ is non-degenerate and if $c(\gamma)$ is quadratic, then $\frac{\partial \bar{\gamma}}{\partial \lambda} < 0$.*

5 Measurement Error and Cash Flow Effect on Investment

The model developed in this paper focuses on three variables that are often used in empirical studies of investment, specifically, the investment-capital ratio, γ , the value of a unit of capital, v , which is Tobin's q , and cash flow per unit of capital, Φ . This model, like most existing models, uses the first-order condition for optimal investment, $1 + c'(\gamma) = v(\phi)$ (equation 21), to draw a tight

link between γ and v . This link is often described by saying that v is a sufficient statistic for γ , meaning that if an observer knows the adjustment cost function and the value of v , then the value of γ can be computed in a straightforward manner without any additional information or knowledge of the values of any other variables. Indeed, if the adjustment cost function, $c(\gamma)$, is quadratic, the marginal adjustment cost function is linear, and optimal γ is a linear function of v .

The empirical literature has a long history of finding that v is not a sufficient statistic for γ . In particular, at least since work of Fazzari, Hubbard, and Petersen (1988), researchers have found that in a regression of γ on v and Φ , estimated coefficients on both v and Φ tend to be positive and statistically significant. The finding of a positive significant coefficient on cash flow is often interpreted as evidence that firms face financing constraints or some other imperfection in financial markets. This interpretation of financial frictions, as they are sometimes known, is bolstered by the finding that in firms that one might suspect to be more likely to face these frictions, the cash flow effects tend to be more substantial. For instance, as the argument goes, firms that are growing rapidly may encounter more substantial financial frictions and cash flow coefficients are often larger for such firms.¹⁰

In this section, I will offer a different interpretation of the cash flow coefficients. I will demonstrate that if v is measured with error, then the coefficient of v is biased toward zero and, more importantly, the coefficient on cash flow, Φ , will be positive, even though in the absence of measurement error in v , the coefficient on Φ would be zero. The fact that measurement error in v can affect the coefficient estimates in this way has been pointed out by Erickson and Whited (2000) and Gilchrist and Himmelberg (1995) and others, though the particular simple expressions in the paper appear to be new. The finding that measurement error in v can lead to a positive cash flow coefficient does not use the particular model in this paper, other than the result that v and Φ are positively correlated with each other. However, the model in this paper is used in the next step, showing that cash flow coefficients are larger for firms that have higher growth rates. Although the literature interprets this finding as further evidence of financial frictions, the model here has no financial frictions whatsoever, and yet leads to the same finding. Therefore, the finding of positive cash flow coefficients, including larger coefficients for firms that are growing more rapidly, does not necessarily show that financial frictions are important or operative.

To isolate measurement error from specification error, I assume that the adjustment cost function is quadratic so that optimal γ is a linear function of v . As before, the quadratic adjustment cost function is $c(\gamma) = \theta\gamma^2$ so the first-order condition for optimal γ in equation (21) implies that

$$\gamma = \frac{v - 1}{2\theta}. \tag{44}$$

¹⁰For instance, Deveraux and Schiantarelli (1990) state "The perhaps surprising result from table 11.7 is that the coefficient on cash flow is greater for firms operating in growing sectors." (p. 298).

Assume that the manager of the firm can observe v , Φ , and γ without error, but people outside the firm, including the econometrician, can only view these variables with classical measurement error. Specifically, the econometrician observes the value of a unit of capital as $\tilde{q} = v + \varepsilon_q$, the investment-capital ratio as $\tilde{\gamma} = \gamma + \varepsilon_\gamma = \frac{v-1}{2\theta} + \varepsilon_\gamma$, and cash flow as $\tilde{c} = \Phi + \varepsilon_c$, where the observation errors ε_q , ε_γ , and ε_c , are mean zero, mutually independent, and independent of v , Φ , and γ . Erickson and Whited (2000) offer a useful taxonomy of reasons for measurement error in q , and except for differences between marginal q and average q (which are non-existent in the model presented here), those reasons could apply here.

Consider a linear regression of $\tilde{\gamma}$ on \tilde{q} and \tilde{c} , after all variables have been de-meanned. Let b_q and b_c be the plims of the estimated coefficients on \tilde{q} and \tilde{c} , respectively, so

$$\begin{bmatrix} b_q \\ b_c \end{bmatrix} = \begin{bmatrix} \text{Var}(\tilde{q}) & \text{Cov}(\tilde{q}, \tilde{c}) \\ \text{Cov}(\tilde{q}, \tilde{c}) & \text{Var}(\tilde{c}) \end{bmatrix}^{-1} \begin{bmatrix} \text{Cov}(\tilde{q}, \tilde{\gamma}) \\ \text{Cov}(\tilde{c}, \tilde{\gamma}) \end{bmatrix}. \quad (45)$$

The first two rows of the variance-covariance matrix, A , of $(\tilde{q}, \tilde{c}, \tilde{\gamma})$ conveniently display the variances and covariance in equation (45), where

$$A = \begin{bmatrix} \text{Var}(v) + \text{Var}(\varepsilon_q) & \text{Cov}(v, \Phi) & \frac{1}{2\theta} \text{Var}(v) \\ \text{Cov}(v, \Phi) & \text{Var}(\Phi) + \text{Var}(\varepsilon_c) & \frac{1}{2\theta} \text{Cov}(v, \Phi) \\ \frac{1}{2\theta} \text{Var}(v) & \frac{1}{2\theta} \text{Cov}(v, \Phi) & \frac{1}{4\theta^2} \text{Var}(v) + \text{Var}(\varepsilon_\gamma) \end{bmatrix}. \quad (46)$$

Substituting the relevant second moments from equation (46) into equation (45), and performing the indicated matrix inversion and matrix multiplication yields

$$\begin{bmatrix} b_q \\ b_c \end{bmatrix} = \frac{\frac{1}{2\theta}}{[\text{Var}(v) + \text{Var}(\varepsilon_q)] [\text{Var}(\Phi) + \text{Var}(\varepsilon_c)] - [\text{Cov}(v, \Phi)]^2} \times \begin{bmatrix} [\text{Var}(\Phi) + \text{Var}(\varepsilon_c)] \text{Var}(v) - \text{Cov}(v, \Phi) \text{Cov}(v, \Phi) \\ [\text{Var}(v) + \text{Var}(\varepsilon_q)] \text{Cov}(v, \Phi) - \text{Cov}(v, \Phi) \text{Var}(v) \end{bmatrix}. \quad (47)$$

Define $s_q^2 \equiv \frac{\text{Var}(\varepsilon_q)}{\text{Var}(v)}$ as the variance of the measurement error in \tilde{q} , normalized by $\text{var}(v)$, which is the variance of the true value of q , $s_c^2 \equiv \frac{\text{Var}(\varepsilon_c)}{\text{Var}(\Phi)}$ as the variance of the measurement error in cash flow normalized by the variance of the true value of cash flow, and $r^2 = \frac{[\text{Cov}(v, \Phi)]^2}{\text{Var}(\Phi) \text{Var}(v)}$ as the squared correlation between the true values of q and cash flow. Dividing both the numerators and denominators of b_q and b_c in equation (47) by $\text{Var}(\Phi) \text{Var}(v)$ yields

$$\begin{bmatrix} b_q \\ b_c \end{bmatrix} = \frac{\frac{1}{2\theta}}{1 - r^2 + s_c^2 + s_q^2 + s_q^2 s_c^2} \begin{bmatrix} 1 - r^2 + s_c^2 \\ s_q^2 \frac{\text{Cov}(v, \Phi)}{\text{Var}(\Phi)} \end{bmatrix}. \quad (48)$$

Equation (48) shows the impact of measurement error in q . If q is perfectly measured, then

$s_q^2 = 0$ and, regardless of whether cash flow is measured with error, equation (48) immediately yields $b_q = \frac{1}{2\theta}$ and $b_c = 0$. Thus, if q is perfectly measured, b_q equals the derivative of the optimal value of γ with respect to v in the first-order condition in equation (44). In addition, the estimated effect of cash flow on investment, b_c , is zero. However, if q is measured with error, so that $s_q^2 > 0$, then, b_q , the estimated coefficient on q is smaller than $\frac{1}{2\theta}$, the true derivative of γ with respect to v . Moreover, if $s_q^2 > 0$, then b_c , the estimated coefficient on cash flow can be nonzero; in fact, if q and cash flow are positively correlated, the estimated cash flow coefficient, b_c , is positive. Much of the investment literature interprets a significantly positive coefficient on cash flow in a regression of investment on q and cash flow as evidence of financing constraints.¹¹ Yet equation (48) demonstrates that measurement error in q will lead to a positive coefficient on cash flow, provided that q and cash flow are positively correlated, even if there are no financial frictions. This argument is not restricted to the particular specification of the firm in this model, and has been made less formally by, for example, Gilchrist and Himmelberg (1995).¹² The model in this paper allows the analysis to go one step further and to account for differences in the estimated cash flow coefficients for firms with different growth rates, as I discuss next.

Proponents of the view that positive cash flow coefficients are evidence of financing constraints bolster their view by showing that firms that are likely to face binding financing constraints are more likely to exhibit positive cash flow coefficients. For instance, they argue that firms that are growing more quickly are more likely to face binding financing constraints. Empirical evidence that rapidly growing firms have positive cash flow coefficients is then presented as evidence of financing constraints. However, the model in this paper offers an alternative interpretation. Equation (48) shows that the cash flow coefficient is proportional to $\frac{Cov(v, \Phi)}{Var(\Phi)}$, which is the population regression coefficient of v on Φ . The analog of this coefficient in our model is $\frac{dv(\Phi)}{d\Phi}$, which equals $\frac{\partial c'(\gamma(\Phi, \bar{v}, r + \delta, \lambda))}{\partial \Phi}$ because $v(\Phi) = 1 + c'(\gamma(\Phi, \bar{v}, r + \delta, \lambda))$. The following proposition formally presents a case in which a firm that grows more rapidly will have a higher cash flow coefficient in the model presented here, in which financial frictions are absent.

Proposition 9 *Consider two firms with identical quadratic adjustment cost functions but with different unconditional distributions of Φ , $F_1(\Phi)$ and $F_2(\Phi)$, and different unconditional values of capital \bar{v}_1 and \bar{v}_2 . If $F_2(\Phi)$ first-order stochastically dominates $F_1(\Phi)$, then*

1. $\gamma(\Phi, \bar{v}_2, r + \delta, \lambda) > \gamma(\Phi, \bar{v}_1, r + \delta, \lambda)$
2. $\int_G \gamma(\Phi, \bar{v}_2, r + \delta, \lambda) dF_2(\Phi) > \int_G \gamma(\Phi, \bar{v}_1, r + \delta, \lambda) dF_1(\Phi)$
3. $\frac{dv_2(\Phi)}{d\Phi} > \frac{dv_1(\Phi)}{d\Phi}$ and

¹¹Exceptions include Abel and Eberly (2011), Altı (2003), Cooper and Ejarque (2003), Gilchrist and Himmelberg (1995) and Gomes (2001).

¹²Gilchrist and Himmelberg, p. 544, state "More generally, anything that systematically reduces the signal-to-noise ratio of Tobin's Q (for example, measurement error or 'excess volatility' of stock prices) will shift explanatory power away from Tobin's Q toward cash flow, thus making such firms appear to be financially constrained when in fact they are not."

$$4. \int_G \frac{dv_2(\Phi)}{d\Phi} dF_2(\Phi) > \int_G \frac{dv_1(\Phi)}{d\Phi} dF_1(\Phi).$$

Proposition 9 states the firm with distribution $F_2(\Phi)$ is the faster-growing firm, whether the speed of growth is measured by the investment-capital ratio at any given value of Φ (statement 1) or by the unconditional expectation of the investment-capital ratio (statement 2). This proposition also states that the firm with the distribution $F_2(\Phi)$ has the higher value of $\frac{dv(\Phi)}{d\Phi}$ for a given value of Φ (statement 3) and the higher unconditional expected value of $\frac{dv(\Phi)}{d\Phi}$ (statement 4). Therefore, the firm with the distribution $F_2(\Phi)$ has the higher value of $\frac{Cov(v,\Phi)}{Var(\Phi)}$ and hence the higher cash flow coefficient. To summarize, the firm that is growing more rapidly has the larger coefficient on cash flow, even though there are no financial frictions in this model.

6 Concluding Remarks

This paper develops a model of a competitive firm with constant returns to scale to provide a tractable and useful stochastic framework to analyze the behavior and interrelationships of optimal investment, q , and cash flow that are widely studied in the empirical literature. As first shown by Hayashi (1982), average q and marginal q are identically equal in this framework. Within the class of models for which average q and marginal q are equal, the model presented here places only one additional restriction on technology, namely that adjustment costs, which are a function of investment and the capital stock, are additively separable from the production function for output, which is a function of capital and labor. For convenience, the model specifies a constant discount rate and a constant depreciation rate of capital. Finally, the analysis of the stochastic model is greatly facilitated by the simple Markov regime-switching specification for the marginal operating profit capital.

The model developed here is tractable enough to analyze various aspects of optimal investment behavior in a framework that is consistent with empirical analyses that use average q to measure marginal q and that specify the investment-capital ratio as a function of q . A closed-form solution for optimal investment and q is derived only for the case in which the marginal operating profit of capital is known to be constant and the cost of adjustment function is quadratic. When the marginal operating profit of capital follows a Markov regime-switching process, I present analytic expressions for the optimal investment-capital ratio and the value of a unit of capital up to a single undetermined constant.

After demonstrating various properties of optimal investment and q , I use the model to analyze the effects of various changes in stochastic environment facing the firm. Relative to the situation in which the marginal operating profit of capital is constant with certainty, the introduction of uncertainty in the form of a Markov regime-switching process increases the expected value of Tobin's q . For states of the world with a low marginal operating profit of capital, the introduction of the Markov regime-

switching process increases the optimal investment-capital ratio, and for states of the world with a high marginal operating profit of capital, the introduction of the Markov regime-switching process reduces the optimal investment-capital ratio. A favorable shift in the unconditional distribution of the marginal operating profit of capital, in the sense of first-order stochastic dominance, increases the expected value of a unit of capital, and shifts the distribution of the optimal investment-capital ratio in a first-order stochastically dominating way. In addition, a mean-preserving spread in the unconditional distribution of the marginal operating profit of capital increases the average value of a unit of capital, as in the existing literature, though the channel of the effect is different than in previous studies. Finally, I show that an increase in the persistence of regimes increases the average value of a unit of capital.

To address the common empirical finding of a positive coefficient on cash flow in a regression of the investment-capital ratio on q and cash flow, I introduce classical measurement error. Consistent with existing arguments, I show that measurement error in q can lead to a positive coefficient on cash flow. However, I use the model to go a step further and demonstrate that the model predicts that the coefficient on cash flow will be larger for firms that grow more rapidly. Proponents of the importance of financing constraints point to the positive coefficient on cash flow as evidence of the importance of these constraints. Moreover, they argue that larger cash flow coefficients for firms likely to be constrained, such as rapidly-growing firms, bolster the interpretation that positive cash flow coefficients indicate the importance of financing constraints. However, the model presented here has no financing constraints at all, yet in the presence of classical measurement error, it predicts coefficients on cash flow that are both positive and are larger for firms that grow more rapidly.

References

- Abel, Andrew B., "Optimal Investment under Uncertainty," American Economic Review, 73, 1 (March 1983), 228-233.
- Abel, Andrew B. and Janice C. Eberly, "How Q and Cash Flow Affect Investment without Frictions: An Analytic Explanation," The Review of Economic Studies, 78, 4 (October 2011), 1179-1200.
- Alti, Aydogan, "How Sensitive is Investment to Cash Flow When Financing is Frictionless?" The Journal of Finance, 58, 2 (April 2003), 707-722.
- Caballero, Ricardo J., "On the Sign of the Investment-Uncertainty Relationship," American Economic Review, 81, 1 (March 1991), 279-288.
- Cooper, Russell and Joao Ejarque, "Financial Frictions and Investment: Requiem in Q," Review of Economic Dynamics, 6, 4 (October 2003), 710-728.
- Deveraux, Michael and Fabio Schiantarelli, "Investment, Financial Factor, and Cash Flow: Evidence from U.K. Panel Data" in Asymmetric Information, Corporate Finance, and Investment, edited by R. Glenn Hubbard, University of Chicago Press, 1990, pp. 279-306.
- Erickson, Timothy and Toni M. Whited, "Measurement Error and the Relationship between Investment and q ," Journal of Political Economy, 108, 5 (October 2000), 1027-1057.
- Gomes, Joao F., "Financing Investment," American Economic Review, 91, 5 (December 2001), 1263-1285.
- Fazzari, Steven, R. Glenn Hubbard, and Bruce C. Petersen, 1988, "Financing Constraints and Corporate Investment," Brookings Papers on Economic Activity, 1988:1, 141-206.
- Gilchrist, Simon and Charles Himmelberg, "Evidence of the Role of Cash Flow for Investment," Journal of Monetary Economics, 36, 3 (December 1995) pp. 541-572.
- Hartman, Richard, "The Effects of Price and Cost Uncertainty on Investment," Journal of Economic Theory, 5,2 (October 1972), 258-266.
- Lucas, Robert E., Jr., "Adjustment Costs and the Theory of Supply," Journal of Political Economy, 75, 4 Part 1 (August 1967), 321-334.
- Lucas, Robert E., Jr., and Edward C. Prescott, "Investment under Uncertainty," Econometrica, 39, 5 (September 1971), 659-681.

Mussa, Michael, "External and Internal Adjustment Costs and the Theory of Aggregate and Firm Investment," Economica, 44, 174 (May 1977), 163-178.

Appendix: Proofs of Lemmas, Propositions and Corollaries

Proof. of Proposition 1: Let $\{K_s^A, I_s^A\}_{s=t}^{s=\infty}$ satisfy the capital accumulation equation in (1) and attain the maximum on the right-hand side of (2). Let $\{K_s^B, I_s^B\}_{s=t}^{s=\infty} = \{\omega K_s^A, \omega I_s^A\}_{s=t}^{s=\infty}$ for an arbitrary $\omega > 0$ and note that $\{K_s^B, I_s^B\}_{s=t}^{s=\infty}$ satisfies the capital accumulation equation in (1). Then $V_t(\omega K_t^A) = V_t(K_t^B) \geq E_t \left\{ \int_t^\infty \pi_s(K_s^B, I_s^B) M(t, s) ds \right\} = E_t \left\{ \int_t^\infty \pi_s(\omega K_s^A, \omega I_s^A) M(t, s) ds \right\} = \omega E_t \left\{ \int_t^\infty \pi_s(K_s^A, I_s^A) M(t, s) ds \right\} = \omega V_t(K_t^A)$. Since $V_t(\omega K_t) \geq \omega V_t(K_t)$ for any $\omega > 0$ and any $K_t > 0$, we have $V_t(\frac{1}{\omega} K_t^B) \geq \frac{1}{\omega} V_t(K_t^B)$, which implies $\omega V_t(K_t^A) = \omega V_t(\frac{1}{\omega} K_t^B) \geq V_t(K_t^B) = V_t(\omega K_t^A)$. Therefore, $V_t(\omega K_t^A) \geq \omega V_t(K_t^A) \geq V_t(\omega K_t^A)$, which implies $V_t(\omega K_t^A) = \omega V_t(K_t^A)$. ■

Proof. of Lemma 1: Use the definition of $H(\gamma, \Phi, \rho)$ to obtain $H(\rho, \Phi, \rho) = \Phi - \rho - c(\rho) \leq \Phi - (r + \delta) - c(r + \delta) < 0$, where the first inequality follows from the fact that $\gamma + c(\gamma)$ is strictly increasing in γ for $\Phi \in G$ and the assumption that $\rho \geq r + \delta$; the second inequality follows from the fact that $\Phi \in G$. Use the definition of $H(\gamma, \Phi, \rho)$ to obtain $H(\gamma^m, \Phi, \rho) = \Phi - \gamma^m - (\rho - \gamma^m) - c(\gamma^m) - (\rho - \gamma^m) c'(\gamma^m)$. Recall from the definition of γ^m that $1 + c'(\gamma^m) = 0$, so $H(\gamma^m, \Phi, \rho) = \Phi - \gamma^m - c(\gamma^m) > 0$, where the inequality follows from the fact that $\Phi \in G$. Therefore, since $H(\gamma^m, \Phi, \rho) > 0 > H(\rho, \Phi, \rho)$, and the function $H(\gamma, \Phi, \rho)$ is continuous in γ , there is at least one root of $H(\gamma, \Phi, \rho) = 0$ in (γ^m, ρ) . Since $H_\gamma(\gamma, \Phi, \rho) = -(\rho - \gamma) c''(\gamma) < 0$ throughout this interval, the root is unique. Since $H_\gamma(\gamma, \Phi, \rho) = -(\rho - \gamma) c''(\gamma) < 0$ for $\gamma \leq \gamma^m$ and since $H(\gamma^m, \Phi, \rho) > 0$, $H(\gamma, \Phi, \rho) > 0$ for all $\gamma < \gamma^m$. Hence, there are no real roots of $H(\gamma, \Phi, \rho) = 0$ less than γ^m . Since $H(\rho, \Phi, \rho) < 0$ and $\lim_{\gamma \rightarrow \infty} H(\gamma, \Phi, \rho) > 0$, there is at least one root in (ρ, ∞) ; that root is unique because $H_\gamma(\gamma, \Phi, \rho) = -(\rho - \gamma) c''(\gamma) > 0$ for all $\gamma \in (\rho, \infty)$. ■

Proof. of Proposition 2: The optimal value of γ is a root of $H(\gamma, \Phi, \rho) = 0$. Lemma 1 states that this equation has two roots, which I denote as $\gamma_1 < \gamma_2$. Using the fact that $H(\gamma_i, \Phi, \rho) = 0$, for $i = 1, 2$, yields $\Phi - \gamma_i - c(\gamma_i) = (\rho - \gamma_i) [1 + c'(\gamma_i)]$, $i = 1, 2$. The left hand side of this equation is the cash flow, net of investment costs, at γ_i . Since $1 + c'(\gamma) > 0$ for $\gamma > \gamma^m$, cash flow, net of investment costs, is positive at $\gamma_1 \in (\gamma^m, \rho)$ and negative at $\gamma_2 \in (\rho, \infty)$. Therefore, γ_1 is the optimal value of the investment-capital ratio. ■

Proof. of Corollary 1: Apply the implicit function theorem to $H(\gamma^c, \Phi, \rho) = 0$ and use the facts that $\gamma^c < r + \delta \leq \rho$, $1 + c'(\gamma^c) > 0$, and $c''(\gamma^c) > 0$ to obtain $\frac{\partial \gamma^c(\Phi, \rho)}{\partial \Phi} = -\frac{H_\Phi}{H_\gamma} = \frac{1}{(\rho - \gamma^c) c''(\gamma^c)} > 0$ and $\frac{\partial \gamma^c(\Phi, \rho)}{\partial \rho} = -\frac{H_\rho}{H_\gamma} = -\frac{1 + c'(\gamma^c)}{(\rho - \gamma^c) c''(\gamma^c)} < 0$. ■

Proof. of Corollary 2: Differentiate the first-order condition for optimal investment, which holds at all points of time, $v = 1 + c'(\gamma^c)$, with respect to Φ to obtain $\frac{\partial v}{\partial \Phi} = c''(\gamma^c) \frac{\partial \gamma^c(\Phi, \rho)}{\partial \Phi} = c''(\gamma^c) \frac{1}{(\rho - \gamma^c) c''(\gamma^c)} = \frac{1}{\rho - \gamma^c} > 0$. Differentiate the expression for $\frac{\partial v}{\partial \Phi}$ with respect to Φ to obtain $\frac{\partial^2 v}{(\partial \Phi)^2} = \frac{1}{(\rho - \gamma^c)^2} \frac{\partial \gamma^c}{\partial \Phi} = \frac{1}{(\rho - \gamma^c)^2} \frac{1}{(\rho - \gamma^c) c''(\gamma^c)} = \frac{1}{(\rho - \gamma^c)^3} \frac{1}{c''(\gamma^c)} > 0$. Differentiate $v = 1 + c'(\gamma^c)$, with respect to ρ to obtain $\frac{\partial v}{\partial \rho} = c''(\gamma^c) \frac{\partial \gamma^c(\Phi, \rho)}{\partial \rho} = -c''(\gamma^c) \frac{1 + c'(\gamma^c)}{(\rho - \gamma^c) c''(\gamma^c)} = -\frac{1 + c'(\gamma^c)}{\rho - \gamma^c} < 0$. ■

Proof. of Proposition 3: Differentiate the first-order condition $v = 1 + c_\gamma(\gamma, \varepsilon)$ with respect to ε to obtain $\frac{\partial v}{\partial \varepsilon} = c_{\gamma\gamma}(\gamma, \varepsilon) \frac{\partial \gamma}{\partial \varepsilon} + c_{\gamma\varepsilon}(\gamma, \varepsilon)$. With the adjustment cost function expressed as $c(\gamma, \varepsilon)$, the condition $H(\gamma, \Phi, \rho) = 0$ can be written as $\Phi - \rho - c(\gamma, \varepsilon) - (\rho - \gamma) c_\gamma(\gamma, \varepsilon) = 0$, which implies $c(\gamma, \varepsilon) + (\rho - \gamma) c_\gamma(\gamma, \varepsilon) = \Phi - \rho$. Differentiating this expression with respect to ε yields $c_\varepsilon(\gamma, \varepsilon) + (\rho - \gamma) c_{\gamma\gamma}(\gamma, \varepsilon) \frac{\partial \gamma}{\partial \varepsilon} + (\rho - \gamma) c_{\gamma\varepsilon}(\gamma, \varepsilon) = 0$. Therefore, $c_{\gamma\gamma}(\gamma, \varepsilon) \frac{\partial \gamma}{\partial \varepsilon} + c_{\gamma\varepsilon}(\gamma, \varepsilon) = -\frac{1}{\rho - \gamma} c_\varepsilon(\gamma, \varepsilon)$, which can be substituted into the expression for $\frac{\partial v}{\partial \varepsilon}$ to obtain $\frac{\partial v}{\partial \varepsilon} = -\frac{1}{\rho - \gamma} c_\varepsilon(\gamma, \varepsilon) < 0$ since $\gamma = \gamma^c < \rho$. Therefore, a reduction in ε will increase v . ■

Proof. of Corollary 3: The proof of Proposition 3 states that $c_{\gamma\gamma}(\gamma, \varepsilon) \frac{\partial \gamma}{\partial \varepsilon} + c_{\gamma\varepsilon}(\gamma, \varepsilon) = -\frac{1}{\rho - \gamma} c_\varepsilon(\gamma, \varepsilon)$, which implies $\frac{\partial \gamma}{\partial \varepsilon} = -\frac{1}{c_{\gamma\gamma}(\gamma, \varepsilon)} \left[\frac{1}{\rho - \gamma} c_\varepsilon(\gamma, \varepsilon) + c_{\gamma\varepsilon}(\gamma, \varepsilon) \right] < 0$, since $c_{\gamma\gamma}(\gamma, \varepsilon) > 0$, $c_\varepsilon(\gamma, \varepsilon) \geq 0$, $c_{\gamma\varepsilon}(\gamma, \varepsilon) \geq 0$, $c_\varepsilon(\gamma, \varepsilon) + c_{\gamma\varepsilon}(\gamma, \varepsilon) > 0$, and $\gamma^c < \rho$. ■

Proof. of Lemma 2: Differentiate $H(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, r + \delta + \lambda) = -\lambda\kappa$ with respect to ϕ to obtain $H_\gamma(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho) \frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial \phi} + H_\phi(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho) = 0$. Therefore, $\frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial \phi} = -\frac{H_\phi(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho)}{H_\gamma(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho)}$. Use the facts that $H_\phi(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho) = 1$ and $H_\gamma(\gamma, \phi, \rho) = -(\rho - \gamma) c''(\gamma)$ to obtain $\frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial \phi} = \frac{1}{(\rho - \gamma) c''(\gamma)} > 0$ since optimal $\gamma < \rho$.

Differentiate $H(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho) = -\lambda\kappa$ with respect to κ to obtain $H_\gamma(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho) \frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial \kappa} = -\lambda$. Use the fact that $H_\gamma(\gamma, \phi, \rho) = -(\rho - \gamma) c''(\gamma)$ to obtain $\frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial \kappa} = \frac{\lambda}{(\rho - \gamma) c''(\gamma)} > 0$ since optimal $\gamma < \rho$.

Differentiate $H(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, r + \delta + \lambda) = -\lambda\kappa$ with respect to $r + \delta$ to obtain $H_\gamma(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho) \frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial (r + \delta)} + H_\rho(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho) = 0$. Use the facts that $H_\rho(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho) = -(1 + c'(\gamma))$ and $H_\gamma(\gamma, \Phi, \rho) = -(\rho - \gamma) c''(\gamma)$ to obtain $\frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial (r + \delta)} = -\frac{H_\rho(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho)}{H_\gamma(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho)} = -\frac{1 + c'(\gamma)}{(\rho - \gamma) c''(\gamma)} < 0$ since optimal $\gamma < \rho$.

Differentiate $H(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, r + \delta + \lambda) = -\lambda\kappa$ with respect to λ to obtain $H_\gamma(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho) \frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial \lambda} + H_\rho(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho) = -\kappa$. Use the facts that $H_\rho(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho) = -(1 + c'(\gamma))$ and $H_\gamma(\gamma, \Phi, \rho) = -(\rho - \gamma) c''(\gamma)$, to obtain $\frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial \lambda} = -\frac{(\kappa + H_\rho(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho))}{H_\gamma(\gamma(\phi, \kappa, r + \delta, \lambda), \phi, \rho)} = -\frac{1 + c'(\gamma) - \kappa}{(\rho - \gamma) c''(\gamma)}$, which is positive for relatively small values of ϕ , for which $v = 1 + c'(\gamma) < \kappa$, and is negative for relatively large values of ϕ , for which $v = 1 + c'(\gamma) > \kappa$. ■

Proof. of Corollary 4: Use the chain rule to obtain $\frac{\partial c'(\gamma(\phi, \kappa, r + \delta, \lambda))}{\partial x} = \frac{\partial c'(\gamma(\phi, \kappa, r + \delta, \lambda))}{\partial \gamma} \frac{\partial \gamma}{\partial x} = c''(\gamma) \frac{\partial \gamma}{\partial x}$ for $x = \phi, \kappa, r + \delta$, and λ . Also use the chain rule to obtain $\frac{\partial^2 c'(\gamma(\phi, \kappa, r + \delta, \lambda))}{(\partial \phi)^2} = \frac{\partial}{\partial \phi} \frac{\partial c'(\gamma(\phi, \kappa, r + \delta, \lambda))}{\partial \phi} = \frac{\partial}{\partial \phi} \frac{1}{\rho - \gamma} = \frac{1}{(\rho - \gamma)^2} \frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial \phi} = \frac{1}{(\rho - \gamma)^3 c''(\gamma)} > 0$ since optimal $\gamma < \rho$. ■

Proof. of Lemma 3: To prove property (1), use the definitions of $\alpha(\kappa)$ and $\gamma(\phi, \kappa, r + \delta, \lambda)$ to obtain $\alpha(0) = 1 + \int_G c'(\gamma(\Phi, 0, r + \delta, \lambda)) dF(\Phi)$. Equation (42) implies that $\gamma(\Phi, 0, r + \delta, \lambda) > \gamma^m$, so the convexity of $c(\gamma)$ implies $c'(\gamma)$ is strictly increasing and hence that $\alpha(0) > 1 + \int_G c'(\gamma^m) dF(\Phi) = 1 + c'(\gamma^m) = 0$.

To prove property (2), use the definitions of $\alpha(\kappa)$ and $\gamma(\phi, \kappa, r + \delta, \lambda)$ to obtain $\alpha(1 + c'(r + \delta)) = 1 + \int_G c'(\gamma(\Phi, 1 + c'(r + \delta), r + \delta, \lambda)) dF(\Phi)$. The definition of $H(\gamma, \phi, \rho)$ implies that $H(\gamma, \phi, r + \delta + \lambda) = H(\gamma, \phi, r + \delta) - \lambda(1 + c'(\gamma))$. In particular, this equation holds for $\gamma = r + \delta$, so that $H(r + \delta, \phi, r + \delta + \lambda) = H(r + \delta, \phi, r + \delta) - \lambda(1 + c'(r + \delta))$. Since $\Phi \in G \equiv \{\Phi : c(\gamma^m) + \gamma^m < \Phi < c(r + \delta) + r + \delta\}$, $H(r + \delta, \phi, r + \delta) = \phi - (r + \delta) - c(r + \delta) < 0$. Therefore, $H(r + \delta, \phi, r + \delta + \lambda) < -\lambda(1 + c'(r + \delta))$, which, along with $H_\gamma(\gamma, \phi, r + \delta + \lambda) < 0$ for $\gamma < \rho$, implies $\gamma(\phi, 1 + c'(r + \delta), r + \delta, \lambda) < r + \delta$. Therefore, the convexity of $c(\gamma)$ implies $\alpha(1 + c'(r + \delta)) = 1 + \int_G c'(\gamma(\Phi, 1 + c'(r + \delta), r + \delta, \lambda)) dF(\Phi) < 1 + \int_G c'(r + \delta) dF(\Phi) = 1 + c'(r + \delta)$.

To prove property (3), use the definition of $\alpha(\kappa)$ to obtain $\alpha'(\kappa) = \int_G c''(\gamma(\Phi, \kappa, r + \delta, \lambda)) \frac{\partial \gamma(\Phi, \kappa, r + \delta, \lambda)}{\partial \kappa} dF(\Phi)$. Use statement 2 in corollary 4, $\frac{\partial \gamma(\phi, \kappa, r + \delta, \lambda)}{\partial \kappa} = \frac{\lambda}{(\rho - \gamma)c''(\gamma)}$, to obtain $\alpha'(\kappa) = \int_G \frac{\lambda}{r + \delta + \lambda - \gamma(\Phi, \kappa, r + \delta, \lambda)} dF(\Phi)$. Since $\gamma(\Phi, \kappa, r + \delta, \lambda) < r + \delta$, $\frac{\lambda}{r + \delta + \lambda - \gamma(\Phi, \kappa, r + \delta, \lambda)} < 1$. Therefore, $\alpha'(\kappa) < \int_G dF(\Phi) = 1$. ■

Proof. of Proposition 4: The function $\alpha(\kappa)$, which is continuous over the domain $[0, 1 + c'(r + \delta)]$, and has three properties listed in Lemma 3. Therefore, there exists a unique positive value of $\kappa < 1 + c'(r + \delta)$ that satisfies $\alpha(\kappa) = \kappa$. For that value of κ , $\int_G [1 + c'(\gamma(\Phi, \kappa, \rho))] dF(\Phi) = \kappa = \bar{v}$. ■

Proof. of Proposition 5: The definition of $H(\gamma, \phi, \rho)$ implies that $H(\gamma, \phi, r + \delta + \lambda) = H(\gamma, \phi, r + \delta) - \lambda(1 + c'(\gamma))$ for any γ and ϕ , so that in particular $H(\gamma(\widehat{\phi}, \bar{v}, r + \delta, \lambda), \widehat{\phi}, r + \delta + \lambda) = H(\gamma(\widehat{\phi}, \bar{v}, r + \delta, \lambda), \widehat{\phi}, r + \delta) - \lambda(1 + c'(\gamma(\widehat{\phi}, \bar{v}, r + \delta, \lambda))) = H(\gamma(\widehat{\phi}, \bar{v}, r + \delta, \lambda), \widehat{\phi}, r + \delta) - \lambda \bar{v}$. Recall that $\gamma(\widehat{\phi}, \bar{v}, r + \delta, \lambda)$ is defined so that $H(\gamma(\widehat{\phi}, \bar{v}, r + \delta, \lambda), \widehat{\phi}, r + \delta + \lambda) = -\lambda \bar{v}$. Therefore, $H(\gamma(\widehat{\phi}, \bar{v}, r + \delta, \lambda), \widehat{\phi}, r + \delta) = 0$, but since $H(\gamma(\widehat{\phi}, \bar{v}, r + \delta, \lambda), \widehat{\phi}, r + \delta) = 0$ and $H(\gamma, \widehat{\phi}, r + \delta)$ is strictly decreasing in γ for $\gamma < \rho$, we have that $\gamma(\widehat{\phi}, \bar{v}, r + \delta, \lambda) = \gamma(\widehat{\phi}, 0, r + \delta, 0) = \gamma^c(\widehat{\phi}, r + \delta)$, that is, if $\Phi = \widehat{\phi}$, so the optimal value of γ is the same under the Markov regime-switching process as under certainty.

Once again use the fact that the definition of $H(\gamma, \phi, \rho)$ implies that $H(\gamma, \phi, r + \delta + \lambda) = H(\gamma, \phi, r + \delta) - \lambda(1 + c'(\gamma))$ for any γ and ϕ . Evaluating both sides of this equation at $\gamma = \gamma(\phi, 0, r + \delta, 0)$, i.e., the optimal value of γ under certainty yields $H(\gamma(\phi, 0, r + \delta, 0), \phi, r + \delta + \lambda) = H(\gamma(\phi, 0, r + \delta, 0), \phi, r + \delta) - \lambda(1 + c'(\gamma(\phi, 0, r + \delta, 0))) = -\lambda(1 + c'(\gamma(\phi, 0, r + \delta, 0)))$ since $H(\gamma(\phi, 0, r + \delta, 0), \phi, r + \delta) = 0$. If $\phi < \widehat{\phi}$, then $c'(\gamma(\widehat{\phi}, 0, r + \delta, 0)) > c'(\gamma(\phi, 0, r + \delta, 0))$ so $H(\gamma(\phi, 0, r + \delta, 0), \phi, r + \delta + \lambda) > -\lambda(1 + c'(\gamma(\widehat{\phi}, 0, r + \delta, 0))) = -\lambda(1 + c'(\gamma(\widehat{\phi}, \bar{v}, r + \delta, \lambda))) = -\lambda \bar{v} = H(\gamma(\phi, \bar{v}, r + \delta, \lambda), \phi, r + \delta + \lambda)$. Therefore, since $H(\gamma, \widehat{\phi}, r + \delta)$ is strictly decreasing in γ for $\gamma < \rho$, $\gamma(\phi, 0, r + \delta, 0) < \gamma(\phi, \bar{v}, r + \delta, \lambda)$. A similar argument proves that if $\phi > \widehat{\phi}$, then $\gamma(\phi, \bar{v}, r + \delta, \lambda) < \gamma(\phi, 0, r + \delta, 0)$. ■

Proof. of Corollary 8: The proof will proceed by contradiction. First suppose that $\bar{v} \geq 1 + c'(r + \delta)$. Then there is some $\phi_H \in G$ for which $v(\phi_H) \geq 1 + c'(r + \delta)$, which implies $\gamma(\phi_H, \bar{v}, r + \delta, \lambda) \geq r + \delta$.

But $\gamma^c(\phi, r + \delta) < r + \delta$ for all $\phi \in G$, so $\gamma(\phi_H, \bar{v}, r + \delta, \lambda) > \gamma^c(\phi_H, r + \delta)$. Therefore, Proposition 5 implies that $\phi_H < \hat{\phi}$. Since $\gamma(\phi, \bar{v}, r + \delta, \lambda)$ is increasing in ϕ , $\gamma(\phi_h, \bar{v}, r + \delta, \lambda) > \gamma(\phi_H, \bar{v}, r + \delta, \lambda)$ for any $\phi_h \geq \hat{\phi}$. But Proposition 5 implies that $\gamma(\phi_h, \bar{v}, r + \delta, \lambda) < \gamma^c(\phi_h, r + \delta)$, which is less than $r + \delta$. However, $\gamma(\phi_h, \bar{v}, r + \delta, \lambda) < r + \delta$ contradicts $\gamma(\phi_h, \bar{v}, r + \delta, \lambda) > \gamma(\phi_H, \bar{v}, r + \delta, \lambda) \geq r + \delta$. Therefore, \bar{v} cannot be greater than or equal to $1 + c'(r + \delta)$.

Now suppose that $\bar{v} \leq 0$. Then there is some $\phi_L \in G$ for which $v(\phi_L) \leq 0$, which implies $\gamma(\phi_L, \bar{v}, r + \delta, \lambda) \leq \gamma^m$, where, recall that γ^m is defined so that $1 + c'(\gamma^m) = 0$. But $\gamma^c(\phi, r + \delta) > \gamma^m$ for all $\phi \in G$, so $\gamma(\phi_L, \bar{v}, r + \delta, \lambda) < \gamma^c(\phi_L, r + \delta)$. Therefore, Proposition 5 implies that $\phi_L > \hat{\phi}$. Since $\gamma(\phi, \bar{v}, r + \delta, \lambda)$ is increasing in ϕ , $\gamma(\phi_l, \bar{v}, r + \delta, \lambda) < \gamma(\phi_L, \bar{v}, r + \delta, \lambda)$ for any $\phi_l \leq \hat{\phi}$. But Proposition 5 implies that $\gamma(\phi_l, \bar{v}, r + \delta, \lambda) > \gamma^c(\phi_l, r + \delta)$, which is greater than zero. However, $\gamma(\phi_l, \bar{v}, r + \delta, \lambda) > 0$ contradicts $\gamma(\phi_l, \bar{v}, r + \delta, \lambda) < \gamma(\phi_L, \bar{v}, r + \delta, \lambda) \leq 0$. Therefore, \bar{v} cannot be less than zero. ■

Proof. of Proposition 6: Suppose that $\kappa^* = \bar{v}_1$, which implies that $\bar{v}_1 = \int_G [1 + c'(\gamma(\Phi, \kappa^*, r + \delta, \lambda))] dF_1(\Phi)$. Since $c'(\gamma(\Phi, \kappa^*, r + \delta, \lambda))$ is strictly increasing in Φ (Statement 1 of Corollary 4), the assumption that $F_2(\Phi)$ strictly first-order stochastically dominates $F_1(\Phi)$ implies that $\kappa^* = \bar{v}_1 = \int_G [1 + c'(\gamma(\Phi, \kappa^*, r + \delta, \lambda))] dF_1(\Phi) < \int_G [1 + c'(\gamma(\Phi, \kappa^*, r + \delta, \lambda))] dF_2(\Phi) \equiv \alpha_2(\kappa^*)$, using the definition in (43). Therefore, Corollary 6 implies that $\bar{v}_2 > \kappa^* = \bar{v}_1$.

Define $\omega(\gamma^*, \kappa, r + \delta, \lambda)$ to be the value of Φ for which $\gamma(\Phi, \kappa, r + \delta, \lambda) = \gamma^*$. Since $\gamma(\Phi, \kappa, r + \delta, \lambda)$ is strictly increasing in Φ and strictly increasing in κ , it follows that $\omega(\gamma^*, \kappa, r + \delta, \lambda)$ is strictly increasing in γ^* and is strictly decreasing in κ . Note that $\Gamma_1(\gamma^*) = F_1(\omega(\gamma^*, \bar{v}_1, r + \delta, \lambda)) \geq F_2(\omega(\gamma^*, \bar{v}_1, r + \delta, \lambda)) \geq F_2(\omega(\gamma^*, \bar{v}_2, r + \delta, \lambda)) = \Gamma_2(\gamma^*)$, where the first inequality follows from the assumption that $F_2(\Phi)$ first-order stochastically dominates $F_1(\Phi)$, and the second inequality follows from the facts that $\bar{v}_2 > \bar{v}_1$, $\omega(\gamma^*, \kappa, r + \delta, \lambda)$ is strictly decreasing in κ , and $F_2(\Phi)$ is increasing. Since $F_2(\Phi)$ strictly first-order stochastically dominates $F_2(\Phi)$, the inequality $\Gamma_1(\gamma^*) \geq \Gamma_2(\gamma^*)$, which holds for all γ^* , holds strictly for some γ^* . Therefore, $\Gamma_2(\gamma)$ strictly first-order stochastically dominates $\Gamma_1(\gamma)$. ■

Proof. of Proposition 7 Suppose that initially $\bar{v} = \kappa^*$. Since statement 5 of Corollary 4 states that $\frac{\partial^2 c'(\gamma(\phi, \kappa, r + \delta, \lambda))}{(\partial \phi)^2} = \frac{1}{(\rho - \gamma)^3 c''(\gamma)} > 0$, $c'(\gamma(\phi, \kappa, r + \delta, \lambda))$ is a convex function of ϕ . Therefore, a mean-preserving spread of $F(\Phi)$ increases the value of $\int_G c'(\gamma(\Phi, \kappa^*, r + \delta, \lambda)) dF(\Phi)$, which increases the value of $\alpha(\kappa^*) \equiv 1 + \int_G c'(\gamma(\Phi, \kappa^*, r + \delta, \lambda)) dF(\Phi)$, so that $\alpha(\kappa^*) > \kappa^*$. Corollary 6 implies that $\bar{v} > \kappa^*$. ■

Proof. of Corollary 9: Note that $H(\gamma(\bar{\phi}, 0, r + \delta, 0), \bar{\phi}, r + \delta + \lambda) = H(\gamma(\bar{\phi}, 0, r + \delta, 0), \bar{\phi}, r + \delta) - \lambda(1 + c'(\gamma(\bar{\phi}, 0, r + \delta, 0))) = -\lambda(1 + c'(\gamma(\bar{\phi}, 0, r + \delta, 0)))$, where the first equality follows from the definition of $H(\gamma, \Phi, \rho)$ and the second equality follows from $H(\gamma(\bar{\phi}, 0, r + \delta, 0), \bar{\phi}, r + \delta) = 0$,

which defines $\gamma(\bar{\phi}, 0, r + \delta, 0)$. Since Proposition 7 implies that $\bar{v} > 1 + c'(\gamma(\bar{\phi}, 0, r + \delta, 0))$, we have $H(\gamma(\bar{\phi}, 0, r + \delta, 0), \bar{\phi}, r + \delta + \lambda) > -\lambda\bar{v}$. Therefore, the optimal value of γ under the Markov regime-switching process, $\gamma(\bar{\phi}, \bar{v}, r + \delta, \lambda)$, which satisfies $H(\gamma(\bar{\phi}, \bar{v}, r + \delta, \lambda), \bar{\phi}, r + \delta + \lambda) = -\lambda\bar{v}$, is greater than $\gamma(\bar{\phi}, 0, r + \delta, 0)$ because $H(\gamma, \Phi, \rho)$ is decreasing in γ for $\gamma < \rho$. ■

Proof. of Corollary 10 Recall that optimal investment implies that $\bar{v} = \int_G (1 + c'(\gamma(\Phi, \bar{v}, r + \delta, \lambda))) dF(\Phi)$.

If the adjustment cost function $c(\gamma)$ is quadratic in γ , then the marginal adjustment cost function, $c'(\gamma)$ is linear in γ , say $a\gamma + b$, so that $\int_G (1 + c'(\gamma(\Phi, \bar{v}, r + \delta, \lambda))) dF(\Phi) = \int_G (1 + a\gamma(\Phi, \bar{v}, r + \delta, \lambda) + b) dF(\Phi) = 1 + a\bar{\gamma} + b$, where $\bar{\gamma} \equiv \int_G \gamma dF(\Phi)$ is the expected value of γ . Therefore, $\bar{v} = 1 + a\bar{\gamma} + b$, and since Proposition 7 implies that a mean-preserving spread of $F(\Phi)$ increases \bar{v} , it also increases $\bar{\gamma}$. ■

Proof. of Proposition 8: Since $\alpha(\kappa) \equiv 1 + \int_G c'(\gamma(\Phi, \kappa, r + \delta, \lambda)) dF(\Phi)$, we have $\frac{\partial \alpha(\kappa)}{\partial \lambda} = \int_G \frac{\partial c'(\gamma(\Phi, \kappa, r + \delta, \lambda))}{\partial \lambda} dF(\Phi)$. Use statement 4 of Corollary 4 to obtain $\frac{\partial \alpha(\kappa)}{\partial \lambda} = \int_G \frac{\kappa - (1 + c'(\gamma(\Phi, \kappa, r + \delta, \lambda)))}{\rho - \gamma(\Phi, \kappa, r + \delta, \lambda)} dF(\Phi)$. Let $\kappa^* = \bar{v}$ and define ϕ^* as the unique value of Φ for which $1 + c'(\gamma(\phi^*, \kappa^*, r + \delta, \lambda)) = \kappa^*$, so $\kappa^* - (1 + c'(\gamma(\Phi, \kappa^*, r + \delta, \lambda))) > 0$ if $\Phi < \phi^*$ and $\kappa^* - (1 + c'(\gamma(\Phi, \kappa^*, r + \delta, \lambda))) < 0$ if $\Phi > \phi^*$. Since $\gamma(\Phi, \kappa^*, r + \delta, \lambda)$ is increasing in Φ , $\frac{1}{\rho - \gamma(\Phi, \kappa^*, r + \delta, \lambda)} < \frac{1}{\rho - \gamma(\phi^*, \kappa^*, r + \delta, \lambda)}$ if $\Phi < \phi^*$, and $\frac{1}{\rho - \gamma(\Phi, \kappa^*, r + \delta, \lambda)} > \frac{1}{\rho - \gamma(\phi^*, \kappa^*, r + \delta, \lambda)}$ if $\Phi > \phi^*$. Therefore, $\frac{\kappa^* - (1 + c'(\gamma(\Phi, \kappa^*, r + \delta, \lambda)))}{\rho - \gamma(\Phi, \kappa^*, r + \delta, \lambda)} \leq \frac{\kappa^* - (1 + c'(\gamma(\phi^*, \kappa^*, r + \delta, \lambda)))}{\rho - \gamma(\phi^*, \kappa^*, r + \delta, \lambda)}$ with strict inequality for $\Phi \neq \phi^*$. Therefore, if $F(\Phi)$ is non-degenerate, then $\frac{\partial \alpha(\kappa^*)}{\partial \lambda} < \int_G \frac{\kappa^* - (1 + c'(\gamma(\Phi, \kappa^*, r + \delta, \lambda)))}{\rho - \gamma(\phi^*, \kappa^*, r + \delta, \lambda)} dF(\Phi) = \frac{1}{\rho - \gamma(\phi^*, \kappa^*, r + \delta, \lambda)} \int_G [\kappa^* - (1 + c'(\gamma(\Phi, \kappa^*, r + \delta, \lambda)))] dF(\Phi) = 0$. Since an increase in λ reduces $\alpha(\kappa)$ for all κ , Corollary 6 implies that an increase in λ also reduces \bar{v} . ■

Proof. of Proposition 9: Proposition 6 states that $\bar{v}_2 > \bar{v}_1$ and statement 2 of Lemma 2 is that $\gamma(\phi, \kappa, r + \delta, \lambda)$ is increasing in κ . Therefore, $\gamma(\Phi, \bar{v}_2, r + \delta, \lambda) > \gamma(\Phi, \bar{v}_1, r + \delta, \lambda)$, which proves statement 1. Statement 1 implies $\int_G \gamma(\Phi, \bar{v}_2, r + \delta, \lambda) dF_2(\Phi) > \int_G \gamma(\Phi, \bar{v}_1, r + \delta, \lambda) dF_2(\Phi)$ and statement 1 of Lemma 2 that $\gamma(\phi, \kappa, r + \delta, \lambda)$ is increasing in ϕ implies that $\int_G \gamma(\Phi, \bar{v}_1, r + \delta, \lambda) dF_2(\Phi) > \int_G \gamma(\Phi, \bar{v}_1, r + \delta, \lambda) dF_1(\Phi)$. Therefore, $\int_G \gamma(\Phi, \bar{v}_2, r + \delta, \lambda) dF_2(\Phi) > \int_G \gamma(\Phi, \bar{v}_1, r + \delta, \lambda) dF_1(\Phi)$, which proves statement 2.

Since $v(\Phi) = 1 + c'(\gamma(\Phi, \bar{v}, r + \delta, \lambda))$, $\frac{dv_i(\Phi)}{d\Phi} = \frac{\partial c'(\gamma(\phi, \bar{v}_i, r + \delta, \lambda))}{\partial \phi}$, $i = 1, 2$. Statement 1 of corollary 4 is $\frac{\partial c'(\gamma(\phi, \kappa, r + \delta, \lambda))}{\partial \phi} = \frac{1}{\rho - \gamma}$, so $\frac{dv_i(\Phi)}{d\Phi} = \frac{1}{\rho - \gamma(\phi, \bar{v}_i, r + \delta, \lambda)}$. Therefore, statement 1 implies that $\frac{dv_2(\Phi)}{d\Phi} > \frac{dv_1(\Phi)}{d\Phi}$, which proves statement 3.

Statement 1 of Lemma 2 is that $\gamma(\phi, \kappa, r + \delta, \lambda)$ is increasing in ϕ , so that $\frac{1}{\rho - \gamma(\Phi, \bar{v}_2, r + \delta, \lambda)}$ is increasing in Φ . Therefore, $\int_G \frac{dv_2(\Phi)}{d\Phi} dF_2(\Phi) = \int_G \frac{1}{\rho - \gamma(\Phi, \bar{v}_2, r + \delta, \lambda)} dF_2(\Phi) > \int_G \frac{1}{\rho - \gamma(\phi, \bar{v}_2, r + \delta, \lambda)} dF_1(\Phi)$. Since $\gamma(\phi, \kappa, r + \delta, \lambda)$ is increasing in κ , $\int_G \frac{1}{\rho - \gamma(\phi, \bar{v}_2, r + \delta, \lambda)} dF_1(\Phi) > \int_G \frac{1}{\rho - \gamma(\phi, \bar{v}_1, r + \delta, \lambda)} dF_1(\Phi) = \int_G \frac{dv_1(\Phi)}{d\Phi} dF_1(\Phi)$. Putting together the inequalities in the two preceding sentences implies $\int_G \frac{dv_2(\Phi)}{d\Phi} dF_2(\Phi) > \int_G \frac{dv_1(\Phi)}{d\Phi} dF_1(\Phi)$, which proves statement 4. ■