USC FBE MACROECONOMICS AND INTERNATIONAL FINANCE WORKSHOP
presented by Dmitriy Stolyarov
FRIDAY, October 29, 2004
3:30 pm – 5:00 pm, **Room: HOH-601K**

The Role of Owned Ideas in Stock Market Run-ups*

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PRELIMINARY AND INCOMPLETE,
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October 5, 2004

Abstract

During the late 1990s the market value of businesses grew at 15% per year. It is hard to reconcile such fast growth with the observed rates of investment in physical capital and R&D. We therefore propose a model in which new ideas are privately owned, but discovering them does not require resources. In our model it is possible that market value rises very rapidly without drastic changes in factor prices or resource allocation. We examine possible scenarios for the rapid market run-up of the 1990s, including optimistic assessment of future productivity growth, a shift in the innovation process and a change in patent policy. At this point, plausible changes can explain no more than a third of the observed 1995-99 market run-up. This exercise suggests a method of testing for stock market bubbles.

JEL codes O16, O41, O34

1 Introduction

The sharp run up, and subsequent precipitous decline, in the U.S. stock market in the late 1990s raises a question of whether human emotions sometimes drive price changes in financial markets or whether economic fundamentals are always the root cause. The purpose of this paper is to present a model in which rapid, rational increases in stock prices are possible and to examine (i) whether there are plausible limits to the rate of increase and (ii) what changes in other variables we might expect concurrently with sudden, large, but conceptually justifiable, price movements. Since changes in investment do not seem large enough to explain recent stock market rises and declines, we consider a framework in which innovative ideas occur through random luck but are privately owned. The model may help us to understand what stock market participants were thinking during the late 1990s, and we use simulations to attempt to assess whether the observed market run-up was conceivably consistent with calibrated parameter values.

*Stolyarov gratefully acknowledges the financial support from the National Science Foundation (NSF).

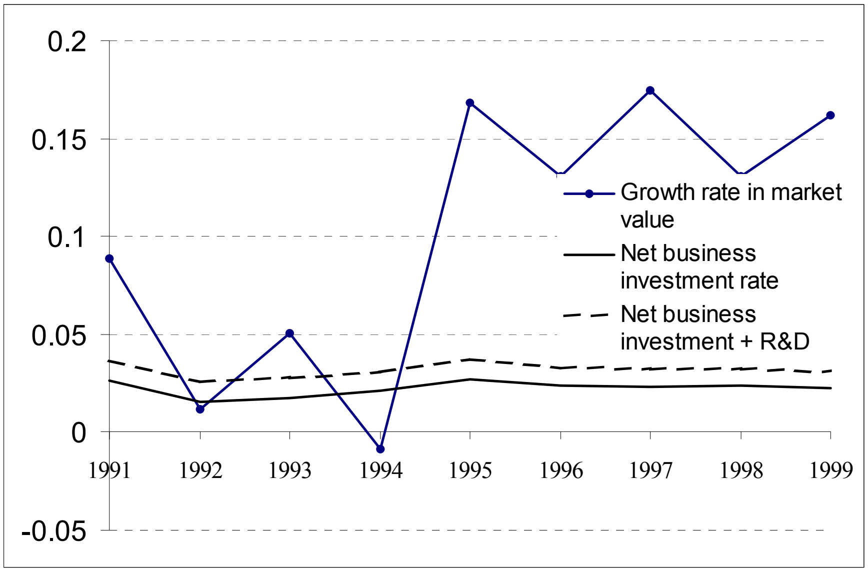

Figure 1: Growth in market value versus investment rate.

From 1994 to 1999 the market value of U.S. businesses more than doubled in real terms, climbing at an average rate of more than 15% per year.[1] According to the neoclassical growth model, the market value of businesses equals the size of the capital stock. Therefore, the rate of stock market increase should equal the rate of net investment. Yet, empirical evidence does not seem to bear this relationship out: Figure 1 shows that through the 1990s the NIPA rate of net business investment was only about 2.5 percent of market value.[2] In an endogenous growth model (e.g. Grossman and Helpman, 1991, Romer, 1990) or a model with intangible capital (e.g. Laitner and Stolyarov, 2003), stock prices also incorporate investment in private knowledge. Because the National Income and Product Accounts may omit knowledge investment, counting it as an intermediate good (Howitt, 1996), perhaps very large investments in knowledge after 1994 explain stock market gains. If that were true, one would presume that knowledge investment would be correlated with private R&D spending; however, the dashed line in Figure 1 shows that in the 1990s the R&D investment rate stayed essentially constant.[3] Although rising costs of adjustment could drive up the marginal cost of new investment and increase the price of existing capital, presumably an increase in gross investment rate would be needed to stimulate higher adjustment costs. But, the share of gross investment in GDP during the 1990s was between 0.15 and 0.17 — not much different from the share's long-run average of 0.155. In sum, it seems hard to reconcile the stock market run-up of 1995-99 with evidence on investment rates.

This paper therefore turns from a model in which investment in either physical capital or knowledge determines the rate of change in stock market values to a model in which a natural flow of inspired ideas can affect market prices. In particular, this paper focuses on a model in which innovative ideas occur exogenously and randomly but in which entrepreneurs and/or existing businesses seize control of the new ideas and appropriate concomitant prof-

[1] Source: U.S. Flow of Funds, http://www.federalreserve.gov/releases/Z1/Current/data.htm

[2] Business investment is total investment less owner-occupied housing. See Appendix 1 for more details.

[3] Source: National Science Foundation. Assuming that depreciation rate on R&D investment is zero, this investment rate can be measured with the ratio of R\&D spending to the market value of businesses.

its. Table 1 maps our model's place in the existing growth literature. The well-known Solow (1956, 1960) growth models assume that technological progress is exogenous and that private agents do not own new ideas. In that setup, the benefits from a new technology quickly filter through the economy and spill to labor. Models with intangible investment (Laitner and Stolyarov, 2003) or intentional R&D (Grossman and Helpman, 1991) make two changes: new knowledge originates from intentional investment, and investors gain ownership rights to knowledge. The present paper considers an intermediate case: we assume that new knowledge arises from luck and inspiration, but fortunate agents or firms appropriate the new knowledge, at least for a time. For expositional convenience, we refer to any proprietary knowledge as a *patent*. The last century provides numerous examples of inventors and entrepreneurs who perceived needs and opportunities, often at young ages, and developed new products and/or ideas from which they derived fortunes seemingly totally incommensurate with their inputs of time and money. If this is the case, market values need not follow investment rates.

	Origin of Technological Progress	
	Luck	R&D or knowledge investment
Knowledge is public	Solow (1956, 1960)	No incentives for investment
Knowledge is private	this paper	Romer (1990), Grossman and Helpman (1991), Laitner and Stolyarov (2003)

Table 1: Classification of growth models

Although our framework might seem to remove all limitations on changes in stock market valuations, the model produces a series of implications for observable variables that discipline one's choice of parameters and restrict simulated growth rates. We develop a general framework with exogenous, disembodied technological progress that occurs via independent Poisson processes in numerous sectors of the economy. All agents are *ex ante* identical, but randomly selected individuals, or businesses, appropriate the new ideas every period. A fortunate agent attains a patent, sells it to a business (whose shares are traded on the stock market), and uses the proceeds to finance his consumption and saving; a fortunate business makes a discovery and receives a patent, and its existing shareholders benefit.

Comparative–static exercises generate this paper's results. (1) If the arrival rate of new ideas slows down, our model predicts that the stock market should rise. A drawback to this as a possible explanation of recent developments is that the same change causes a slowdown in productivity growth, which seems at odds with many commentators' views of the late 1990s. (2) If the magnitude of improvement from each new idea increases, again the model implies that the stock market should rise. In this case, however, we find that it is difficult to provide calibrations for which our simulations match the magnitude and speed of observed common stock price changes. (3) While our most basic setup assumes successive innovation, in which a new idea simply replaces the one that preceded it, in practice new products sometimes depend on multiple, complementary patents. Historical examples include the sewing machine, the airplane and the automobile, and more recently, the DVD (e.g., Lerner, Tirole and Strojwas, 2003). Similarly, innovations often use the same fundamental idea and improve upon one another. For example, Visicalc, Lotus 1-2-3 and Microsoft Excel are all sequential improvements on the same basic idea of an electronic spreadsheet.[4] The market value of patents on such *fundamental ideas* depends not only on the current cost savings that they generate but also on their owner's ability to extract surplus from complementary innovations in the future. In other words, a third scenario that can generate a rapid rise in the stock market is an optimistic perception of the value of fundamental ideas — which we model as an increase in the prevalence and longevity of *fundamental* patents. Such an increase could arise from a demand shift toward industries where ideas are highly complementary, such as semiconductors and computer hardware and software (Bessen and Maskin, 2000). Or, it could follow from a change in the legal policy that affords protection to previously unpatentable areas or allows broader patents. Arguably both of the latter changes in policy in fact emerged during the 1980s and 1990s: court outcomes and administrative changes in the 1980s made computer programs and semiconductor chip designs patentable subject matter (Hunt, 1999), and patentability for computer implemented business methods (such as Amazon.com one-click shopping and Priceline.com *name your own price* auction) followed in the 1990s after seminal court decisions. Nevertheless, our numerical analysis shows that our model matches the 1995-1999 stock market run on if the probability that the next innovation is based on the same idea permanently rose an implausibly large amount.

We conclude that although our framework suggests that a steep run–up in the stock market could have a rational basis, our analysis, at this point, arguably does not provide a plausible, economics–based explanation for the magnitude of stock price changes in the late 1990s.

2 Successive innovation

The production side of the economy is a generalization of the quality ladder model in Grossman and Helpman (1991).[5] Final output is an aggregate of a large number of intermediate goods

$$Y_t = \left(\sum_j x_{jt}^{\eta} \right)^{\frac{1}{\eta}}, \eta < 1 \tag{1}$$

where x_j is the quantity of intermediate good j and p_{jt} is its price. Final goods sector consists of competitive firms that maximize

$$\left(\sum_j x_{jt}^{\eta} \right)^{\frac{1}{\eta}} - \sum_j p_{jt} x_{jt}$$

Profit maximization implies the following demand curve for intermediate good j:

$$\left(\frac{Y_t}{x_{jt}} \right)^{1-\eta} = p_{jt} \tag{2}$$

[4] Part of this fundamental idea is actually patented. Any spreadsheet program uses the natural order recalculation method, U.S. Patent No. 4,398,249, granted 1983. This patent is owned by Refac, a litigation company. In 1989 Refac sued six major spreadsheet publishers, including Lotus, Microsoft, and Ashton-Tate, for patent infringement.

[5] Barro and Sala-i-Martin (1999, Ch 7) have an alternative formulation that leads to the same aggregate production function for output.

Agents can produce intermediate good j from capital and labor using different technologies indexed by $n \leq N_{jt}$, where N_{jt} is the total number of technological innovations in industry j so far. In contrast to the endogenous growth literature, we assume that technological innovations arise from inspiration (rather than R&D investment) and that they arrive in an exogenous and random fashion. The agent who "discovers" a technology owns it, and patents protect his property rights indefinitely. It is in the owner's interest to license his technology to only one producer. At this point, we make the following assumption about the nature of the innovation process:

Successive innovation assumption *Each technology n is a stand-alone alternative production method and does not rely on know-how from previously discovered technologies. A producer that wants to produce intermediate good x_j with technology n needs to license only one patent — from the owner of technology n in that industry.*

In any industry j, N_{jt} is a Poisson variable with arrival rate λ. We assume that arrivals are independent in different industries; thus, N_{jt} are independent random variables. Newer technologies have higher TFP levels. In particular, we assume that in any industry, technology n has TFP level z^n, where $z > 1$ is a constant step in the quality ladder that does not depend on j or time. The output in industry j from k_{nj} units of physical capital, ℓ_{nj} units of labor, and technology n is

$$x_{jt} = \sum_{n \leq N_{jt}} z^n k_{njt}^{\alpha} l_{njt}^{1-\alpha}.$$

The corresponding marginal cost function for technology n (in any industry) is

$$c_{nt} = \frac{1}{z^n} \left(\frac{R_t}{\alpha}\right)^{\alpha} \left(\frac{W_t}{1-\alpha}\right)^{1-\alpha} = \frac{c_t}{z^n},$$

where R_t is the rental fee on physical capital, and W_t is the wage. Let $Z_{jt} \equiv z^{N_{jt}}$ denote the TFP level that corresponds to the most advanced technology in industry j.

At any point in time, several producers can make the same intermediate good using technologies with different marginal costs. We assume that producers engage in Bertrand price competition. In a Bertrand equilibrium, the industry leader — the firm employing the most advanced technology — produces all the output of the given intermediate good. The leader charges either the unconstrained monopoly price $p_{jt} = \frac{1}{\eta} \cdot \frac{c_t}{Z_{jt}}$ if it exists and is below the closest rival's marginal cost; otherwise, the leader charges a limit price equal to the closest rival's marginal cost, $p_{jt} = z \cdot \frac{c_t}{Z_{jt}}$. That is,

$$p_{jt} = \frac{1}{\max\left(\eta, \frac{1}{z}\right)} \frac{c_t}{Z_{jt}} = m \frac{c_t}{Z_{jt}},$$

where m is a constant markup.[6] Industry output is

$$x_{jt} = Z_{jt} K_{jt}^{\alpha} L_{jt}^{1-\alpha}, \tag{3}$$

where $K_{jt} \equiv k_{N_{jt},j}$, $L_{jt} \equiv l_{N_{jt},j}$.

Poisson processes in different industries induce a distribution of technology levels by sector. To start the analysis, assume that at time $t = 0$ all industries have $N = 0$ and the

[6] Under limit pricing, the industry leader is willing to pay the most for a new patent, because buying this patent enables the leader to raise his markup from z to z^2. However, inh this section we assume that antitrust policy prevents the leader from accumulating patents.

same initial technology.[7] Let $f(N,t)$ denote the fraction of industries that have exactly N innovations at time t, and let J be total the number of industries. It turns out that aggregate output Y depends on the technologies in different sectors only through the quality index

$$Z_t = \left(\sum_j (Z_{jt})^{\frac{\eta}{1-\eta}} \right)^{\frac{1-\eta}{\eta}} = \left(J \sum_{N=0}^{\infty} f(N,t) \, z^{N \frac{\eta}{1-\eta}} \right)^{\frac{1-\eta}{\eta}}. \tag{4}$$

Without loss of generality, we can normalize $J = 1$ in (4). Although output and profit in each industry grow stochastically, Z_t and aggregate output both have a deterministic growth rate. Let K_t be the physical capital stock at date t, L_t the labor supply, and Π_t aggregate profits of intermediate good producers. The following proposition derives an aggregate production function and shows that quality index (4), which scales the aggregate production function, grows at a constant rate depending on λ, z and η.

Proposition 1
The aggregate production function is

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha}. \tag{5}$$

$$Z_t = \exp\left(\gamma \frac{1-\eta}{\eta} t\right) \text{ where}$$

$$\gamma = \lambda \left(z^{\frac{\eta}{1-\eta}} - 1 \right). \tag{6}$$

Letting $m = 1/\max\left\{\eta, \frac{1}{z}\right\}$, the division of national income between physical capital, labor and monopoly profits is

$$R_t K_t = \frac{\alpha}{m} Y_t, \; W_t L_t = \frac{1-\alpha}{m} Y_t, \; \Pi_t = \frac{m-1}{m} Y_t \tag{7}$$

Proof: See Appendix.

We also have

Corollary: Profits of the leader in industry j at time t equal

$$\pi_{jt} = \Pi_t Z_t^{-\frac{\eta}{1-\eta}} \cdot \left(z^{N_{jt}}\right)^{\frac{\eta}{1-\eta}} \equiv \pi_t (N_{jt}). \tag{8}$$

Proof: See Appendix.

Let w_{jt} denote the value of a patent in industry j at time t. It equals the expected present value of profits of the current industry leader until the moment when another innovation arrives and the leader is priced out of the market. Accordingly, for an industry with current technology $Z_j = z^N$, we have

$$w_{jt} = w_t(N) = \int_t^{\infty} d\tau \lambda e^{-\lambda(\tau - t)} \int_t^{\tau} e^{-\int_t^s r_\nu d\nu} \cdot \pi_s(N) \, ds \tag{9}$$

Differentiating with respect to t,

$$\frac{d}{dt} w_t(N) = (\lambda + r_t) w_t(N) - \pi_t(N). \tag{10}$$

[7] Assume that technology $z^0 = 1$ is proprietary, but there is a universally known technology $1/z$ that is freely available. Then initially every industry has a leader and a potential competitor.

The aggregate value of patents in the economy is

$$V_t = \sum_j w_{jt} = \sum_{N=0}^{\infty} f(N,t)\, w_t(N). \tag{11}$$

The next proposition derives the law of motion for the aggregate value of patents.

Proposition 2:

$$\dot{V}_t = \gamma V_t + (r_t + \lambda) V_t - \Pi_t \tag{12}$$

Proof: See Appendix.

The interpretation of this expression is as follows. The first term gives the *net* growth in patent wealth, where γ is defined in (6).[8] The second term is the (gross of depreciation) rate of return on current patent wealth, and the third term is payouts to owners of intellectual property.

For what follows, it is convenient to define Δ to be the depreciation rate on patent wealth. In (12), depreciation rate on a patent equals the flow probability that it is replaced by another innovation, i.e. $\Delta = \lambda$.

Equilibrium. Assume that households save a constant fraction σ of their total (gross of depreciation) income flow. In addition to labor income and capital income, whose sum is $Y_t - \Pi_t$, households receive returns on existing patent wealth in the amount of $(r_t + \Delta) V_t$ and flow of newly created patent wealth, γV_t. Depreciation payments on V add another ΔV_t. Consequently, the economy's saving flow is

$$S_t = \sigma (Y_t - \Pi_t + (r_t + \Delta) V_t + \gamma V_t + \Delta V_t) = \sigma \left(Y_t + \dot{V}_t + \Delta V_t \right)$$

Saving finances the growth in the economy's total net worth, $K_t + V_t$. Then saving equals physical investment plus the gross of depreciation flow of new patent wealth. The market clearing condition for the economy is

$$\dot{K}_t + \delta K_t + \dot{V}_t + \Delta V_t = \sigma (Y_t - \Pi_t + (r_t + \Delta) V_t + \gamma V_t + \Delta V_t) \tag{13}$$

This saving function captures in reduced form the consumer side of an overlapping generations economy with heterogenous agents. Lucky agents would like to borrow against the future monopoly profits and consume, and therefore aggregate consumption rises with the returns on patent wealth:

$$C_t = Y_t - I_t = (1-\sigma) Y_t + (1-\sigma) \left(\dot{V}_t + \Delta V_t \right),$$

$$I_t = \sigma Y_t - (1-\sigma) \left(\dot{V}_t + \Delta V_t \right).$$

Physical investment, on the other hand, can be crowded out by paper wealth, as the expression above shows.

Equations (12) and (13) together with (5) and the expression for the interest rate

$$r_t = \frac{\alpha}{m} \frac{Y_t}{K_t} - \delta$$

[8] Note that if $\eta < 0$, γ is negative, because then demand for intermediate goods is inelastic and revenue falls with output. Then leader's profits shrink as cost falls and output expands.

determine the equilibrium time path for this economy.

Let

$$k_t = \frac{K_t}{Z_t^{\frac{1}{1-\alpha}} L_t},\ v_t = \frac{V_t}{Z_t^{\frac{1}{1-\alpha}} L_t},\ y_t = k_t^{\alpha}$$

denote the detrended variables.

Let

$$g \equiv \frac{1}{1-\alpha}\frac{\dot{Z}}{Z} = \frac{1}{1-\alpha}\frac{1-\eta}{\eta}\gamma, \tag{14}$$

$$s = \sigma + (1-\sigma)\frac{m-1}{m},$$

and

$$n = \frac{\dot{L}_t}{L_t}$$

From (12) and (13),

$$\frac{\dot{V}_t}{V_t} = \frac{\dot{v}}{v} + n + g = (\gamma + r_t + \Delta) - \frac{m-1}{m}\frac{y}{v}$$

$$\frac{\dot{K}_t}{K_t} = \frac{\dot{k}}{k} + n + g = sk^{\alpha-1} - \delta - (1-\sigma)(\gamma + r_t + 2\Delta)\frac{v}{k}$$

Then we have the analogs of (12)-(13) in the detrended variables

$$\dot{v} = (\gamma + r_t + \Delta - (n+g))\, v - \frac{m-1}{m}k^{\alpha}, \tag{15}$$

$$\dot{k} = sk^{\alpha} - (\delta + n + g)\, k - (1-\sigma)(\gamma + r_t + 2\Delta)\, v \tag{16}$$

The phase diagram. The isoclines for (15)-(16) are given by

$$\dot{v} = 0\colon\ v = \frac{\frac{m-1}{m}k^{\alpha}}{(\gamma + r + \Delta - (n+g))} \tag{17}$$

$$\dot{k} = 0\colon\ v = \frac{sk^{\alpha} - (\delta + n + g)\, k}{(1-\sigma)(\gamma + r + 2\Delta)} \tag{18}$$

Figure 2 illustrates the phase diagram, which is a saddle with stable arm running southwest to northeast. The following proposition establishes that the nontrivial stationary state is unique.

Proposition 3 There is a unique $(k_*, v_*) > 0$ that solves (17)-(18)
Proof: See Appendix.

Note that in the conventional analysis of perfect foresight models (e.g., Blanchard and Kahn [1980]), the saddlepoint in Figure 2 is desirable: k_t is a predetermined variable and v_t is a jump variable; thus, at initial time $t = 0$, k_0 is given but v_0 is not. If we assume that nonexplosive paths for the detrended variables are the only empirically relevant ones, picking v_0 to locate (k_0, v_0) on the saddle's stable arm yields a unique equilibrium. A source point in Figure 3 would imply "instability;" a sink point would imply "indeterminacy."

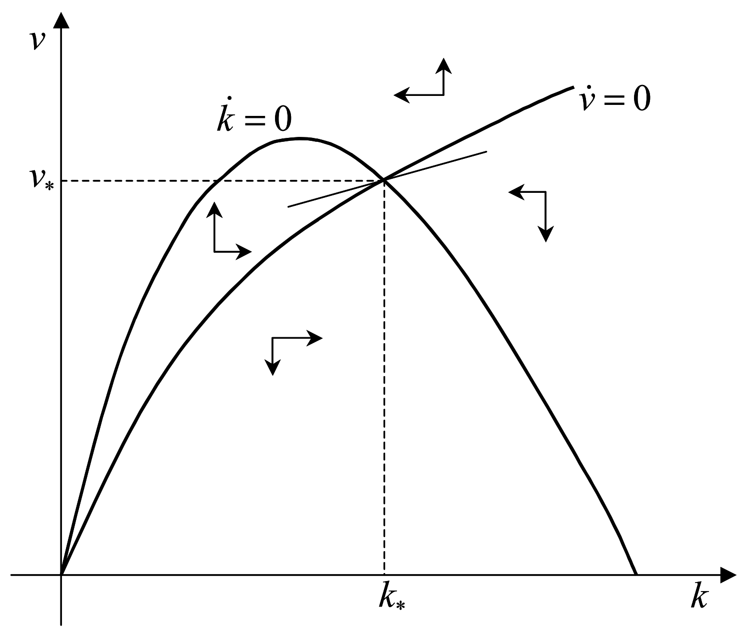

Figure 2: The phase diagram.

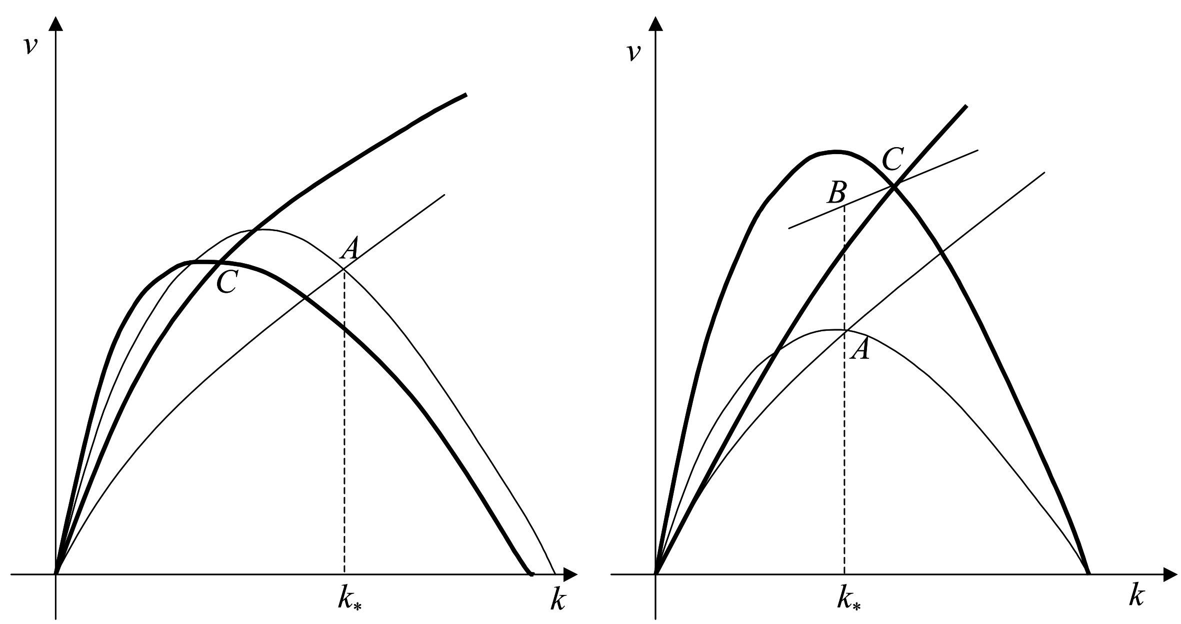

Figure 3: A permanent increase in z (left panel). A permanent decrease in Δ (right panel).

2.1 Comparative Statics

We are interested in the changes in the model's parameters that lead to an increase in the value of patents.

PERMANENT INCREASE IN Z. Suppose that initially the economy rests in a stationary state but that the technology step z increases to z_1 at $t = 0$. Let the increase be a complete surprise, and suppose that it is permanent. Let $\eta < \frac{1}{z}$ so that our limit pricing story holds. Then the increase in z has two effects: it increases the markup and also increases trend growth. In the new steady state, k is lower, because the trend growth is higher, but $\frac{v}{k}$ is higher, because the share of monopoly profits in the economy rises permanently. The left panel of Figure 3 depicts the initial steady state (point A) and the new steady state (point C).

Unfortunately, the dynamics are complicated, because markups do not rise from z to z_1 all at once, but do so gradually, as innovations with a bigger step size happen in individual industries. Initially the frontier TFP level in industry j is $[z]^{N_{j0}}$; over time, however, it changes to $[z]^{N_{j0}} \cdot [z_1]^{(N_{jt}-N_{j0})}$. Nevertheless, from any TFP distribution at time 0 (that is summarized by the aggregate quality index Z_0, according to Proposition1), we can simulate the adjustment path from initial condition that corresponds to the old stationary state k_* to the new stationary state consistent with z_1. Roughly speaking, over time the phase portrait of Figure 2 must reemerge, and the equilibrium must follow its stable arm. We address the details of this dynamics in Section 4.2.

Discussion. After the shock hits, a given industry faces a demand that grows more rapidly. In particular, as patents expire in other industries, the upward steps in technology are bigger than before — creating spillovers for all industries, in the form of stronger outward shifts in their demand curves than before.

There may be one insight here to the late 1990s: if z rises, existing patents can benefit from demand spillovers — despite the fact that new high–z inventions are owned by inventors who capture their direct profits. Thus the model implies that the stocks of existing companies should have risen rapidly through demand spillovers even though these companies did not experience any TFP growth of their own.

DECREASE IN Δ. The right panel of Figure 3 depicts the effect of a permanent decrease in the depreciation rate of patents, Δ. On impact, v_t rises to point B because existing patents henceforth last longer. As the economy converges to its new stationary state at point C, the interest rate falls and the value of patents rises further.

Since $\Delta = \lambda$ in (12), the only factor that could have decreased Δ is a slowdown in the rate of innovation. The latter seems to contradict public perceptions that productivity growth in the U.S. accelerated after 1995. In this light, we will not dwell on this comparative static case; nevertheless, the model illustrates the perhaps counterintuitive point that an era of slow technological progress can lead to higher stock prices. In Section 3, we develop a more general model of innovation where the rate of depreciation on patents can be different from the rate of innovation.

3 Fundamental innovations

The model of section 2 assumed that new technologies simply replace the old. However, it seems possible to argue that some ideas are more "fundamental" than others in the sense that subsequent innovations must, or will, utilize the same fundamental idea. Then the fundamental ideas not only survive the introduction of subsequent inventions, but may also appreciate in value. An example is Microsoft's computer operating system, which many generations of personal computers and computer software relied upon.

We model fundamental ideas as follows. Assume that, as before, technological progress is governed by the same Poisson process and each innovation is a fixed step up the productivity ladder. But now the innovation can be of two types: it either improves the existing fundamental idea (with probability θ) or replaces it with a new fundamental idea (with probability $1-\theta$). Therefore, θ is also the probability that a fundamental idea survives a new invention, and when $\theta = 1$ each existing patent lives forever. The model of section 2, by contrast, assumed that $\theta = 0$, so that an existing patent never survived an innovation, and each innovation was fundamental.

If the innovation is fundamental, its operation requires licensing one patent for that innovation, just as in the model of section 2. In contrast, when the innovation is, say, the n-th improvement on some original fundamental idea, then the producer needs to license not one, but n patents, from the owners of all the previous improvements. Then the fundamental patent and the ones that follow operate together as a "patent pool."[9] For tractability, throughout this section we assume that $\eta > \frac{1}{z}$, so when the fundamental idea changes, limit pricing against competing patent pools (that use older fundamental ideas) is never an issue. Then, regardless of its size, the patent pool operates with a constant (unconstrained monopoly) markup $m = \frac{1}{\eta}$. In Section 5, we relax this assumption and allow markups to grow with pool size.

If we want to determine the division of the pool's profit among the patent holders, we need additional assumptions. We require that the division of profits be a core allocation, i.e. there is no coalition of producers that can form an alternative patent pool that includes the fundamental idea and some subsequent improvements. We will consider two polar cases. (a) One is where each innovator gets the payoff equal to his contribution to the pool's profit. The other case (b) is when the owner of the fundamental patent usurps the pool's entire profit. The distribution of profits between innovators affects patent value, because it determines the rate of return on patents. Fundamental patents have two effects: on patent replacement rate and on growth rate of profits to the patent owner.

Case (a) *Innovators paid according to contribution* The aggregate resource flow to the owners of the pool that includes the frontier technology N is $\pi(N)$. Let technology N be based on some earlier fundamental idea $N_f \leq N$. Then the pool that uses technology N consists of $N - N_f + 1$ patents. Patents in the pool are indexed by $n = N_f, ..., N$, where patent N_f is the fundamental idea. Let $u(n, N)$ be the flow payoff to the owner of patent n when the most advanced technology is N. We specify the payoffs to all the owners of patents in the pool as follows.

$$\begin{aligned} u(N_f, N) &= \pi(N_f) \\ u(n, N) &= \pi(n) - \pi(n-1), \; n = N_f + 1, ..., N. \end{aligned} \tag{19}$$

In words, each innovator gets the payoff equal to the additional profit flow that his innovation generates for as long as the pool is not replaced. With this distribution, flow payoffs to patent owners do not change as new patents are added to the pool, and the most recent innovator captures all the additional payoff that his patent generates.

[9] In an industry which currently has $N \geq 1$ innovations, operating the best technology requires a pool of $n \leq N$ patents, where n is a random variable distributed according to $\theta^{n-1}(1-\theta)/(1-\theta^N)$.

Case (b) *Fundamental patent gets all.* Since the fundamental innovation constitutes a blocking patent for all the subsequent technologies in the patent pool, its owner can extract all the surplus that future innovations generate. Each time a new patent is added to the existing pool, the owner of fundamental patent gets additional profit:

$$\begin{aligned} u\left(N_f, N\right) &= \pi\left(N\right) \\ u\left(n, N\right) &= 0,\ n = N_f + 1, ..., N. \end{aligned} \tag{20}$$

The following Proposition derives the analogs of (12) for different distribution of payoffs.

Proposition 4:

$$\dot{V}_t = \gamma V_t + \left(r_t + \Delta\left(\theta\right)\right) V_t - \Pi_t,$$

where

$$\Delta\left(\theta\right) = \begin{cases} \lambda\left(1-\theta\right) & \text{in case (a)} \\ \lambda\left(1-\theta\right) - \gamma\theta & \text{in case (b)} \end{cases} \tag{21}$$

Proof: See Appendix.

The intuition the expression (21) is as follows. When the innovators are paid according to their contribution (case (a)),.every idea in the current patent pool generates a profit flow until the pool is replaced by a new fundamental idea. In this case, the depreciation rate on patents equals the flow probability of replacement, $\lambda\left(1-\theta\right)$. In case (b), fundamental patents are the only ones that generate positive profits. The arrival of a non-fundamental improvement makes the fundamental patent appreciate, because its owner captures additional profits from the innovation that is added to the pool. The rate of depreciation is now the difference between the flow probability that a fundamental idea is replaced, $\lambda\left(1-\theta\right)$, and the expected growth rate in the payout to the owner of the fundamental idea, $\theta\gamma$.

In case (a), the aggregate value of patents V only includes the patents that are currently active, and all profits from future patents go to future innovators. By contrast, in case (b), the owner of the fundamental idea is entitled to profits from all future improvements, which is why current V also includes the expected present value of improvements that have not yet appeared.

The equilibrium of the model with fundamental innovations is still described by (15)-(16), although the rate of depreciation on patent wealth Δ now depends on the probability θ that a fundamental idea survives through the next innovation and the distribution of payoffs among the owners of the patent pool. Note that for a given value of θ, the lower rate of depreciation on patents and the higher V corresponds to the situation where all the surplus from innovations goes to the owner of the fundamental idea.

4 Results

In this section , we evaluate numerically the effect on an increase in θ and z and attempt to quantitatively account for the levels of the stock market in the 1990s.

4.1 Calibration

We calibrate the parameters of the model to fit three long-run statistics for the US economy: growth rate of GDP per worker $g_y = 0.02$, investment to output ratio $i = 0.137$ and market value to GDP ratio $\mu = 1.62$.

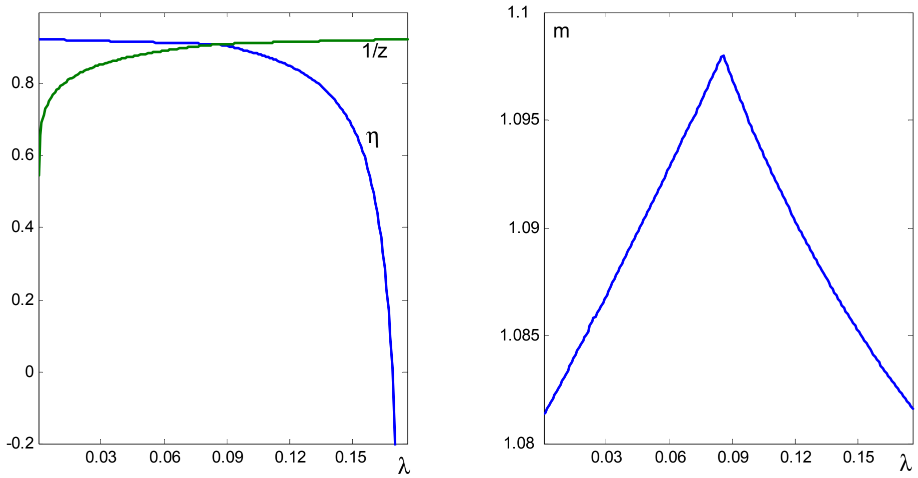

Figure 4: Calibrated η, $1/z$ and m for different values of λ.

We set conventional parameters $\alpha = 0.33$, $\delta = 0.07$ and $n = 0.01$ and experiment with different values of λ. It is left to pick k^* z, η and σ. For a fixed λ, we have four calibration equations. The first equation matches the economy's steady state investment to GDP ratio to the US long-run average $i = 0.137$.

$$i = (\delta + n + g_y)\, k_*^{1-\alpha} \tag{22}$$

Equation (22) pins down k_*. Given k_*, z and η can be uniquely determined from

$$\lambda \left(z^{\frac{\eta}{1-\eta}} - 1 \right) \frac{1-\eta}{\eta} \frac{1}{1-\alpha} = g_y, \tag{23}$$

$$\frac{v_*}{y_*} = \frac{\frac{m(\eta,z)-1}{m(\eta,z)}}{(\gamma + r_* + \Delta - (n+g))} + k_*^{1-\alpha} = \mu, \tag{24}$$

where

$$\gamma = g_y (1-\alpha) \frac{\eta}{1-\eta}, \; r_* = \frac{\alpha}{m(\eta, z)} k_*^{\alpha-1} - \delta$$

Equation (23) uses (14) and sets the economy's steady state growth rate of output per worker equal to g_y. Equation (24) uses (17) to match the economy's market value to output ration with the US long-run average. The remaining unknown, σ, can be found from

$$0 = \left(\sigma + (1-\sigma) \frac{m(\eta, z) - 1}{m(\eta, z)} \right) y_* - (\delta + n + g)\, k_* - (1-\sigma)(\gamma + r_* + 2\Delta)\, v_*. \tag{25}$$

Equation (25) picks the value of σ that makes k_* satisfy (18).

Figure 4 shows the calibration results for different values of λ. For $\lambda \leq \lambda_0 = 0.09$ there exists a solution with $\eta > \frac{1}{z}$ and $m = \frac{1}{\eta}$, and for $\lambda \in \left[\lambda_0, \bar{\lambda}\right]$ there exists a solution with $m = z$ (left panel of Figure 4). For $\lambda > \bar{\lambda} = 0.18$ equation (23) has no solutions. Calibrated markups are in a tight $[1.08, 1.1]$ range for any feasible value of λ. Consequently, values of σ are also in a tight range $[0.142, 0.180]$.

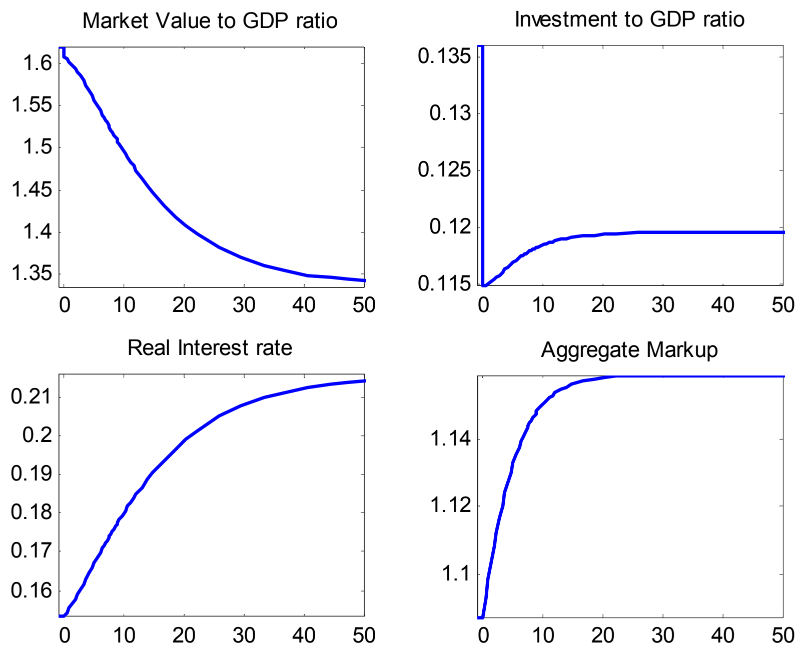

Figure 5: The effect of a rise in step size z.

4.2 A permanent increase in growth rate

A permanent increase in step size z leads to a permanently higher trend growth and a permanently higher share of monopoly profits. However, in a particular industry, the markup rises only after an innovation occurs and the current patent is replaced. This is why existing patents do not benefit from higher step size, except through demand spillovers from other industries that had received an innovation.

For a fixed λ, we choose a pair of z and $z_1 > z$ such that trend growth g corresponding to step size z is 2 percent, and trend growth corresponding to step size z_1 is 4 percent. The implied values of z and z_1 do not vary much with λ, and we report the results for $\lambda = 0.135$, which is completely representative of all simulations in the $\lambda \in \left[\lambda_0, \bar{\lambda}\right]$ range. The non-autonomous aggregate equations of motion that govern the transition path are derived in Appendix 3.

Figure 5 shows the simulation results. The value of existing patents falls on impact, because firms that produce with them lose access to capital and labor, as input prices are bid by innovators in other industries that charge higher markups. Physical investment share in GDP falls sharply, because when the change in z occurs, the rate of growth in newly created patent wealth rises abruptly. To accommodate this higher flow of new patent wealth, saving has to be shifted away from physical investment. Over time, the lower investment flow reduces capital intensity. In the long run $\frac{V}{K}$ rises but $\frac{K}{Y}$ falls. The net effect is that market value to GDP ratio, $\frac{V+K}{Y}$, falls in the long run.

The simulation confirms the intuition that slower innovation can be associated with rising market value, because if innovations occurs slowly, less resources are tied up in the ownership of V and more resources are available for physical investment.

4.3 A change in the innovation process

We assume that at time $t = 0$ agents suddenly realize that some existing ideas are going to survive through the next innovations. Suppose that the probability that an idea survives through the next innovation permanently rises from 0 (its value in the basic model of Section 2) to $\theta > 0$. Note that a change in θ does not affect the economy's production possibilities, because innovations arrive exactly as before. The only effect of a rise in θ operates through distribution of monopoly profits. With higher θ, a lucky agent who discovers a fundamental idea captures a larger fraction of the future monopoly profits in a particular market.

Let the owner of the fundamental idea usurp the entire profit of the patent pool, that is, let the depreciation rate on patents be $\Delta = \lambda(1-\theta) - \gamma\theta$, which corresponds to case (b) in Proposition 4. The impact of the increase in θ is higher for higher values of λ. The analysis in Section 3 applies only when $\eta > \frac{1}{z}$. We therefore pick the highest possible $\lambda = \lambda_0$ that is consistent with this restriction. We also think that the plausible upper bound for the expected life of the fundamental idea is no more than $T = 20$ years. This implies that

$$\theta = 1 - \frac{1}{\lambda T} \approx 0.42.$$

Figure 6 (bold lines) shows the simulation results. Market value to GDP ratio rises to 1.81

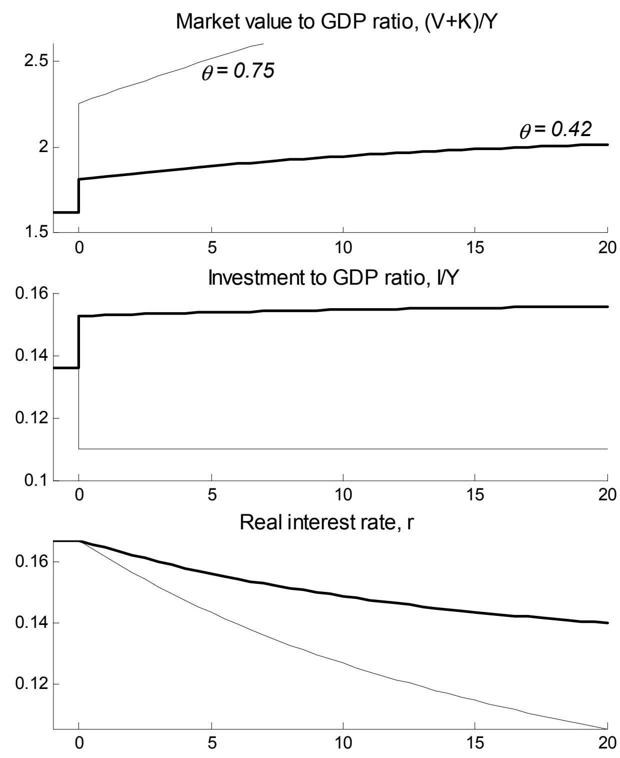

Figure 6: Impact of a change in the innovation process, $\theta = 0.42$ and $\theta = 0.75$.

on impact and reaches 1.9 within 5 years. In principle, it is possible to match the market value to GDP ratio that rose from 1.6 to over 2.5 late 1990s. The thin lines on Figure 6 correspond to $\theta = 0.75$. Then market value to GDP ratio rises to 2.25 on impact and reaches 2.5 within 5 years. The shock of this magnitude does not look plausible, because the implied average lifetime of a fundamental idea is 47 years.

5 Fundamental ideas with rising markups

The effect of a given change in θ on the depreciation rate of patents Δ is stronger for higher values of λ. We can therefore expect that when innovations arrive faster, fundamental ideas will be more valuable, because the expected profit from future improvements will accrue faster. However, in Section 3 we assumed that the markup is constant $m = \frac{1}{\eta}$, so our choice of λ was limited to the range $[0, \lambda_0]$ that is consistent with this assumption. For $\lambda > \lambda_0$, the markup is chosen by limit pricing against the competing patent pool (based on the previous fundamental idea):

$$m = \frac{1}{\max\left(\eta, \frac{1}{z^n}\right)}. \tag{26}$$

As the pool size grows, it gains more and more cost advantage over the closest competitor, and therefore the markup rises according to z^n, where n is the current pool size, but only up to a point. When z^n reaches $\frac{1}{\eta}$, the markup ceases to depend on pool size and stays at $\frac{1}{\eta}$ for as long as the pool is active.

The aggregation of an economy with markups that depend on pool size becomes difficult, as we need to keep track of the joint distribution of industries by number of innovations N as well as by patent pool size $n \leq N$. In general, the analytical expression for the aggregate production function no longer exists. Nevertheless, we can carry out the aggregation analytically for an important special case when the distribution of markups has a two-point support $\left\{z, \frac{1}{\eta}\right\}$. Then all pools of size $n = 1$ have a markup z and all pools of size $n > 1$ have markup $\frac{1}{\eta} > z$. The aggregate equations of motion that correspond to the transition path of the economy after a shock to θ are derived in Appendix 3.

For the simulation , we pick the highest value of λ that is consistent with the assumed markup rule (that is, the highest value of $\lambda = 0.1227$ for which $\eta \in [1/z^2, 1/z]$). Picking a higher value of λ also relaxes the constraint on the plausible upper bound for θ. If we continue to assume that the average life of the fundamental idea T is 20 years, then we can set $\theta = 1 - 1/\lambda T = 0.59$. Figure 7 shows the results. The $\frac{M}{Y}$ ratio jumps from 1.62 to 1.87 on impact and reaches 1.97 within 5 years. The dynamics of investment is governed by two opposing forces. A rise in value of patents reduces investment, but a fall in r and Δ makes return on patent wealth smaller, which increases investment. A sharp rise in V on impact lowers investment, but then it gradually grows as higher θ makes the value of Δ permanently lower and r keeps falling.

Of course, it is also possible to perform simulations for even higher values of λ, if for tractability we keep the same two-point distribution of markups (although this mark-up rule is no longer optimal). Note that picking a higher λ makes the calibrated value of η lower and the corresponmding maximum markup $\frac{1}{\eta}$ higher. Higher markups permanently increase the share of monoploy profits and provide a drag on TFP. If λ is high enough, the share of monoploy profits becomes so high that the economy goes to a steady state with lower investment and lower k. For example, setting $\lambda = 0.15$ implies that the share of monopoly profits goes from 0.08 to 0.2 in the long run. When $\theta = 1 - 1/\lambda T = 2/3$, market value to GDP ratio jumps to 2.07 on impact, reaches 2.26 in 5 years, tops out at 2.3 and heads lower in the long run.

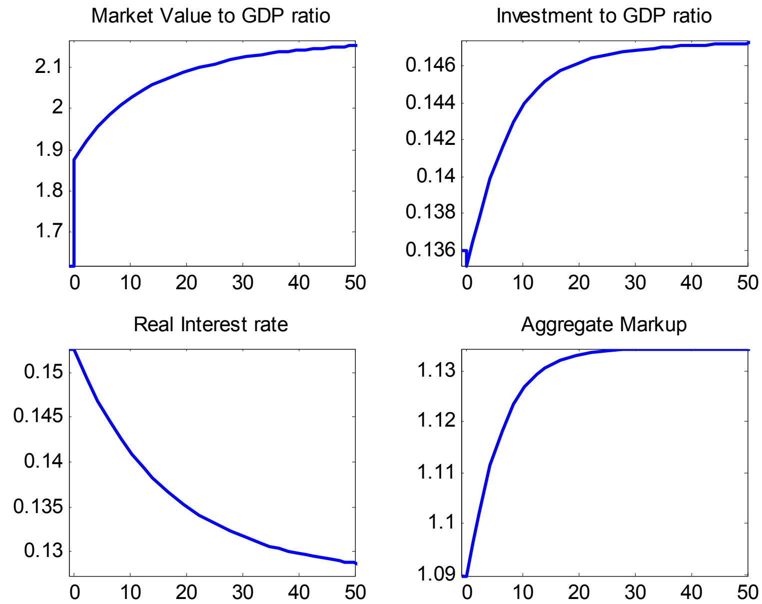

Figure 7: Impact of a chnage in θ from 0 to 0.59, when markups are gorwing.

6 Conclusion

Faster rate of progress can lead to lower market value to GDP ratio. A higher rate of progress raises the value of ideas, but this also diverts funds from physical investment. Patent policy has a potentially important impact on aggregate makret value and physical investment.

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Appendix 1: Data Construction

We assume that all tenant-occupied housing is counted in the market value of business. Therefore we need to construct a measure of private investment that excludes owner-occupied housing, but includes tenant-occupied housing. Gross private domestic investment (T 1.1.5 L6) less residential investment (T 1.1.5 L11) multiplied by the share of owner-occupied housing in residential investment equals business investment. To get net investment, subtract capital consumption of domestic business (T 7.5 L3).

The share of owner-occupied housing in residential investment can be imputed from the share of owner-occupied housing services in total housing services (T 7.4.5. L4 divided by T 7.4.5. L1).

Figure 8 depicts the share of investment in GDP for 1952-2002. The distance between the top line and the middle line on Figure 8 is depreciation of private capital (including owner-occupied housing). The distance between the middle line and the bottom line is net investment in owner-occupied housing.

Appendix 2: Proofs

Proof of Proposition 1: Omitting the time subscript for more compact notation and using the expression for the industry production function (3), the demands for capital and labor

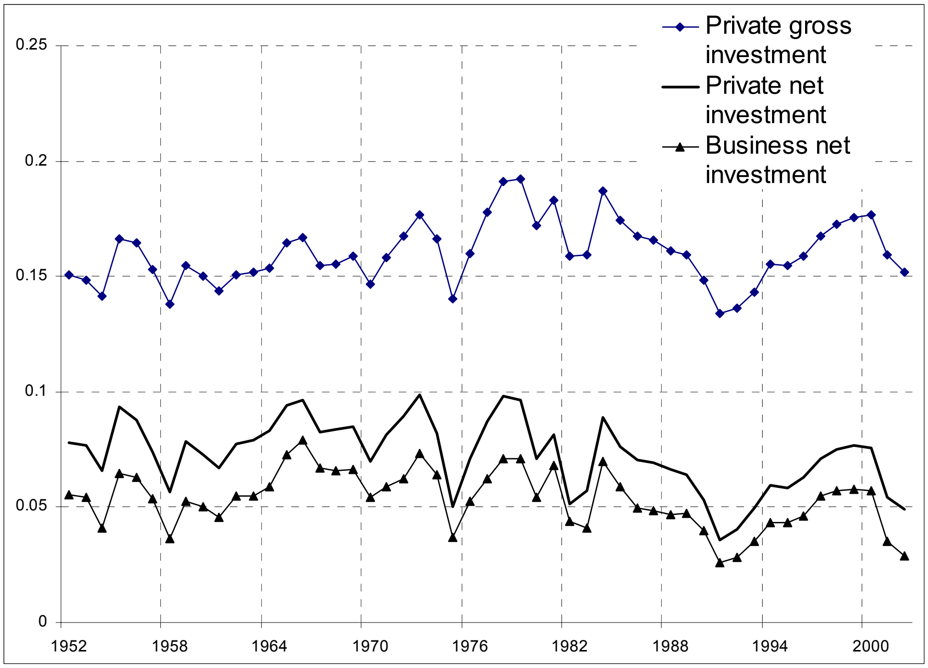

Figure 8: Different measures of investment as a share of GDP.

are given by

$$R = \alpha \frac{x_j}{K_j},\; W = (1-\alpha)\frac{x_j}{L_j} \tag{27}$$

Then it is immediate that every industry has the same capital-labor ratio

$$k = \frac{K_j}{L_j} = \frac{K}{L}.$$

Let $l_j = L_j/L$ denote the fraction of the total labor force employed in industry j. Then, using the expression for aggregate output (1) and also(3),

$$Y = \left(\sum_j x_j^\eta\right)^{\frac{1}{\eta}} = \left(\sum_j (Z_j k^\alpha l_j L)^\eta\right)^{\frac{1}{\eta}} = K^\alpha L^{1-\alpha}\left(\sum_j (Z_j l_j)^\eta\right)^{\frac{1}{\eta}}$$

From (3),

$$\frac{x_j}{x_i} = \frac{Z_j k^\alpha l_j L}{Z_i k^\alpha l_i L}.$$

Then using the above expression and (2),

$$\frac{l_j}{l_i} = \frac{\frac{x_j}{x_i}}{\frac{Z_j}{Z_i}} = \frac{\left(\frac{p_i}{p_j}\right)^{\frac{1}{1-\eta}}}{\frac{Z_j}{Z_i}} = \left(\frac{Z_j}{Z_i}\right)^{\frac{\eta}{1-\eta}}.$$

Solving these linear equations for l_j together with $\sum_j l_j = 1$ we have

$$l_j = \frac{Z_j^{\frac{\eta}{1-\eta}}}{\sum_k Z_k^{\frac{\eta}{1-\eta}}}.$$

Substituting this into the expression for output,

$$Y = K^{\alpha}L^{1-\alpha}\left(\sum_j (Z_j l_j)^{\eta}\right)^{\frac{1}{\eta}} = K^{\alpha}L^{1-\alpha}\frac{1}{\sum_k Z_k^{\frac{\eta}{1-\eta}}}\left(\sum_j Z_j^{\frac{\eta}{1-\eta}}\right)^{\frac{1}{\eta}} =$$

$$= K^{\alpha}L^{1-\alpha}\left(\sum_j Z_j^{\frac{\eta}{1-\eta}}\right)^{\frac{1-\eta}{\eta}} = ZK^{\alpha}L^{1-\alpha}.$$

It is left to derive the expression for Z_t. Our assumption of an independent Poisson process with identical arrival rate λ in each of the many industries implies, by law of large numbers, that

$$f(N,t) = \frac{(\lambda t)^N e^{-\lambda t}}{N!}.$$

Then (4) becomes

$$Z_t = \left(\sum_{N=0}^{\infty} f(N,t)\, z^{N\frac{\eta}{1-\eta}}\right)^{\frac{1-\eta}{\eta}} = \left(e^{-\lambda t}\sum_{N=0}^{\infty}\frac{\left(\lambda t z^{\frac{\eta}{1-\eta}}\right)^N}{N!}\right)^{\frac{1-\eta}{\eta}} =$$

$$= \left(\exp\left(\lambda t z^{\frac{\eta}{1-\eta}} - \lambda t\right)\right)^{\frac{1-\eta}{\eta}} = \exp\left(\lambda\left(z^{\frac{\eta}{\eta-1}} - 1\right)\frac{1-\eta}{\eta}t\right).$$

It is left to prove (7). From (3),

$$\frac{x_j}{l_j} = Z_j k^{\alpha}, \text{ and} \sum_j \frac{x_j}{Z_j} = K^{\alpha}L^{1-\alpha}$$

From the zero profit condition in the final goods sector,

$$ZK^{\alpha}L^{1-\alpha} = Y = \sum_j p_j x_j = mc\sum_j \frac{x_j}{Z_j} = mcK^{\alpha}L^{1-\alpha},$$

which implies that in equilibrium $mc = Z$.

From (27), the market clearing conditions for labor and capital read

$$RK = \alpha X,\, WL = (1-\alpha)X\,,$$

where

$$X = \sum_j x_j.$$

This implies that in equilibrium

$$c = \left(\frac{R}{\alpha}\right)^{\alpha}\left(\frac{W}{1-\alpha}\right)^{1-\alpha} = \frac{X}{K^{\alpha}L^{1-\alpha}},$$

so that

$$X = \frac{Y}{m}.$$

Therefore,

$$WL = \frac{1-\alpha}{m}Y,\ RK = \frac{\alpha}{m}Y$$

and

$$\Pi = Y - WL - RK = \frac{m-1}{m}Y.$$

■

Proof of Corollary to Proposition 1

Omitting the time subscript for more compact notation, we can write leader's current profit as

$$\pi_j = \left(p_j - \frac{c}{Z_j}\right)x_j = (m-1)\,c\frac{x_j}{Z_j}$$

Using demand function (2) and using the fact that $cm = Z$,

$$x_j = Y \cdot (cm)^{-\frac{1}{1-\eta}}\,Z_j^{\frac{1}{1-\eta}} = Y \cdot \left(\frac{Z_j}{Z}\right)^{\frac{1}{1-\eta}},$$

$$\pi_j = (m-1)\,c\frac{x_j}{Z_j} = \frac{m-1}{m}Y \cdot (cm)^{-\frac{\eta}{1-\eta}}\,Z_j^{\frac{\eta}{1-\eta}} = \frac{m-1}{m}Y \cdot \left(\frac{Z_j}{Z}\right)^{\frac{\eta}{1-\eta}}.$$

■

Proof of Proposition 2: Similarly to (11), aggregate profits of intermediate goods producers can be written as

$$\Pi_t = \sum_{N=0}^{\infty} f(N,t)\,\pi_t(N) \tag{28}$$

Also note that combining (8) and (9), we can write for any N

$$w_t(N) = z^{\frac{\eta}{1-\eta}} w_t(N-1)$$

which implies that

$$w_t(N) = w_t(0) \cdot z^{N\frac{\eta}{1-\eta}},$$

$$\pi_t(N) = \pi_t(0) \cdot z^{N\frac{\eta}{1-\eta}}$$

Accordingly, (28) and (11)can be written as

$$\Pi_t = \pi_t(0) \cdot \sum_{N=0}^{\infty} f(N,t)\,z^{N\frac{\eta}{1-\eta}} = \pi_t(0) \cdot e^{\gamma t},$$

$$V_t = w_t(0) \cdot e^{\gamma t}.$$

Differentiating the last expression and using (10), we get

$$\dot{V} = \gamma V_t + \frac{dw_t(0)}{dt} \cdot e^{\gamma t} = \gamma V_t + (\lambda + r_t)\,w_t(0) \cdot e^{\gamma t} - \pi_t(0) \cdot e^{\gamma t} =$$

$$= \gamma V_t + (\lambda + r_t)\,V_t - \Pi_t.$$

■

Proof of Proposition 3: (to be added)

Proof of Proposition 4:

Case (a) Let $w_t(n)$ be the present value of receiving the flow payoff $\pi_t(n)$ for as long the current pool stays valuable, i.e. until the first arrival of a technology that replaces the pool.

$$w_t(n) = \int_t^\infty d\tau \lambda e^{-\lambda(\tau-t)} \left(\int_t^\tau ds e^{-\bar{r}(s,t)} \cdot \pi_s(n) + \theta e^{-\bar{r}(\tau,t)} w_\tau(n) \right) \tag{29}$$

Differentiating,

$$\frac{dw_t(n)}{dt} = -\lambda\theta w_t(n) + \lambda w_t(n) + \int_t^\infty d\tau \lambda e^{-\lambda(\tau-t)} \frac{d}{dt} \left(\int_t^\tau ds e^{-\bar{r}(s,t)} \cdot \pi_s(n) + \theta e^{-\bar{r}(\tau,t)} w_\tau \right)$$

$$= -\lambda\theta w_t + \lambda w_t - \int_t^\infty d\tau \lambda e^{-\lambda(\tau-t)} \pi_t(n) + r_t \int_t^\infty d\tau \lambda e^{-\lambda(\tau-t)} \left(\int_t^\tau ds e^{-\bar{r}(s,t)} \cdot \pi_s(n) + \theta e^{-\bar{r}(\tau,t)} w_\tau \right) =$$

$$= -\lambda\theta w_t(n) + (\lambda + r_t) w_t(n) - \pi_t(n)$$

Let $n \leq N_j$ be a technology in a patent pool that includes N_j and let $U_t(n, N_{jt})$ denote the expected present value of payoffs to the owner of patent n in a pool that includes patent N_{jt}. Then, computing this expected value from (19) and (29), we get

$$U_t(N_f, N_{jt}) = w_t(N_f)$$
$$U_t(n, N_{jt}) = w_t(n) - w_t(n-1), \; n = N_f + 1, ..., N_{jt}.$$

Then, the current value of the patent pool is

$$\sum_{n \leq N_{jt}} U_t(n, N_{jt}) = w_t(N_{jt}).$$

The aggregate value of all the patent pools in the economy is

$$V_t = \sum_{N=0}^\infty f(N,t) w_t(N) = w_t(0) \cdot e^{\gamma t}$$

Corollary to Proposition 2

$$\dot{V}_t = \gamma V_t + (r_t + \lambda(1-\theta)) V_t - \Pi_t \tag{30}$$

This expression is the same as (12), except that now the flow rate of depreciation of patent value is $\Delta = \lambda(1-\theta)$, which is the flow probability of arrival of a new fundamental idea.

Case (b)

The owner of fundamental patent gets the entire profit from the pool. Let $\bar{w}_t(N)$ be the expected present value of profits from a patent pool in an industry that currently has N innovations:

$$\bar{w}_t(N) = \int_t^\infty d\tau \lambda e^{-\lambda(\tau-t)} \left(\int_t^\tau ds e^{-\bar{r}(s,t)} \cdot \pi_s(N) + \theta e^{-\bar{r}(\tau,t)} \bar{w}_\tau(N+1) \right) =$$

$$= \int_t^\infty d\tau \lambda e^{-\lambda(\tau-t)} \left(\int_t^\tau ds e^{-\bar{r}(s,t)} \cdot \pi_s(N) + \theta z^{\frac{\eta}{1-\eta}} e^{-\bar{r}(\tau,t)} \bar{w}_\tau(N) \right), \tag{31}$$

Differentiating this expression yields

$$\frac{d\bar{w}_t(N)}{dt} = \left(-\lambda\theta z^{\frac{\eta}{1-\eta}} + \lambda + r\right)\bar{w}_t(N) - \pi_t(N) =$$
$$= (\lambda(1-\theta) - \theta\gamma + r)\,\bar{w}_t(N) - \pi_t(N)$$

The aggregate value of patents equals the total value of all existing patent pools

$$V_t = \sum_{N=0}^{\infty} f(N,t)\,\bar{w}_t(N) = \bar{w}_t(0)\cdot e^{\gamma t}$$

According to Proposition 2, the differential form of this expression is

$$\dot{V}_t = \gamma V_t + (r_t + \lambda(1-\theta) - \theta\gamma)\,V_t - \Pi_t \tag{32}$$

■

Appendix 3: Aggregation of economies with time-varying markups

As Proposition 1 shows, the distribution of industry TFP levels at time t is summarized by the aggregate quality index Z_t. Therefore, without loss of generality, we can set the time of the shock to $t = 0$, $N_{j0} = 0$ for every j, and capture any initial distribution of industry TFP levels before the shock by appropriately choosing the value of Z_0.

For convenience, let the old step size be z and the new step size be $z_1 > z$. The old markup equals z. The new markup equals $m = 1/\max\{\eta, 1/z_1\} > z$. Let $\beta = z_0/m < 1$.

Step 1 *Aggregate production function*

$$Y = Z(t)\,\varphi(t)\,K^{\alpha}L^{1-\alpha}$$

Here $Z(t)$ is defined in (4). We will now derive $\varphi(t)$ in closed form.

Assume that at time $t = 0$ every industry has a leader and a competitor. Then an industry with $N = 0$ innovations charges a price $\bar{p} = cz$, produces output

$$\bar{x} = Y(cz)^{-\frac{1}{1-\eta}}.$$

and employs a fraction of the labor force equal to $\bar{l}$, so that

$$\bar{x} = k^{\alpha}\bar{l}L = \bar{l}K^{\alpha}L^{1-\alpha}. \tag{33}$$

Let $p(N)$, $x(N)$ and $l(N)$ be the price, output and fraction of labor force employed by an industry with N innovations. Then, for any $N > 0$

$$p(N) = m\frac{c}{z_1^N} = \frac{1}{\beta}\frac{\bar{p}}{z_1^N}.$$

From (2),

$$\left(\frac{x(N)}{\bar{x}}\right)^{\eta} = \left(\frac{\bar{p}}{p(N)}\right)^{\frac{\eta}{1-\eta}} = \beta^{\frac{\eta}{1-\eta}}\cdot z_1^{N\frac{\eta}{1-\eta}},\ N > 0 \tag{34}$$

and

$$\frac{p(N)\,x(N)}{\bar{p}\bar{x}} = \left(\frac{\bar{p}}{p(N)}\right)^{\frac{\eta}{1-\eta}} = \beta^{\frac{\eta}{1-\eta}} \cdot z_1^{N\frac{\eta}{1-\eta}},\ N > 0 \tag{35}$$

Since every firm optimally chooses the same capital-labor ratio, it follows from the expression for the production function (3) that

$$\frac{l(N)}{\bar{l}} = \frac{x(N)}{\bar{x}} \frac{1}{z_1^N} = \beta^{\frac{1}{1-\eta}} \cdot z_1^{N\frac{\eta}{1-\eta}} \tag{36}$$

From the labor market clearing condition,

$$1 = \sum_{N=0}^{\infty} f(N,t)\, l(N) = \bar{l} \cdot \left(f(0,t) \left(1 - \beta^{\frac{1}{1-\eta}}\right) + \beta^{\frac{1}{1-\eta}} \sum_{N=0}^{\infty} f(N,t)\, z_1^{N\frac{\eta}{1-\eta}} \right) = \bar{l} \cdot \Sigma_L,$$

where

$$\Sigma_L = e^{-\lambda t} \left(1 - \beta^{\frac{1}{1-\eta}}\right) + \beta^{\frac{1}{1-\eta}} e^{\gamma t}$$

Using (34) and (33), aggregate output equals

$$Y = \left(\sum_{N=0}^{\infty} f(N,t)\, x^{\eta}(N)\right)^{\frac{1}{\eta}} = \bar{x} \left(f(0,t) \left(1 - \beta^{\frac{\eta}{1-\eta}}\right) + \beta^{\frac{\eta}{1-\eta}} \sum_{N=0}^{\infty} f(N,t)\, z_1^{N\frac{\eta}{1-\eta}} \right)^{\frac{1}{\eta}} = \frac{\Sigma_Y^{\frac{1}{\eta}}}{\Sigma_L} \cdot K^{\alpha} L^{1-\alpha},$$

where

$$\Sigma_Y = e^{-\lambda t} \left(1 - \beta^{\frac{\eta}{1-\eta}}\right) + \beta^{\frac{\eta}{1-\eta}} e^{\gamma t}$$

From the expressions for $\Sigma_L$and Σ_Y,

$$\frac{\Sigma_Y^{\frac{1}{\eta}}}{\Sigma_L} = e^{\gamma t\left(\frac{1-\eta}{\eta}\right)} \cdot \frac{\left(\beta^{\frac{\eta}{1-\eta}} + \left(1 - \beta^{\frac{\eta}{1-\eta}}\right) e^{-(\lambda+\gamma)t}\right)^{\frac{1}{\eta}}}{\beta^{\frac{1}{1-\eta}} + \left(1 - \beta^{\frac{1}{1-\eta}}\right) e^{-(\lambda+\gamma)t}} = Z_t \cdot \varphi(t).$$

Note that

$$\lim_{t\to 0} \varphi(t) = \lim_{t\to\infty} \varphi(t) = 1$$

Finally, we have for the aggregate production function

$$Y = Z(t)\, \varphi(t)\, K^{\alpha} L^{1-\alpha}.$$

■

Step 2 *Goods market equilibrium*
Using the zero profit condition in the final goods sector and (35),

$$Y = \sum_{N=0}^{\infty} f(N,t)\, p(N)\, x(N) = \bar{p}\bar{x} \cdot \Sigma_Y = \beta m c \bar{x} \cdot \Sigma_Y$$

The factor prices (on which c depends) adjust so that

$$c\bar{x} = \frac{Y}{\beta m \cdot \Sigma_Y}.$$

Step 3 *Aggregate monopoly profits, aggregate markup and division of national income.* Using the expression (36), we can write the aggregate monopoly profit as

$$\Pi_t = \sum_{N=0}^{\infty} f(N,t)\left(p(N)\,x(N) - \frac{c}{z_1^N}x(N)\right) = Y - \frac{c\bar{x}}{\bar{l}}\sum_{N=0}^{\infty} f(N,t)\,l(N) =$$

$$= Y - c\bar{x}\cdot\Sigma_L = Y\left(1 - \frac{\Sigma_L}{\beta m\cdot\Sigma_Y}\right)$$

Note that

$$\lim_{t\to 0}\left(\frac{\Sigma_L}{\Sigma_Y}\right) = 1 \text{ and } \lim_{t\to\infty}\left(\frac{\Sigma_L}{\Sigma_Y}\right) = \beta.$$

Therefore, the share of monopoly profits in output grows over time from $1 - 1/(\beta m)$ to $1 - 1/m$. Define the aggregate markup

$$m(t) = \frac{\beta m \Sigma_Y}{\Sigma_L}$$

Then,

$$RK = \alpha(Y - \Pi) = \frac{\alpha}{m(t)}Y,$$

$$WL = (1-\alpha)(Y - \Pi) = \frac{1-\alpha}{m(t)}Y,$$

Step 4 *Aggregate value of patents*

Let $\pi_t(N)$ be the time t profit in an industry with a fundamental idea and N innovations. Then

$$\pi_t(0) = (\beta m - 1)\,c\bar{x}$$

and for all $N > 0$

$$\pi_t(N) = (m-1)\,c\frac{x(N)}{z_1^N} = (m-1)\,c\bar{x}\beta^{\frac{1}{1-\eta}}\cdot z_1^{N\frac{\eta}{1-\eta}} = \frac{m-1}{\beta m - 1}\beta^{\frac{1}{1-\eta}}\cdot z_1^{N\frac{\eta}{1-\eta}}\cdot\pi_t(0).$$

For more compact notation, let

$$\omega = \frac{m-1}{\beta m - 1}\beta^{\frac{1}{1-\eta}}$$

Since the value of any patent obeys (9), we can write

$$w_t(N) = \omega\cdot z_1^{N\frac{\eta}{1-\eta}}\cdot w_t(0).$$

The aggregate value of patents equals

$$V_t = \sum_{N=0}^{\infty} f(N,t)\,w_t(N) = w_t(0)(1-\omega)\,e^{-\lambda t} + w_t(0)\,\omega\sum_{N=0}^{\infty} f(N,t)\,z_1^{N\frac{\eta}{1-\eta}} =$$

$$= w_t(0)\left((1-\omega)\,e^{-\lambda t} + \omega e^{\gamma t}\right)$$

where $w_t(0)$ obeys the differential equation (10)

$$\frac{dw_t(0)}{dt} = (\lambda + r)\,w_t(0) - \pi_t(0) =$$

$$= (\lambda + r) w_t(0) - \frac{(\beta m - 1)}{\beta m} \frac{Y}{\Sigma_Y}. \quad (37)$$

Differentiating the expression for V_t and using (37) yields the differential equation for V:

$$\dot{V}_t = (\lambda + r) V + \frac{-\lambda (1-\omega) e^{-\lambda t} + \gamma \omega e^{\gamma t}}{(1-\omega) e^{-\lambda t} + \omega e^{\gamma t}} V - \pi_t(0) \left((1-\omega) e^{-\lambda t} + \omega e^{\gamma t} \right) =$$

$$= (\lambda + r) V + \frac{-\lambda (1-\omega) e^{-\lambda t} + \gamma \omega e^{\gamma t}}{(1-\omega) e^{-\lambda t} + \omega e^{\gamma t}} V - \frac{(\beta m - 1)}{\beta m} \frac{Y}{\Sigma_Y} \left((1-\omega) e^{-\lambda t} + \omega e^{\gamma t} \right).$$

Note that

$$\lim_{t \to \infty} \frac{(1-\omega) e^{-\lambda t} + \omega e^{\gamma t}}{\Sigma_Y} = \frac{m-1}{\beta m - 1} \beta,$$

so that the last term tends to $\frac{m-1}{m} Y$ as $t \to \infty$.

Step 6: *The boundary value problem.*

Overall we have a system of two first-order non-autonomous differential equations for K and V:

$$\dot{K}_t + \delta K_t = \sigma Y_t - (1-\sigma) (\dot{V}_t + \lambda V)$$

and

$$\dot{V}_t + \lambda V = (2\lambda + r_t + \psi(t)) V_t - \kappa(t) \cdot Y_t,$$

where

$$Y_t = Z(t) \varphi(t) K^\alpha L^{1-\alpha},$$

$$\psi(t) = \frac{-\lambda (1-\omega) e^{-\lambda t} + \gamma \omega e^{\gamma t}}{(1-\omega) e^{-\lambda t} + \omega e^{\gamma t}}, \ \kappa(t) = \frac{(\beta m - 1)}{\beta m \cdot \Sigma_Y} \left((1-\omega) e^{-\lambda t} + \omega e^{\gamma t} \right),$$

$$r_t = \frac{\alpha}{m(t)} \frac{Y}{K} - \delta, \ m(t) = \frac{\beta m \Sigma_Y}{\Sigma_L}$$

Using detrended variables

$$k_t = \frac{K_t}{Z_t^{\frac{1}{1-\alpha}} L_t}, \ v_t = \frac{V_t}{Z_t^{\frac{1}{1-\alpha}} L_t}, \ y_t = k_t^\alpha$$

so that $\frac{Y_t}{K_t} = \varphi(t) k^\alpha$, we have

$$\frac{\dot{K}_t}{K_t} = \frac{\dot{k}}{k} + n + g = -\delta + \sigma \varphi(t) k^{\alpha - 1} - (1-\sigma) \left[(2\lambda + r_t + \psi(t)) \frac{v}{k} - \kappa(t) \varphi(t) k^{\alpha-1} \right]$$

$$\frac{\dot{V}_t}{V_t} = \frac{\dot{v}}{v} + n + g = \lambda + r_t + \psi(t) - \kappa(t) \varphi(t) \frac{k^\alpha}{v}$$

We have to specify two boundary conditions:

$$\begin{aligned} k(0) &= k_*, \\ v(\infty) &= v'_*, \end{aligned}$$

where k_* corresponds to the old steady state with step size z and v'_* corresponds to the new steady state with step size z_1.

Fundamental ideas and rising markups

Assume that

$$\frac{1}{z^2} < \eta < \frac{1}{z}. \tag{38}$$

Suppose that at time $t = 0$ every industry has a leader and a competitor, and therefore markup equals z in every industry. Let the probability that an idea survives through the next innovation permanently rise from 0 to $\theta > 0$. This shock induces a certain distribution of patent pools by size. The pool size will determine the markup according to the markup rule (26), with larger pools charging higher markups. Assumption (38) simplifies the analysis, because markups have a simple two-point distribution on $\left\{z, \frac{1}{\eta}\right\}$. In particular, under this assumption all pools of size 1 will charge a markup equal to z,and all pools of size greater than 1 will charge the markup equal to $\frac{1}{\eta} > z$. Let $f(m, N, t)$ be the joint density for markups and the number of innovations, i.e. the fraction of industries that at time t have N innovations and charge markup $m = \left\{z, \frac{1}{\eta}\right\}$.

In an industry with $N = 0$ innovations, all pools have to be of length 1, and therefore all of these industries have $m = z$:

$$f(z, 0, t) = f(0, t)$$

Out of all industries with $N > 0$ innovations, fraction $1 - \theta$ have patent pools of length 1, because the probability that the N-th innovation was a (new) fundamental idea equals $1 - \theta$. The rest have pools that are longer than 1. Therefore, for all $N > 0$

$$\begin{aligned} f(z, N, t) &= (1-\theta) f(N, t), \\ f\left(\frac{1}{\eta}, N, t\right) &= \theta f(N, t). \end{aligned}$$

We can now aggregate the economy.

Step 1 *Aggregate production function*

$$Y = Z(t)\varphi(t) K^{\alpha} L^{1-\alpha}$$

Here $Z(t)$ is defined in (4). We will now derive $\varphi(t)$ in closed form.

Let $p(N, m)$, $x(N, m)$ and $l(N, m)$ be the price, output and fraction of labor force employed by an industry with N innovations and markup m. For convenience, let

$$\begin{aligned} p(0, z) &= cz = \bar{p}, \\ l(0, z) &= \bar{l}, \\ x(0, z) &= \bar{l} K^{\alpha} L^{1-\alpha} = \bar{x} \end{aligned}$$

Since the marginal cost in an industry with N innovations equals $\frac{c}{z^N}$,

$$p(N, z) = z\frac{c}{z^N},\; p\left(N, \frac{1}{\eta}\right) = \frac{1}{\eta}\frac{c}{z^N}$$

From (2),

$$\left(\frac{x(N, m)}{\bar{x}}\right)^{\eta} = \left(\frac{\bar{p}}{p(N, m)}\right)^{\frac{\eta}{1-\eta}} = \begin{cases} z^{N\frac{\eta}{1-\eta}}, & m = z \\ (z\eta)^{\frac{\eta}{1-\eta}} \cdot z^{N\frac{\eta}{1-\eta}}, & m = \frac{1}{\eta} \end{cases}$$

and

$$\frac{p(N,m)\,x(N,m)}{\bar{p}\bar{x}} = \left(\frac{\bar{p}}{p(N,m)}\right)^{\frac{\eta}{1-\eta}} = \begin{cases} z^{N\frac{\eta}{1-\eta}}, & m = z \\ (z\eta)^{\frac{\eta}{1-\eta}} \cdot z^{N\frac{\eta}{1-\eta}}, & m = \frac{1}{\eta} \end{cases}$$

Since every firm optimally chooses the same capital-labor ratio, it follows from the expression for the production function (3) that

$$\frac{l(N,m)}{\bar{l}} = \frac{x(N,m)}{\bar{x}}\frac{1}{z^N} = \begin{cases} z^{N\frac{\eta}{1-\eta}}, & m = z \\ (z\eta)^{\frac{1}{1-\eta}} \cdot z^{N\frac{\eta}{1-\eta}}, & m = \frac{1}{\eta} \end{cases}$$

From the labor market clearing condition,

$$1 = \sum_m \sum_{N=0}^{\infty} f(N,m,t)\,l(N,m) = \bar{l}\cdot\left(f(0,t) + \sum_{N=1}^{\infty} f(N,t)\,z^{N\frac{\eta}{1-\eta}}\left(1-\theta+\theta(z\eta)^{\frac{1}{1-\eta}}\right)\right) = \bar{l}\cdot\Sigma_L,$$

where

$$\begin{aligned} \Sigma_L &= \rho_L \cdot e^{-\lambda t} + (1-\rho_L)\,e^{\gamma t}, \\ \rho_L &= \theta\left(1-(z\eta)^{\frac{1}{1-\eta}}\right) \end{aligned}$$

Aggregate output equals

$$Y = \left(\sum_m \sum_{N=0}^{\infty} f(N,m,t)\,x^{\eta}(N,m)\right)^{\frac{1}{\eta}} = \bar{x}\left(f(0,t) + \sum_{N=1}^{\infty} f(N,t)\,z^{N\frac{\eta}{1-\eta}}\left(1-\theta+\theta(z\eta)^{\frac{\eta}{1-\eta}}\right)\right)^{\frac{1}{\eta}} =$$

$$= \bar{x}\Sigma_Y^{\frac{1}{\eta}} = \frac{\Sigma_Y^{\frac{1}{\eta}}}{\Sigma_L} K^{\alpha} L^{1-\alpha}$$

where

$$\begin{aligned} \Sigma_Y &= \rho_Y \cdot e^{-\lambda t} + (1-\rho_Y)\cdot e^{\gamma t}, \\ \rho_Y &= \theta\left(1-(z\eta)^{\frac{\eta}{1-\eta}}\right) < \rho_L. \end{aligned}$$

From the expressions for $\Sigma_L$and Σ_Y,

$$\frac{\Sigma_Y^{\frac{1}{\eta}}}{\Sigma_L} = e^{\gamma t\left(\frac{1-\eta}{\eta}\right)} \cdot \frac{\left(1-\rho_Y+\rho_Y e^{-(\lambda+\gamma)t}\right)^{\frac{1}{\eta}}}{1-\rho_L+\rho_L e^{-(\lambda+\gamma)t}} = Z_t \cdot \varphi(t)\,.$$

Note that

$$\begin{aligned} \lim_{t\to 0} \varphi(t) &= 1 \\ \lim_{t\to\infty} \varphi(t) &= \frac{(1-\rho_Y)^{\frac{1}{\eta}}}{1-\rho_L} \end{aligned}$$

Finally, we have for the aggregate production function

$$Y = Z(t)\,\varphi(t)\,K^{\alpha}L^{1-\alpha}.$$

Step 2 *Goods market equilibrium*
Using the zero profit condition in the final goods sector,

$$Y = \sum_m \sum_{N=0}^{\infty} f(N, m, t)\, p(N, m)\, x(N, m) = \bar{p}\bar{x} \cdot \Sigma_Y = zc\bar{x} \cdot \Sigma_Y$$

The factor prices (on which c depends) adjust so that

$$c\bar{x} = \frac{Y}{z \cdot \Sigma_Y}.$$

Step 3 *Aggregate monopoly profits, aggregate markup and division of national income.*
The aggregate monopoly profit equals

$$\Pi_t = \sum_m \sum_{N=0}^{\infty} f(N, m, t) \left(p(N, m)\, x(N, m) - \frac{c}{z^N} x(N, m) \right) = Y - \frac{c\bar{x}}{\bar{l}} \sum_m \sum_{N=0}^{\infty} f(N, m, t)\, l(N, m) =$$

$$= Y - c\bar{x} \cdot \Sigma_L = Y \left(1 - \frac{\Sigma_L}{z \cdot \Sigma_Y} \right)$$

Note that

$$\lim_{t \to 0} \left(\frac{\Sigma_L}{\Sigma_Y} \right) = 1 \text{ and } \lim_{t \to \infty} \left(\frac{\Sigma_L}{\Sigma_Y} \right) = \frac{1 - \rho_L}{1 - \rho_Y} < 1.$$

Therefore, the share of monopoly profits in output grows over time. Define the aggregate markup

$$m(t) = \frac{z\Sigma_Y}{\Sigma_L}$$

Then

$$m(0) = z,\ m(\infty) = z\frac{1 - \rho_Y}{1 - \rho_L} > z,$$

$$RK = \alpha (Y - \Pi) = \frac{\alpha}{m(t)} Y,$$

$$WL = (1 - \alpha)(Y - \Pi) = \frac{1 - \alpha}{m(t)} Y,$$

Step 4 *Aggregate value of patents*
Let $\pi_t(N, m)$ be the time t profit in an industry with N innovations and markup m. Then,

$$\pi_t \left(N, \frac{1}{\eta} \right) = \left(\frac{1}{\eta} - 1 \right) c \frac{x\left(N, \frac{1}{\eta}\right)}{z^N} = \left(\frac{1}{\eta} - 1 \right) (z\eta)^{\frac{1}{1-\eta}} \cdot z^{N \frac{\eta}{1-\eta}} \cdot c\bar{x},$$

$$\pi_t(N, z) = (z - 1)\, c \frac{x(N, m)}{z^N} = (z - 1) \cdot z^{N \frac{\eta}{1-\eta}} \cdot c\bar{x} = \omega \cdot \pi_t \left(N, \frac{1}{\eta} \right),$$

where

$$\omega = \frac{z - 1}{\left(\frac{1}{\eta} - 1 \right) (z\eta)^{\frac{1}{1-\eta}}} < 1 \text{ because } z < \frac{1}{\eta}$$

Accordingly, let $w_t(N, m)$ be the value of a fundamental idea in an industry with with N innovations and markup m. Note that $w_t\left(N, \frac{1}{\eta}\right)$ satisfies (31), because once the markup reaches the value $\frac{1}{\eta}$, it never changes again for as long as the pool is not replaced by another fundamental idea:

$$w_t\left(N,\frac{1}{\eta}\right)=\int_t^{\infty}d\tau\lambda e^{-\lambda(\tau-t)}\left(\int_t^{\tau}dse^{-\bar{r}(s,t)}\cdot\pi_s\left(N,\frac{1}{\eta}\right)+\theta e^{-\bar{r}(\tau,t)}w_\tau\left(N+1,\frac{1}{\eta}\right)\right)$$

The pool of length 1 has a value of $w_t(N,z)$. Upon the first Poisson arrival, with probability θ, the pool's size grows by 1, so the markup rises to $\frac{1}{\eta}$ and the value of the pool becomes $w\left(N+1,\frac{1}{\eta}\right)$. Accordingly, we can write

$$w_t(N,z)=\int_t^{\infty}d\tau\lambda e^{-\lambda(\tau-t)}\left(\int_t^{\tau}dse^{-\bar{r}(s,t)}\cdot\omega\pi_s\left(N,\frac{1}{\eta}\right)+\theta e^{-\bar{r}(\tau,t)}w_\tau\left(N+1,\frac{1}{\eta}\right)\right)$$

Since

$$\pi_t\left(N+1,\frac{1}{\eta}\right)=z^{\frac{\eta}{1-\eta}}\pi_t\left(N,\frac{1}{\eta}\right),$$

it follows from the above value equations that

$$w_t\left(N+1,\frac{1}{\eta}\right)=z^{\frac{\eta}{1-\eta}}w_t\left(N,\frac{1}{\eta}\right),$$

$$w_t(N+1,z)=z^{\frac{\eta}{1-\eta}}w_t(N,z).$$

The aggregate value of patents

$$V_t=\sum_m\sum_{N=0}^{\infty}f(N,m,t)\cdot w_t(N,m)=$$

$$=f(0,t)\,w_t(0,z)+\sum_{N=1}^{\infty}f(N,t)\,z^{N\frac{\eta}{1-\eta}}\left((1-\theta)\,w_t(N,z)+\theta w_t\left(N,\frac{1}{\eta}\right)\right)=.$$

$$=f(0,t)\,w_t(0,z)-f(0,t)(1-\theta)\,w_t(0,z)-\theta f(0,t)\,w_t\left(0,\frac{1}{\eta}\right)+$$

$$+\sum_{N=0}^{\infty}f(N,t)\,z^{N\frac{\eta}{1-\eta}}\left((1-\theta)\,w_t(N,z)+\theta w_t\left(N,\frac{1}{\eta}\right)\right)=$$

$$=-\theta\left(w_t\left(0,\frac{1}{\eta}\right)-w_t(0,z)\right)+\sum_{N=0}^{\infty}f(N,t)\,z^{N\frac{\eta}{1-\eta}}w_t\left(N,\frac{1}{\eta}\right)-$$

$$-(1-\theta)\sum_{N=0}^{\infty}f(N,t)\,z^{N\frac{\eta}{1-\eta}}\left(w_t\left(N,\frac{1}{\eta}\right)-w_t(N,z)\right).$$

Letting

$$u_t(N)=w_t\left(N,\frac{1}{\eta}\right)-w_t(N,z)=\int_t^{\infty}d\tau\lambda e^{-\lambda(\tau-t)}\int_t^{\tau}dse^{-\bar{r}(s,t)}\cdot(1-\omega)\,\pi_s\left(N,\frac{1}{\eta}\right),$$

we can rewrite V_t as

$$V_t=-\theta u_t(0)\,e^{-\lambda t}+w_t\left(0,\frac{1}{\eta}\right)e^{\gamma t}-(1-\theta)\,u_t(0)\,e^{\gamma t}$$

Let

$$\bar{V}_t = w_t\left(0, \frac{1}{\eta}\right)e^{\gamma t},$$

which is what the aggregate value of patents would have been if the markup were constant over time and equal $\frac{1}{\eta}$, and let

$$U_t = u_t\left(0\right)e^{\gamma t},$$

which measures the aggregate difference in value between a pool with a low markup and a pool with a high markup. Then,

$$V_t = \bar{V}_t - U_t\left(1 - \theta + \theta e^{-(\lambda+\gamma)t}\right)$$

Differentiating $\bar{V}_t$ and U_t, we get two differential equations that are similar to (32) and (12) respectively:

$$\frac{d\bar{V}_t}{dt} = \gamma\bar{V}_t + \left(r_t + \lambda - \theta\left(\lambda+\gamma\right)\right)\bar{V}_t - \pi_t\left(0, \frac{1}{\eta}\right)e^{\gamma t}$$

$$\frac{dU_t}{dt} = \gamma U_t + \left(r_t + \lambda\right)U_t - \left(1-\omega\right)\pi_t\left(0, \frac{1}{\eta}\right)e^{\gamma t},$$

where

$$\pi_t\left(0, \frac{1}{\eta}\right)e^{\gamma t} = \frac{z-1}{\omega}c\bar{x}e^{\gamma t} = \frac{z-1}{\omega z}\frac{e^{\gamma t}}{\Sigma_Y}Y = \frac{z-1}{\omega z}\frac{1}{1-\rho_Y+\rho_Y\cdot e^{-(\lambda+\gamma)t}}Y.$$

Step 5: *The boundary value problem.*

We have a system of three first-order non-autonomous differential equations for $K, \bar{V}$ and U:

$$\dot{K}_t + \delta K_t = \sigma Y_t - \left(1-\sigma\right)\left(\dot{V}_t + \Delta V_t\right)$$

$$\frac{d\bar{V}_t}{dt} = \gamma\bar{V}_t + \left(r_t + \lambda - \theta\left(\lambda+\gamma\right)\right)\bar{V}_t - \kappa\left(t\right)Y$$

$$\frac{dU_t}{dt} = \gamma U_t + \left(r_t + \lambda\right)U_t - \left(1-\omega\right)\kappa\left(t\right)Y$$

where

$$Y_t = Z\left(t\right)\varphi\left(t\right)K^{\alpha}L^{1-\alpha},$$

$$V_t = \bar{V}_t - U_t\left(1-\theta+\theta e^{-(\lambda+\gamma)t}\right),$$

$$\dot{V}_t = \frac{d\bar{V}_t}{dt} - \frac{dU_t}{dt}\left(1-\theta+\theta e^{-(\lambda+\gamma)t}\right) + \theta\left(\lambda+\gamma\right)e^{-(\lambda+\gamma)t}U_t,$$

$$\dot{V}_t + \Delta V_t = \left(\frac{d\bar{V}_t}{dt} + \Delta\bar{V}_t\right) - \left(1-\theta+\theta e^{-(\lambda+\gamma)t}\right)\left(\frac{dU_t}{dt} + \Delta U_t\right) + \theta\left(\lambda+\gamma\right)e^{-(\lambda+\gamma)t}U_t,$$

$$\Delta = \lambda - \theta\left(\lambda+\gamma\right), \ \omega = \frac{z-1}{\left(\frac{1}{\eta}-1\right)\left(z\eta\right)^{\frac{1}{1-\eta}}},$$

$$\kappa\left(t\right) = \frac{z-1}{\omega z}\frac{1}{\Sigma_Y}, \ \varphi\left(t\right) = \frac{\left(1-\rho_Y+\rho_Y e^{-(\lambda+\gamma)t}\right)^{\frac{1}{\eta}}}{1-\rho_L+\rho_L e^{-(\lambda+\gamma)t}}$$

$$r_t = \frac{\alpha}{m(t)}\frac{Y_t}{K_t} - \delta,\ m(t) = \frac{z\Sigma_Y}{\Sigma_L}$$

Using detrended variables

$$k_t = \frac{K_t}{Z_t^{\frac{1}{1-\alpha}} L_t}, v_t = \frac{V_t}{Z_t^{\frac{1}{1-\alpha}} L_t}, \bar{v}_t = \frac{\bar{V}_t}{Z_t^{\frac{1}{1-\alpha}} L_t}, u_t = \frac{U_t}{Z_t^{\frac{1}{1-\alpha}} L_t}, y_t = k_t^{\alpha},$$

so that $\frac{Y_t}{K_t} = \varphi(t) k^{\alpha}$, we have

$$\frac{\dot{K}_t}{K_t} = \frac{\dot{k}}{k} + n + g = -\delta + \sigma\frac{Y}{K} - (1-\sigma)\left[\frac{\dot{V} + \Delta V}{K}\right]$$

$$\frac{\dot{U}}{U} = \frac{\dot{u}}{u} + n + g = (\gamma + r_t + \lambda) - (1-\omega)\kappa(t)\frac{Y}{U}$$
$$\dot{u} = (\gamma + r_t + \lambda - n - g)u - (1-\omega)\kappa(t)\varphi(t) k^{\alpha}$$

Similarly

$$\frac{d\bar{v}}{dt} = (\gamma + r_t + \Delta - n - g)\bar{v} - \kappa(t)\varphi(t) k^{\alpha}$$

$$\frac{\dot{V}}{K} = \frac{1}{k_t}\left(\frac{d\bar{v}}{dt} + (n+g)\bar{v}_t - (\dot{u}_t + (n+g)u_t)\left(1 - \theta + \theta e^{-(\lambda+\gamma)t}\right) + \theta(\lambda+\gamma)e^{-(\lambda+\gamma)t}u_t\right)$$

$$\frac{V}{K} = \frac{1}{k_t}\left(\bar{v}_t - u_t\left(1 - \theta + \theta e^{-(\lambda+\gamma)t}\right)\right)$$

The steady state values of u_* and $\bar{v}_*$ can be immediately expressed through k_*^{α} and r_*:

$$u_* = \frac{(1-\omega)\kappa(\infty)\varphi(\infty)}{(\gamma + r_* + \lambda - n - g)}k_*^{\alpha},$$
$$\bar{v}_* = \frac{\kappa(\infty)\varphi(\infty)}{(\gamma + r_* + \Delta - n - g)}k_*^{\alpha}.$$

It is left to derive the equation for k_*. Alonmg the balanced growth path, V, $\bar{V}$, U, and K all grow at rate $n+g$. Then

$$\dot{V}_t + \Delta V_t = (n+g+\Delta)\bar{V}_t - \left(1 - \theta + \theta e^{-(\lambda+\gamma)t}\right)(n+g+\Delta)U_t + \theta(\lambda+\gamma)e^{-(\lambda+\gamma)t}U_t =$$

$$= (n+g+\Delta)V_t + \theta(\lambda+\gamma)e^{-(\lambda+\gamma)t}U_t \underset{t\to\infty}{\to} (n+g+\Delta)V_t \underset{t\to\infty}{\to} (n+g+\Delta)\left(\bar{V}_t - (1-\theta)U_t\right).$$

Substituting this expression into the capital accumulation equation, we get, in steady state,

$$(n+d+g)k_* = \sigma\varphi(\infty)k_*^{\alpha} - (1-\sigma)(n+g+\Delta)(\bar{v}_* - (1-\theta)u_*)$$

This is a quadratic equation in r_*, and its positive root corresponds to the steady state.