Numerical Solution of Dynamic Economic Models with Heterogeneous Agents *

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Abstract

This paper presents an algorithm for the computation of sequential competitive equilibria in dynamic models with heterogeneous agents. The algorithm builds on a convergent operator defined over an expanded set of state variables for which a Markovian equilibrium solution is known to exist. We apply this algorithm to a stochastic growth economy and two exchange economies with incomplete financial markets.

1 Introduction

The aim of this paper is to provide an algorithm for the numerical solution of dynamic economic models in which the first welfare theorem may not hold because of the presence of incomplete financial markets, incomplete agents' participation, externalities, taxes and other market frictions. These models are widely used in macroeconomic applications to analyze the effects of various economic policies, the evolution of wealth and income distribution, and the variability of asset prices. There are two main analytical issues that arise in the computation of equilibria for these economies: *(i) Non-existence of Markov equilibria.* Even though the model may have a recursive structure, a Markovian equilibrium may not exist – or no Markov equilibrium may be continuous – over the natural space of state variables. Hence, to compute the set of equilibrium solutions it may be convenient to expand arbitrarily the state space. *(ii) Non-convergence of the algorithm.* Backward

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iteration over a candidate equilibrium function may not converge to a Markovian equilibrium solution. Contractive arguments underlying dynamic programming methods usually break down for non-optimal economies.

The existence of Markov equilibria for non-optimal economies does not seem to have a simple answer. Under regular conditions on the individual decision problem, examples of non-existence of a Markovian solution have been found in one-sector models with taxes and externalities [Santos (2002)], in exchange economies with incomplete financial markets [Krebs (2004) and Kubler and Schmedders (2002)], and in overlapping generations economies [Kubler and Polemarchakis (2004)]. Another strand of the literature has been concerned with the existence of Markov equilibria in these economies. General results on existence of a continuous solution rely on certain monotonicity conditions on the equilibrium dynamics [e.g., see Bizer and Judd (1989), Coleman (1991), and Datta, Mirman and Reffett (2002)]. For the canonical one-sector growth model with taxes and externalities, these conditions follow from fairly mild restrictions on the primitives, but monotone dynamics are much harder to obtain in multi-sector models with heterogeneous agents and incomplete financial markets. Duffie *et al*. (1994) lift the continuity requirement and expand the state space to include additional endogenous variables such as asset prices and individual consumptions. Following Blume (1982), they then demonstrate the existence of a stationary Markovian equilibrium solution for a wide class of discrete-time infinite-horizon models with exogenous short-sale constraints on the trading of assets.

From all these results we may conclude that Markov equilibria may not exist in discrete-time competitive-markets economies under general conditions. Some monotonicity properties on the equilibrium dynamics guarantee the existence of a continuous Markovian solution. In these paper we develop some analytical methods to show the existence of a Markov equilibrium over an expanded state space that includes some endogenous variables. The existence of this Markovian equilibrium solution is closely related to the existence of a sequential competitive equilibrium. The Markovian equilibrium solution may not be continuous if there is a multiplicity of sequential competitive equilibria.

The computation of competitive equilibria for economies with heterogeneous agents has been of considerable interest in macroeconomics and finance [e.g., Castaneda, Diaz-Gimenez and Rios-Rull (2003), Krusell and Smith (1998), Heaton and Lucas (1998), Marcet and Singleton (1992), and Rios-Rull (1999)]. In a recent paper, Kubler and Schmedders (2003) develop an algorithm that implements numerically the methods of Duffie *et al*. (1994), and hence it deals explicitly with the aforementioned problem of non-existence of a Markov equilibrium. This algorithm, however, does not guarantee convergence to an equilibrium function, and it may not be computationally efficient as the state space includes a large number of state variables. Hence, our main objective in this research is to construct a convergent numerical procedure that is computationally efficient by the choice of a minimal set of state variables over which a Markov equilibrium is known to exist.

There are therefore three major considerations that guide the construction of our algorithm: Existence of a Markov equilibrium, convergence to an equilibrium solution, and computational efficiency by conditioning over a small state space that may include some endogenous variables. In sequential competitive equilibria, Bellman's optimality principle does not generally hold. Agents' expectations about future prices must be consistent with individual optimization behavior and market-clearing conditions. This consistency property usually translates into an additional fixed-point problem which is not present in dynamic optimization programs. Hence, backward induction over an initial candidate solution does not ensure convergence to an equilibrium function. To circumvent the convergence problem, we follow a procedure outlined by Kydland and Prescott (1980) and enlarge the state space with the agents' shadow values of investment. Then, we apply a recursive argument over sets of candidate solutions along the lines of Abreu, Pierce and Stacchetti (APS, 1990) who provide some recursive techniques for the characterization of sequential perfect equilibria in dynamic games. The APS approach has been extended to several macroeconomic policy settings [e.g., Atkenson (1991), Phelan and Stacchetti (2001) and Fernandez-Villaverde and Tsyvinski (2004)]. Building on these methods, Judd, Yeltekin and Conklin (2003) present an algorithm for the computation of equilibria in supergames. Also, Marcet and Marimon (1998) develop an Euler-based method for the computation of dynamic contracts that includes the shadow values of the incentive constraints as state variables. All these researchers are concerned with the characterization and computation of equilibria in game-theoretical settings. Miao (2003) provides a recursive characterization of sequential competitive equilibria for the model of Krusell and Smith (1998). Following APS (1990), his state space includes expected continuation utilities over the set of sequential competitive equilibria. This choice of the state space, however, does not seem appropriate for the computation of equilibrium solutions in the present framework. In this paper we redefine the state space to include the equilibrium shadow values of investment. Then, we combine results on existence of sequential competitive equilibria with recursive methods to obtain a Markovian equilibrium formulation. Later, we propose a numerical implementation of the algorithm and evaluate its performance.

2 An Overview of the Computational Method

We first provide an informal description of our computational method, which will be applied to three stochastic economies. Time is discrete, $t = 0, 1, 2, \cdots$. The exogenous state variables follow a Markov process $(z_t)_{t\geq 0}$ over a finite set $\mathbf{Z} \subset \mathbf{R}$ with N elements. Let $\pi(z'|z)$ be the transition probability assumed to be non-degenerate for all $z, z' \in \mathbf{Z}$. Let $z^t = (z_1, z_2, ..., z_t) \in \mathbf{Z}^t$ denote a history of shocks, often referred as a date-event. We write $z^t|z^{t-1}$ to denote that z^t is an immediate successor of z^{t-1}. There is an initial state, $z^0 = z_0$, which is known to all agents in the economy.

The state of the system includes a vector of exogenous shocks z and a vector of predetermined endogenous variables y such as agents' holdings of physical and human capital, and the distribution of financial wealth. Let $\mathbf{S}$ be the space of all values $s = (y, z)$. Define a correspondence $s \longmapsto V^*(s)$, where $V^*(s)$ is the set of all the agents' equilibrium shadow values m from investing in each asset. From the correspondence V^* we can generate recursively the set of sequential competitive equilibria for a given economy. The problem is then to compute V^* as a fixed point of an operator B.

The following steps are involved in the computation of a Markovian equilibrium solution.

(i) Definition of operator $B : W \longmapsto W$. This operator links state variables to future equilibrium states. It embodies all equilibrium conditions such as agents' optimization and market-clearing conditions from any initial value z^0 to all immediate successor nodes z^1. The operator B is analogous to the expectations correspondence defined in Duffie *et al*. (1994), albeit it is defined over a smaller set of endogenous variables. We shall prove that the operator B satisfies a self-generation property similar to APS (1990). That is, if $W \subset B(W)$, then $W \subset V^*$.

(ii) V^* *is a fixed point of* B. This follows from the self-generation property of V^*. In fact, V^* is the largest fixed point of B. Under regular assumptions, the set of sequential competitive equilibria lies in a compact set. Hence, in all our applications the fixed point V^* is non-empty and compact valued.

(iii) Convergence of $B^n(W)$ *to* V^* *as* $n \to \infty$. The set mapping B is monotone and preserves compactness. Hence, if $B(W) \subset W$, then $V^* \subset B^n(W)$ for all n. Moreover, $\lim_{N\to\infty} B^n(W) = V^*$. Again, these results are proved using the self-generation property of B.

(iv) Selection of Markovian Competitive Equilibria. From operator B over correspon-

dence V^* we define a function $f(s,m)$ that describes the law of motion of equilibrium values. Function $f(s,m)$ may not be continuous, but it is derived from an operator with good convergence properties.

3 Stochastic Growth

In this section we consider a one-sector stochastic growth model with heterogeneous households. Production of the aggregate good is subject to an external effect from the average stock of physical capital. Analytically, this production externality is isomorphic to some tax or market distortion on capital. This simple framework bears on several macroeconomic applications that make extensive use of computational methods.

Our analysis is motivated by research in several areas. First, reliable computational methods for dynamic models with taxes and externalities [cf. Bizer and Judd (1989) and Coleman (1991)] are only valid for monotone equilibrium dynamics, but monotone dynamics are hard to obtain in models with heterogeneous agents and multiple goods especially in the presence of externalities, non-linear taxation and government expenditures. Second, stochastic growth models with heterogeneous agents are commonly used to assess the effects of social security and fiscal policies on the distribution of income, wealth, and consumption [e.g., see Castaneda, Diaz-Gimenez and Rios-Rull (2003), Krusell and Smith (1998)]. These models are also useful to evaluate the welfare cost of cyclical fluctuations and the impact of counter cyclical policies [e.g., see Imrohoroglu (1989)].

3.1 The Model Economy

There are I households and a continuum of identical firms. Each household i has preferences for consumption and rents capital and labor to the production sector. Labor is supplied inelastically. The representative firm produces aggregate output using a constant returns to scale technology in its own inputs. Total factor productivity of the firm depends on the realization of the vector of shocks z and on the average quantity of capital $\overline{K}$.

A consumption plan for agent i is a sequence of functions $(c_t^i)_{t\geq 0}$ with $c_t^i : \mathbf{Z}^t \to \mathbf{R}_+$. Preferences over these consumption plans are represented by the expected utility function:

$$E\left[\sum_{t=0}^{\infty} \beta^t u^i \left(c_t^i(z^t), z_t\right)\right], \ \beta \in (0,1) \tag{1}$$

where E is the expectations operator defined by the probabilities of occurrence $\pi\left(z^t|z_0\right)$ for all states z^t.

Assumption 1 *For each $z \in \mathbf{Z}$ the one-period utility $u^i(\cdot, z) : \mathbf{R}_+ \to \mathbf{R}$ is increasing, strictly concave, and twice continuously differentiable. Moreover, u^i is bounded above and unbounded below.*

All consumption plans$\left(c_t^i\left(z^t\right)\right)_{t\geq 0}$ must be financed from labor income, capital returns and profits. At each date-event z^t, for a given rental rate r_t and wage w_t household i rents $k_t^i \geq 0$ units of capital and supplies inelastically $e_t^i(z^t) > 0$ units of labor. The labor endowment $e_t^i : \mathbf{Z}^t \to \mathbf{R}_{++}$ follows a stationary process $e^i(z_t) > 0$ as it only depends on the current realization z_t. For simplicity, we abstract from leisure considerations and we assume that capital depreciates fully at each date-event. Each household is subject to the following sequence of budget constraints

$$k_{t+1}^i + c_t^i = w_t e_t^i + r_t k_t^i + \pi_t, \;\; k_t^i \geq 0, k_0^i \text{ given.} \tag{2}$$

where π_t are profits from the production sector that accrue to the household.

The production sector is made up of a continuum of identical production units that have access to a constant returns to scale technology. Hence, without loss of generality we shall focus on the problem of a representative firm. After observing the current shock z the firm hires K units of capital and L units of labor. The total quantity produced of the single aggregate good is given by a production function $A(z_t, \overline{K}_t)F\left(K_t, L_t\right)$, where $A(z_t, \overline{K}_t)$ is the firm's total factor productivity and $F\left(K_t, L_t\right)$ is the direct contribution of the firm's inputs to the production process. Total factor productivity $A(z_t, \overline{K}_t)$ depends on the realization of the vector of shocks z and on the average stock of capital $\overline{K}$ in the economy that the firm takes as given. At each date-event, one-period profits of the representative firm are then defined as

$$\pi_t = max_{\{K,L\}} A(z_t, \overline{K}_t)F\left(K_t, L_t\right) - r_t K_t - w_t L_t \tag{3}$$

We shall maintain the following assumptions on functions A and F:

Assumption 2 *For each z, function $A(z, \cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ is continuous. There exist positive numbers A^{max} and A^{min} such that $A^{max} \geq A(z, \overline{K}) \geq A^{min}$ for all $(z, \overline{K})$.*

Assumption 3 *$F : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ is increasing, concave, linearly homogeneous, and continuously differentiable. For each positive K and L, $\lim_{K\to 0} D_1 F\left(K, L\right) = \infty$, $\lim_{L\to 0} D_2 F\left(K, L\right) = \infty$, and $\lim_{K\to\infty} D_1 F\left(K, L\right) = 0$.*

The present framework contemplates several deviations from a frictionless world in which a competitive equilibrium may be recast as the solution of an optimal planning program. The vector of shocks z may contain individual uninsurable shocks to labor income and an aggregate shock to production. Households may hold capital to transfer wealth, but they may be unable to smooth out consumption since there is only one single asset and capital holdings must be non-negative. In the production sector, firms must form beliefs about the average amount of physical capital $\overline{K}$, but they cannot coordinate among themselves to influence this aggregate quantity. As illustrated in Greenwood and Huffman (1995) for the purposes of solving the model the production externality is isomorphic to some non-competitive mark-up policies by producers or to a tax on capital holdings.

Definition 1 *A sequential competitive equilibrium (SCE) is a sequence of vectors* $\left(c_t(z^t), k_{t+1}(z^t), K_t(z^t), L_t(z^t), \overline{K}_{t+1}(z^t), w_t(z^t), r_t(z^t)\right)_{t\geq 0}$ *such that*

(i) Constrained utility maximization: For each household i, the sequence $\left(c_t^i, k_{t+1}^i\right)_{t\geq 0}$ *maximizes the objective in (1) subject to the sequence of budget constraints (2).*

(ii) Profit maximization: Each $(K_t(z^t), L_t(z^t))$ *solves problem (3).*

(iii) Markets clearing: For each z^t,

$$\sum_{i=1}^{I} k_{t+1}^i + \sum_{i=1}^{I} c_t^i = A(z_t, \overline{K}_t) F\left(K_t, L_t\right),$$

$$\sum_{i=1}^{I} k_t^i = K_t \text{ and } \sum_{i=1}^{I} e_t^i = L_t.$$

(iv) Consistency of beliefs: For each z^t,

$$\sum_{i=1}^{I} k_t^i(z^t) = \overline{K}_t(z^t).$$

Note that the sequence of equilibrium quantities $\left(K_t(z^t), L_t(z^t), \overline{K}_{t+1}(z^t)\right)_{t\geq 0}$ may be inferred from households' holdings of these factors. Hence, we may refer to a SCE as simply a sequence of vectors $(c_t(z^t), k_{t+1}(z^t), r_t(z^t), w_t(z^t))_{t\geq 0}$. Although there does not seem to be a general proof of existence of equilibrium for infinite-horizon economies with distortions [see, however, Jones and Manuelli (1999)], under Assumptions 1-3 the existence of a SCE can be established by standard methods (see the discussion below). Moreover, one can also show that there are positive constants K^{max} and K^{min} such that for every equilibrium sequence $\left(k_{t+1}^i(z^t)\right)_{t\geq 0}$ if $K^{max} \geq \sum_{i=1}^{I} k_0^i(z^0) \geq K^{min}$ then $K^{max} \geq \sum_{i=1}^{I} k_t^i(z^{t+1}) \geq K^{min}$ for all z^t. Also, the sequence of individual consumptions is uniformly bounded above and below by positive constants. Therefore, at each z^t the set of equilibrium vectors $(c_t(z^t), k_{t+1}(z^t), r_t(z^t), w_t(z^t))_{t\geq 0}$ lies in a compact set separated from zero.

3.2 Computation of SCE

For the present model a Markov equilibrium may not exist or may be discontinuous over the minimal set of state variables $s = (k, z)$. Consequently, it may not be operative to compute competitive equilibria as solutions of some optimal planning program with side constraints.

We now expand arbitrarily the state space so as to include the individual shadow values of investment. In this enlarged state space we prove the existence of a Markov equilibrium. Our approach is in the spirit of Duffie *et al*. (1994), but differs in the choice of the state space; also, our methods seem more amenable to the computation of an equilibrium function.

3.2.1 The Markov equilibrium correspondence V^*

Let $\mathbf{K} = \{k : K^{max} \geq \sum_{i=1}^{I} k^i \geq K^{min}\}$. For any initial distribution of capital k_0 and a given shock z_0, we define the Markov equilibrium correspondence $V^* : \mathbf{K} \times \mathbf{Z} \to \mathbf{R}_+^I$ as

$$V^*(k_0, z_0) = \left\{ \left(..., r_0(z_0) D_1 u^i \left(c_0^i(z_0), z_0\right), ...\right) \in \mathbf{R}^I : (c_t, k_{t+1}, r_t, w_t)_{t\geq 0} \text{ is a SCE} \right\} \quad (4)$$

Therefore, for each vector of state variables (k_0, z_0), this correspondence contains the set of all current equilibrium shadow values $m_0 = (..., r_0 D_1 u^i(c_0^i), ...)$ of investment for each household i.

Proposition 1 *A SCE exists. Therefore, the correspondence V^* is nonempty valued.*

3.2.2 Operator B

For a correspondence $W : \mathbf{K} \times \mathbf{Z} \to \mathbf{R}^I$, define an operator $B : W \longmapsto B(W)$ as follows: For any $(k, z) \in \mathbf{K} \times \mathbf{Z}$, $m = (..., m^i, ...) \in \mathbf{R}^I$ is an element of $B(W)(k, z)$ if there exists

$(c, k_+, r, w, m_+, \lambda) \in \mathbf{R}_+^I \times \mathbf{R}_+^I \times \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_+^{IN} \times \mathbf{R}_+^I$ such that for all i the following *temporary equilibrium conditions* hold:

$$\begin{aligned}
m^i &= rD_1u^i\left(c^i, z\right), && (5)\\
m_+\left(z_+\right) &\in W\left(k_+, z_+\right) \text{ for all } z_+ \in \mathbf{Z}, && (6)\\
D_1u^i\left(c^i, z\right) &= \beta \sum_{z_+ \in \mathbf{Z}} \pi\left(z_+|z\right) m_+^i\left(z_+\right) + \lambda^i, && (7)\\
\lambda^i k_+^i &= 0,\ \lambda^i \geq 0, && (8)\\
c^i + k_+^i &= rk^i + we^i\left(z\right), && (9)\\
r &= A\left(z, \sum_{i=1}^I k^i\right) D_1F\left(\sum_{i=1}^I k^i, \sum_{i=1}^I e^i(z)\right), && (10)\\
w &= A\left(z, \sum_{i=1}^I k^i\right) D_2F\left(\sum_{i=1}^I k^i, \sum_{i=1}^I e^i(z)\right), && (11)\\
\sum_{i=1}^I c^i + \sum_{i=1}^I k_+^i &= A\left(z, \sum_{i=1}^I k^i\right) F\left(\sum_{i=1}^I k^i, \sum_{i=1}^I e^i(z)\right). && (12)
\end{aligned}$$

Therefore, for each state (k, z), the set $B(W)(k, z)$ contains all the shadow values m such that (k, z, m) is consistent with a temporary equilibrium $(c, k_+, r, w, m_+, \lambda)$ satisfying $(k_+, z_+, m_+(z_+)) \in graph(W)$. It is clear from the definition that $B(W) \subset B(W')$ if $W \subset W'$. Moreover, if W has a compact graph then under the above assumptions one can show that $B(W)$ has also a compact graph.

Lemma 1 *Operator B is monotone and preserves compactness.*

Another key property of operator B is self-generation.

Theorem 1 *If $W \subset B(W)$, then $B(W) \subset V^*$.*

Self-generation implies that V^* is a fixed point of B.

Theorem 2 *V^* is a fixed point of operator B, i.e., $V^* = B(V^*)$.*

Our next result shows that operator B ensures convergence to the fixed-point solution.

Theorem 3 *Let W_0 be a compact-valued correspondence such that $W_0 \supset V^*$ and $B(W_0) \subset W_0$. Let $W_n = B(W_{n-1}), n \geq 1$. Then $V^* = \lim_{n\to\infty} W_n$. Moreover, V^* is the largest fixed point of B, i.e., if $W = B(W)$, then $W \subset V^*$.*

3.2.3 Selection of an equilibrium function

From the construction of B, we can select a function $f = (f_c, f_k, f_r, f_w, f_m, f_\lambda)$ that takes each $(k, z, m) \in \text{graph}(B(V^*))$ to some value $f(k, z, m) = (c, k_+, r, w, m_+, \lambda)$ satisfying conditions (5)-(12) as defined by $B(V^*)$. By similar arguments to the proof of Theorem 1 given in the appendix, we can show that if $(c_t(z^t), k_{t+1}(z^t), w_t(z^t), r_t(z^t))_{t\geq 0}$ is generated under f, then this sequence is a SCE. Hence, for the economy described in this section there exists a Markov competitive equilibrium over the space of state variables (k, z, m).

Corollary 1 *Let f be defined as above. For each $z^{t+1}|z^t$, let $(c_t(z^t), k_{t+1}(z^t)$, $m_{t+1}(z^{t+1})$, $r_t(z^t)$, $w_t(z^t)$, $\lambda_t(z^t)) = f(k_t(z^{t-1}), z_t, m_t(z^t))$. Then $(c_t(z^t), k_{t+1}(z^t), r_t(z^t), w_t(z^t))_{t\geq 0}$ is a SCE.*

4 An Asset-Trading Model with Heterogeneous Agents

We consider a pure exchange economy populated by many heterogeneous consumers.

4.1 Environment

There are I agents and a single perishable consumption good. Each agent i has an endowment function $e^i : \mathbf{Z} \to \mathbf{R}_{++}$ determined by a stationary process $e^i(z_t)$ such that $e^i_t(z^t) = e^i(z_t)$ at each node z^t.

A consumption plan for agent i is a sequence of functions $(c^i_t)_{t\geq 0}$ with $c^i_t : \mathbf{Z}^t \to \mathbf{R}_+$. His preferences are represented by the utility function

$$E\left[\sum_{t=0}^{\infty} \beta^t u^i(c_t, z_t)\right], \quad (c_t)_{t\geq 0} \in X_+, \tag{13}$$

where $\beta \in (0,1)$ and u^i satisfies Assumption 1.

There are J assets available for trade. A simple way to introduce the incompleteness of financial markets is to assume that the number of assets is less than the number of states. Without loss of generality we consider that each asset is in unit supply. There is a dividend function $d^j : \mathbf{Z} \to \mathbf{R}_{++}$ such that asset j's dividends $d^j_t(z^t)$ at node z^t satisfies $d^j_t(z^t) = d^j(z_t)$. Asset j's after dividends price at date t is denoted q^j_t. Let the vectors of dividends and prices be $d_t = \left(d^j_t, ..., d^J_t\right)$ and $q_t = \left(q^1_t, ..., q^J_t\right)$.

To rule out Ponzi games, we impose a short-sale constraint. In particular, we assume that agents cannot short any assets. Initially, agent i is endowed with $\theta_0^i \in \mathbf{R}_+^J$ assets. At each z^t the agent can buy c_t^i units of the consumption good and a new portfolio θ_{t+1}^i of assets subject to

$$\begin{aligned} c_t^i + \theta_{t+1}^i \cdot q_t &= e_t^i + \theta_t^i \cdot (q_t + d_t), && (14) \\ \theta_t^i &\geq 0, \text{ all } t, \ \theta_0^i \text{ given.} && (15) \end{aligned}$$

4.2 Equilibrium

Definition 2 *A SCE is a collection* $(c_t(z^t), \theta_{t+1}(z^t), q_t(z^t))_{t\geq 0}$ *such that:*

(i) Given prices $(q_t(z^t))_{t\geq 0}$, *for each* i *the sequence* $\left(c_t^i(z^t), \theta_{t+1}^i(z^t)\right)_{t\geq 0}$ *yields the maximum value for (13) subject to (14)-(15).*

(ii) Markets clear: for each z^t,

$$\begin{aligned} \sum_i \theta_{t+1}^{ji}\left(z^t\right) &= 1, \quad \textit{for every} \quad j && (16) \\ \sum_i \left(c_t^i\left(z^t\right) - e_t^i\left(z^t\right)\right) &= \sum_j d_t^j\left(z^t\right). && (17) \end{aligned}$$

For this economy, Duffie *et al*. (1994) existence of an ergodic invariant distribution over a randomization of the equilibrium correspondence. Krebs (2003) proves that there are no continuous Markov equilibria for which borrowing constraints never bind if the state space is compact.

Define the set of equilibrium marginal utilities

$$V^*(\theta_0, z_0) = \left\{\left(..., \left(q_0^j(z_0) + d_0^j(z_0)\right) D_1 u^i\left(c_0^i(z_0), z_0\right), ...\right) \in \mathbf{R}^{JI} : (c_t, \theta_{t+1}, q_t)_{t\geq 0} \text{ is a SCE}\right\}$$

Proposition 2 V^* *is nonempty and compact-valued.*

4.3 The Operator B

Let $\Theta = \left\{\theta \in \mathbf{R}_+^{JN} : \sum_{i=1}^N \theta^{ji} = 1 \text{ for all } j\right\}$. For a correspondence $W : \mathbf{\Theta} \times \mathbf{Z} \to \mathbf{R}^{JI}$, define an operator $B : W \longmapsto B(W)$ as follows: For any $(\theta, z) \in \mathbf{\Theta} \times \mathbf{Z}$, $(..., m^{ji}, ...) \in \mathbf{R}^{JI}$

is an element of $B(W)(\theta,z)$ if there exists $(c,\theta_+,q,m_+,\lambda) \in \mathbf{R}_+^I \times \mathbf{R}_+^{JI} \times \mathbf{R}_+^J \times \mathbf{R}_+^{JIN} \times \mathbf{R}_+^{JI}$ such that for all i and j the following temporary equilibrium conditions hold:

$$m^{ji} = \left(q^j + d^j(z)\right) D_1 u^i\left(c^i\right), \tag{18}$$

$$\begin{aligned}
\left(..., m_+^{ji}(z_+), ...\right) &\in W(\theta_+, z_+) \text{ for all } z_+ \in \mathbf{Z}, &(19)\\
q^j D_1 u^i\left(c^i\right) &= \sum_{z_+\in\mathbf{Z}} \pi(z_+|z)\, m_+^{ji}(z_+) + \lambda^{ji}, &(20)\\
\lambda^{ji}\theta_+^{ji} &= 0, &(21)\\
c^i + q\cdot\theta_+^i &= (q + d(z))\cdot\theta^i + e^i(z), &(22)\\
\sum_{i=1}^N \theta_+^{ji} &= 1. &(23)
\end{aligned}$$

Lemma 2 *The operator B is monotone and preserves compactness.*

As in the preceding section we get the following results.

Theorem 4 *If W is self-generating, i.e., $W \subset B(W)$, then $B(W) \subset V^*$.*

Theorem 5 *V^* is a fixed point of the operator B, i.e., $B(V^*) = V^*$.*

Theorem 6 *Let W_0 be a compact-valued correspondence such that $W_0 \supset V^*$ and $B(W_0) \subset W_0$. Let $W_n = B(W_{n-1}), n \geq 1$. Then $V^* = \lim_{n\to\infty} W_n$ is the largest fixed point of operator B.*

5 A Stochastic OLG Model

5.1 Environment

We consider a stochastic OLG model similar to that described in Duffie et al (1994). There is a single consumption good. There are I types of agents in each generation. Each agent lives for 2 periods. A type i agent has endowment functions $e_y^i : \mathbf{Z} \to \mathbf{R}_{++}$ and $e_o^i : \mathbf{Z} \to \mathbf{R}_{++}$. The interpretation is that if the current shock is z, then independently of the history of shocks type i agent receives endowment $e_y^i(z)$ when young, and endowment

$e_o^i(z)$ when old. Let $c_{y,t}^i(z^t)$ be the consumption of the type i agent born at date t after history z^t when young, and let $c_{o,t+1}^i(z^{t+1})$ be his consumption when old after history z^{t+1}. His preferences are represented by the utility function

$$U^i\left(c_y^i, c_0^i; z^t\right) = u^i\left(c_{y,t}^i\left(z^t\right)\right) + \beta \sum_{z_{t+1}\in\mathbf{Z}} u^i\left(c_{o,t+1}^i\left(z^{t+1}\right)\right)\pi\left(z_{t+1}|z^t\right),$$

where $\beta \in (0,1)$ is the discount factor, and $u^i : \mathbf{R}_+ \to \mathbf{R}$ is a utility index.

Assumption 4 *For each i, u^i is increasing, strictly concave and continuously differentiable.*

There are J infinitely lived assets available for trade. The total supply of each asset is normalized to 1. The dividend structure is specified by a function $d : \mathbf{Z} \to \mathbf{R}_+^J$, with $d^j(y)$ denoting the dividend paid by asset j when the current shock is z. In each period, the old collect the dividends generated by their portfolios given the current shock, after which the assets are sold ex dividend to the young.

We denote by $\theta_{t+1}^i(z^t)$ a portfolio plan for a type i agent born at date t after history z^t. For simplicity, we assume all agents cannot short sell any assets so that $\theta_{t+1}^i(z^t) \geq 0$.

At date 0 there is an initial generation of old agents who derive utility only for consumption at this date. Each agent i is endowed with e_o^i of the consumption good and a portfolio θ_0^i of assets ($\sum_{i=1}^N \theta_0^{ji} = 1, j = 1, .., J$), $i = 1, ..., I$. Hence, each initial old agent consumes

$$c_{o,0}^i = \left(q_0\left(z^0\right) + d\left(z_0\right)\right)\cdot\theta_0^i + e_o^i. \tag{24}$$

For all other generations, the decision problem of an agent of type i born at date $t \geq 0$ after history z^t is given by

$$\max_{\left(c_{y,t}^i, c_{o,t+1}^i, \theta_{t+1}^i\right)} U^i\left(c_y^i, c_o^i; z^t\right) \tag{25}$$

subject to

$$c_{y,t}^i\left(z^t\right) + q_t\left(z^t\right)\cdot\theta_{t+1}^i\left(z^t\right) = e_y^i\left(z_t\right), \tag{26}$$

$$c_{o,t+1}^i\left(z^{t+1}\right) = \left(q_{t+1}\left(z^{t+1}\right) + d\left(z_{t+1}\right)\right)\cdot\theta_{t+1}^i\left(z^t\right) + e_o^i\left(z_{t+1}\right), \text{ all } z_{t+1}. \tag{27}$$

5.2 Equilibrium

Definition 3 *A SCE is a collection of consumption and portfolio holdings $\left(c_{y,t}^i\left(z^t\right), c_{o,t}^i\left(z^t\right), \theta_{t+1}^i\left(z^t\right)\right)_{i=1}^I$ and prices $q_t\left(z^t\right)$, $z^t \in Z^t$, $t \geq 0$, such that:*

(i) Given prices $(q_t)_{t\geq 0}$, for each agent of type i, of the initial generation $c^i_{o,0}$ is given by (24), and for every agent of type i born at time $t \geq 0$ the consumption-portfolio plan $\left(c^i_{y,t}, c^i_{o,t+1}, \theta^i_{t+1}\right)$ solves the problem (25) subject to (26) and (27), for all t.

(ii) Markets clear: For all z^t

$$\begin{aligned} \sum_{i=1}^{I} \theta^{ji}_{t+1} &= 1, \quad \textit{for all} \quad j \\ \sum_{i=1}^{N} \left(c^i_{y,t} + c^i_{o,t}\right) &= \sum_{i=1}^{N} \left(e^i_{y,t} + e^i_{o,t}\right) + \sum_{j=1}^{J} d^j_t. \end{aligned}$$

Define

$$V^*(\theta_0, z_0) = \left\{\left(\ldots \left(q^j_0(z_0) + d^j_0(z_0)\right) D_1 u^i\left(c^i_o(z_0)\right) \ldots\right) \in \mathbf{R}^{JI} : (c_y, c_o, \theta, q) \text{ is a SCE}\right\}$$

Proposition 3 *Under Assumption 5, V^* is nonempty and compact-valued.*

5.3 The Operator B

Let $\Theta = \left\{\theta \in \mathbf{R}^{JI}_+ : \sum_{i=1}^{N} \theta^{ji} = 1 \text{ for all } j\right\}$. For a correspondence $W : \mathbf{\Theta} \times \mathbf{Z} \rightarrow \mathbf{R}^{JI}$, define an operator $B : W \longmapsto B(W)$ as follows: For any $(\theta, z) \in \mathbf{\Theta} \times \mathbf{Z}$, $(\ldots, m^{ji}, \ldots) \in \mathbf{R}^{JI}$ is an element of $B(W)(\theta, z)$ if there exists $(c_y, c_o, \theta_+, q, m_+, \lambda) \in \mathbf{R}^I_+ \times \mathbf{R}^I_+ \times \mathbf{R}^{JI}_+ \times \mathbf{R}^J_+ \times \mathbf{R}^{JIN}_+ \times \mathbf{R}^{JI}_+$ such that for all i and j the following temporary equilibrium conditions hold:

$$m^{ji} = \left(q^j + d^j(z)\right) D_2 u^i\left(c^i_o\right), \tag{28}$$

$$\begin{aligned} \left(\ldots, m^{ji}_{o+}(z_+), \ldots\right) &\in W(\theta_+, z_+) \text{ for all } z_+ \in \mathbf{Z}, && (29) \\ q_j D_1 u^i\left(c^i_y\right) &= \beta \sum_{z_+ \in \mathbf{Z}} \pi(z_+|z) m^{ji}_+(z_+) + \lambda^{ji}, && (30) \\ \lambda^{ji}\theta^{ji}_+ &= 0, \ \lambda^{ji} \geq 0, && (31) \\ c^i_y + q \cdot \theta^i_+ &= e^i_y(z), && (32) \\ c^i_o &= (q + d(z)) \cdot \theta^i + e^i_o(z), && (33) \\ \sum_{i=1}^{N} \theta^{ji}_+ &= 1. && (34) \end{aligned}$$

Lemma 3 *The operator B is monotone and preserves compactness.*

By the same arguments as in the preceding sections we obtain the following results.

Theorem 7 *If W $W \subset B(W)$, then $B(W) \subset V^*$.*

Theorem 8 *V^* is a fixed point of the operator B.*

Theorem 9 *Let W_0 be a compact-valued correspondence such that $W_0 \supset V^*$ and $B(W_0) \subset W_0$. Define $W_n = B(W_{n-1}), n \geq 1$. Then $V^* = \lim_{n\to\infty} W_n$. Moreover, V^* is the largest fixed point of operator B.*

6 Conclusion

- Our method has some desirable properties over Duffie et al and Kubler and Schmedders.
- Applications: Macro, Finance, Policy Games.

A Appendix: Proofs

Proof of Theorem 1: The proof is by construction. For any $(k_0, z_0, m_0) \in$graph$B(W)$, we shall construct a sequence $\left(c_t\left(z^t\right), k_{t+1}\left(z^t\right), w_t\left(z^t\right), r_t\left(z^t\right)\right)_{t\geq 0}$ such that it constitutes a competitive equilibrium and

$$m_0^i\left(z^0\right) = r_0\left(z^0\right) D_1 u^i\left(c_0^i\left(z^0\right), z_0\right)$$

for all i. To this end, let $f = (f_c, f_k, f_r, f_w, f_m, f_\lambda)$ be a map that, for each $(k, z, m) \in$ graph$(B(W))$, selects a value $f(k, z, m) = (c, k_+, r, w, m_+, \lambda)$ satisfying all the conditions in the definition of $B(W)$.

At $t = 0$, let

$$\begin{aligned}
c_0\left(z^0\right) &= f_c\left(k_0, z_0, m_0\right), \\
k_1\left(z^0\right) &= f_k\left(k_0, z_0, m_0\right), \\
r_0\left(z^0\right) &= f_r\left(k_0, z_0, m_0\right), \\
w_0\left(z^0\right) &= f_w\left(k_0, z_0, m_0\right), \\
\lambda_0^i\left(z^0\right) &= f_\lambda\left(k_0, z_0, m_0\right).
\end{aligned}$$

At $t=1$, let

$$m_1\left(z^1\right)=f_m\left(k_0,z_0,m_0\right)\left(z_1\right) \text{ for all } z_1\in\mathbf{Z}.$$

Since $f_m\left(k_0,z_0,m_0\right)\left(z_1\right)\in W\left(k_1,z_1\right)$ for all $z_1\in\mathbf{Z}$ by (6), $m_1\left(z^1\right)\in W\left(k_1,z_1\right)$ for all $z_1\in\mathbf{Z}$. By the self-generation assumption $W\left(k_1,z_1\right)\subset B\left(W\right)\left(k_1,z_1\right),$ we know that $\left(k_1,z_1,m_1\left(z^1\right)\right)\in\text{graph}B\left(W\right).$ So we can define

$$\begin{aligned}
c_1\left(z^1\right) &= f_c\left(k_1,z_1,m_1\left(z^1\right)\right),\\
k_2\left(z^1\right) &= f_k\left(k_1,z_1,m_1\left(z^1\right)\right),\\
r_1\left(z^1\right) &= f_r\left(k_1,z_1,m_1\left(z^1\right)\right),\\
w_1\left(z^1\right) &= f_w\left(k_1,z_1,m_1\left(z^1\right)\right),\\
\lambda_1\left(z^1\right) &= f_\lambda\left(k_1,z_1,m_1\left(z^1\right)\right).
\end{aligned}$$

At $t=2$, let

$$m_2\left(z^2\right)=f_m\left(k_1,z_1,m_1\left(z^1\right)\right)\left(z_2\right) \text{ for all } z_2\in\mathbf{Z}.$$

Since $f_m\left(k_1,z_1,m_1\left(z^1\right)\right)\left(z_2\right)\in W\left(k_2,z_2\right)$ for all $z_2\in\mathbf{Z}$ by (6), $m_2\left(z^2\right)\in W\left(k_2,z_2\right)$ for all $z_2\in\mathbf{Z}$. By the self-generation assumption $W\left(k_2,z_2\right)\subset B\left(W\right)\left(k_2,z_2\right),$ we know that $\left(k_2,z_2,m_2\left(z^2\right)\right)\in\text{graph}B\left(W\right).$ So we can define

$$\begin{aligned}
c_2\left(z^2\right) &= f_c\left(k_2,z_2,m_2\left(z^2\right)\right),\\
k_3\left(z^2\right) &= f_k\left(k_2,z_2,m_2\left(z^2\right)\right),\\
r_2\left(z^2\right) &= f_r\left(k_2,z_2,m_2\left(z^2\right)\right),\\
w_2\left(z^2\right) &= f_w\left(k_2,z_2,m_2\left(z^2\right)\right),\\
\lambda_2\left(z^2\right) &= f_\lambda\left(k_2,z_2,m_2\left(z^2\right)\right).
\end{aligned}$$

We continue in this fashion and define $\left(c_t\left(z^t\right),k_{t+1}\left(z^t\right),r_t\left(z^t\right),w_t\left(z^t\right),\lambda_t\left(z^t\right)\right)_{t\geq 0}$ and $m_{t+1}\left(z^{t+1}\right),\ t\geq 0.$

By the definition of the operator B and the above construction,

$$\begin{aligned}
D_1u^i\left(c_t^i\left(z^t\right),z_t\right) &= \beta\sum_{z_{t+1}\in\mathbf{Z}}\pi\left(z_{t+1}|z_t\right)m_{t+1}\left(z^{t+1}\right)+\lambda_t^i\left(z^t\right),\\
m_{t+1}^i\left(z^{t+1}\right) &= r_{t+1}\left(z^{t+1}\right)D_1u^i\left(c_{t+1}^i\left(z^{t+1}\right),z_{t+1}\right),\\
\lambda_t^i\left(z^t\right) &\geq 0,\ \lambda_t^i\left(z^t\right)k_{t+1}^i\left(z^t\right)=0.
\end{aligned}$$

Thus, we have the Euler equation

$$D_1u^i\left(c_t^i\left(z^t\right),z_t\right)=\beta\sum_{z_{t+1}\in\mathbf{Z}}\pi\left(z_{t+1}|z_t\right)r_{t+1}\left(z^{t+1}\right)D_1u^i\left(c_{t+1}^i\left(z^{t+1}\right),z_t\right)+\lambda_t^i\left(z^t\right).$$

To show that $(c_t, k_{t+1})_{t\geq 0}$ maximizes utility, we need only to show that the transversality condition $\lim_{T\to\infty} E\left[\beta^T m_T^i k_{T+1}^i\right] = 0$ is satisfied for all i. Since $k_0 \in [K^{\min}, K^{\max}]$, one can deduce that k_{t+1} is uniformly bounded for all t using Assumptions 2-3, as we argued in the main text. By Assumption 1, we can apply an argument similar to Lemma 2 in Phelan and Stacchetti (2001) to deduce that $m_t^i(z^t) = r_t(z^t) D_1 u^i(c_t^i(z^t), z_t)$ is uniformly bounded. Thus, the transversality condition is satisfied.

Further, the definition of the operator B and the above construction imply that $(c_t, k_{t+1}, r_t, w_t)_{t\geq 0}$ satisfies profit maximization, market clearing and consistency of beliefs conditions in Definition 1. Thus, $(c_t, k_{t+1}, r_t, w_t)_{t\geq 0}$ is a SCE. This implies that $(k_0, z_0, m_0) \in \text{graph}V^*$. Thus, $B(W) \subset V^*$.

Proof of Theorem 2: It suffices to show that V^* satisfies the self-generation property, $V^* \subset B(V^*)$. This is because if it is true, one can use the previous theorem to deduce that $B(V^*) \subset V^*$. Hence $V^* = B(V^*)$.

Let $m_0 \in V^*(k_0, z_0)$. Then there exists a SCE $(c_t, k_{t+1}, r_t, w_t)_{t\geq 0}$ such that

$$m_0^i = r_0\left(z^0\right) D_1 u^i\left(c_0^i\left(z^0\right), z_0\right).$$

To show that $m_0 \in B(V^*)(k_0, z_0)$, we only need to show there exists $(c, k_+, r, w, m_+, \lambda) \in \mathbf{R}_+^I \times \mathbf{R}_+^I \times \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_+^{IN} \times \mathbf{R}_+^I$ such that for all i the temporary equilibrium conditions in the definition of the operator B are satisfied. To this end, let $c = c_0, k_+ = k_1, r = r_0, w = w_0, \lambda = \lambda_0$, and $m_+^i(z_1) = r_1(z^1) D_1 u^i(c_1^i(z^1), z_1)$ for all i and all $z_1 \in \mathbf{Z}$. Then by the definition of SCE, conditions (5) and (7)-(12) are satisfied. We only need to check that $(..., r_1(z^1) D_1 u^i(c_1^i(z^1), z_1), ...) \in V^*(k_1, z_1)$ so that (6) is satisfied. Indeed, this is true since $(c_t, k_{t+1}, r_t, w_t)_{t\geq 1}$ constitutes a SCE for the economy starting at date 1 with initial state (k_1, z_1).

Proof of Theorem 3: By Lemma 3, $\{W_n\}$ is a decreasing sequence of compact-valued correspondence. So $W_\infty = \lim_{n\to\infty} W_n = \cap W_n$ is also a compact-valued correspondence. Again by Lemma 1 and Theorems 1-2, $V^* \subset W_\infty$. To complete the proof we only need to show that $W_\infty \subset V^*$. By the self-generation Theorem 1, we only need to show $W_\infty \subset B(W_\infty)$. In fact, notice that $W_n \subset W_{n-1} = B(W_{n-2})$. One can take limits since the temporary equilibrium conditions in the definition of the operator B involve equalities and weak inequalities.

Let $\widetilde{V}^*$ be any fixed point of B. Then since $\widetilde{V}^* = B\left(\widetilde{V}^*\right)$, it follows from Theorem 3 that $\widetilde{V}^* = B\left(\widetilde{V}^*\right) \subset V^*$. Thus V^* is the largest fixed point of B.

Proof of Theorem 4: The proof is similar to that of Theorem 3. For any $(\theta_0, z_0, m_0) \in \text{graph}B(W)$, we shall construct a sequence $(c_t(z^t), \theta_{t+1}(z^t), q_t(z^t))_{t\geq 0}$ such that it constitutes a competitive equilibrium and

$$m_0^{ji}\left(z^0\right) = \left(q_0^j\left(z^0\right) + d_0^j\left(z^0\right)\right) D_1 u^i\left(c_0^i\left(z^0\right)\right)$$

for all i and j. To this end, let $f = (f_c, f_\theta, f_q, f_m, f_\lambda)$ be a map that, for each $(\theta, z, m) \in \text{graph}(B(W))$, selects a value $f(\theta, z, m) = (c, \theta_+, q, m_+, \lambda)$ satisfying all the conditions in the definition of $B(W)$.

At $t = 0$, let

$$\begin{aligned}
c_0\left(z^0\right) &= f_c(\theta_0, z_0, m_0), \\
\theta_1\left(z^0\right) &= f_\theta(\theta_0, z_0, m_0), \\
q_0\left(z^0\right) &= f_q(\theta_0, z_0, m_0), \\
\lambda_0\left(z^0\right) &= f_\lambda(\theta_0, z_0, m_0).
\end{aligned}$$

At $t = 1$, let

$$m_1\left(z^1\right) = f_m(\theta_0, z_0, m_0)(z_1) \text{ for all } z_1 \in \mathbf{Z}.$$

Since $f_m(\theta_0, z_0, m_0)(z_1) \in W(\theta_1, z_1)$ for all $z_1 \in \mathbf{Z}$ by (19), $m_1(z^1) \in W(\theta_1, z_1)$ for all $z_1 \in \mathbf{Z}$. By the self-generation assumption $W(\theta_1, z_1) \subset B(W)(\theta_1, z_1)$, we know that $(\theta_1, z_1, m_1(z^1)) \in \text{graph}B(W)$. So we can define

$$\begin{aligned}
c_1\left(z^1\right) &= f_c(\theta_1, z_1, m_1), \\
\theta_2\left(z^1\right) &= f_\theta(\theta_1, z_1, m_1), \\
q_1\left(z^1\right) &= f_q(\theta_1, z_1, m_1), \\
\lambda_1\left(z^1\right) &= f_\lambda(\theta_1, z_1, m_1).
\end{aligned}$$

At $t = 2$, let

$$m_2\left(z^2\right) = f_m\left(\theta_1, z_1, m_1\left(z^1\right)\right)(z_2) \text{ for all } z_2 \in \mathbf{Z}.$$

Since $f_m(\theta_1, z_1, m_1(z^1))(z_2) \in W(\theta_2, z_2)$ for all $z_2 \in \mathbf{Z}$ by (19), $m_2(z^2) \in W(\theta_2, z_2)$ for all $z_2 \in \mathbf{Z}$. By the self-generation assumption $W(\theta_2, z_2) \subset B(W)(\theta_2, z_2)$, we know that $(\theta_2, z_2, m_2(z^2)) \in \text{graph}B(W)$. So we can define

$$\begin{aligned}
c_2\left(z^2\right) &= f_c(\theta_2, z_2, m_2), \\
\theta_3\left(z^2\right) &= f_\theta(\theta_2, z_2, m_2), \\
q_2\left(z^2\right) &= f_q(\theta_2, z_2, m_2), \\
\lambda_2\left(z^2\right) &= f_\lambda(\theta_2, z_2, m_2).
\end{aligned}$$

We continue in this fashion and define $\left(c_t\left(z^t\right), \theta_{t+1}\left(z^t\right), q_t\left(z^t\right), \lambda_t\left(z^t\right)\right)_{t\geq 0}$ and $m_{t+1}\left(z^{t+1}\right)$, $t \geq 0$.

By the definition of the operator B and the above construction,

$$\begin{aligned} D_1 u^i\left(c_t^i\left(z^t\right)\right) &= \beta \sum_{z_{t+1}\in\mathbf{Z}} \pi\left(z_{t+1}|z_t\right) m_{t+1}^{ji}\left(z^{t+1}\right) + \lambda_t^{ji}\left(z^t\right), \\ m_{t+1}^{ji}\left(z^{t+1}\right) &= \left(q_{t+1}^j\left(z^{t+1}\right) + d_{t+1}^j\left(z^{t+1}\right)\right) D_1 u^i\left(c_{t+1}^i\left(z^{t+1}\right)\right), \\ \lambda_t^{ji}\left(z^t\right) &\geq 0,\ \lambda_t^{ji}\left(z^t\right)\theta_{t+1}^{ji}\left(z^t\right) = 0. \end{aligned}$$

Thus, we have the Euler equation

$$D_1 u^i\left(c_t^i\left(z^t\right), z_t\right) = \beta \sum_{z_{t+1}\in\mathbf{Z}} \pi\left(z_{t+1}|z_t\right)\left(q_{t+1}^j\left(z^{t+1}\right) + d_{t+1}^j\left(z^{t+1}\right)\right) D_1 u^i\left(c_{t+1}^i\left(z^{t+1}\right)\right) + \lambda_t^{ji}\left(z^t\right).$$

By Proposition 3.2 in Duffie et al (1994), the transversality condition is satisfied. Thus, $\left(c_t, \theta_{t+1}\right)_{t\geq 0}$ maximizes utility. Further, the definition of the operator B and the above construction imply that $\left(c_t, \theta_{t+1}, q_t\right)_{t\geq 0}$ all satisfies the market clearing conditions in Definition 7. Thus, $\left(c_t, \theta_{t+1}, q_t\right)_{t\geq 0}$ is a SCE. This implies that $\left(\theta_0, z_0, m_0\right) \in \mathrm{graph}V^*$. Thus, $B\left(W\right) \subset V^*$.

Proof of Theorem 5: Identical to that of Theorem 2.

Proof of Theorem 6: Identical to that of Theorem 3.

Proof of Theorem 7: The proof is similar to that of Theorem 3. For any $\left(\theta_0, z_0, m_0\right) \in \mathrm{graph}B\left(W\right)$, we shall construct a sequence $\left(\left(c_{y,t}^i\left(z^t\right), c_{o,t}^i\left(z^t\right), \theta_{t+1}^i\left(z^t\right)\right)_{i=1}^I,\ q_t\left(z^t\right)\right)_{t\geq 0}$ such that it constitutes a competitive equilibrium and

$$m_0^{ji}\left(z^0\right) = \left(q_0^j\left(z^0\right) + d_0^j\left(z^0\right)\right) D_1 u_o^i\left(c_{o,0}^i\left(z^0\right)\right)$$

for all i and j. To this end, let $f = \left(f_{c_y}, f_{c_o}, f_\theta, f_q, f_m, f_\lambda\right)$ be a map that, for each $\left(\theta, z, m\right) \in \mathrm{graph}(B\left(W\right))$, selects a value $f\left(\theta, z, m\right) = \left(c_y, c_o, \theta_+, q, m_+, \lambda\right)$ satisfying all the conditions in the definition of $B\left(W\right)$.

At $t=0$, let

$$\begin{aligned}
c_{y,0}\left(z^0\right) &= f_{c_y}\left(\theta_0, z_0, m_0\right), \\
c_{o,0}\left(z^0\right) &= f_{c_o}\left(\theta_0, z_0, m_0\right), \\
\theta_1\left(z^0\right) &= f_\theta\left(\theta_0, z_0, m_0\right), \\
q_0\left(z^0\right) &= f_q\left(\theta_0, z_0, m_0\right), \\
\lambda_0\left(z^0\right) &= f_\lambda\left(\theta_0, z_0, m_0\right).
\end{aligned}$$

At $t=1$, let

$$m_1\left(z^1\right) = f_m\left(\theta_0, z_0, m_0\right)\left(z_1\right) \text{ for all } z_1 \in \mathbf{Z}.$$

Since $f_m\left(\theta_0, z_0, m_0\right)\left(z_1\right) \in W\left(\theta_1, z_1\right)$ for all $z_1 \in \mathbf{Z}$ by (19), $m_1\left(z^1\right) \in W\left(\theta_1, z_1\right)$ for all $z_1 \in \mathbf{Z}$. By the self-generation assumption $W\left(\theta_1, z_1\right) \subset B\left(W\right)\left(\theta_1, z_1\right)$, we know that $\left(\theta_1, z_1, m_1\left(z^1\right)\right) \in \text{graph} B\left(W\right)$. So we can define

$$\begin{aligned}
c_{y,1}\left(z^1\right) &= f_{c_y}\left(\theta_1, z_1, m_1\right), \\
c_{o,1}\left(z^1\right) &= f_{c_o}\left(\theta_1, z_1, m_1\right) \\
\theta_2\left(z^1\right) &= f_\theta\left(\theta_1, z_1, m_1\right), \\
q_1\left(z^1\right) &= f_q\left(\theta_1, z_1, m_1\right), \\
\lambda_1\left(z^1\right) &= f_\lambda\left(\theta_1, z_1, m_1\right).
\end{aligned}$$

At $t=2$, let

$$m_2\left(z^2\right) = f_m\left(\theta_1, z_1, m_1\left(z^1\right)\right)\left(z_2\right) \text{ for all } z_2 \in \mathbf{Z}.$$

Since $f_m\left(\theta_1, z_1, m_1\left(z^1\right)\right)\left(z_2\right) \in W\left(\theta_2, z_2\right)$ for all $z_2 \in \mathbf{Z}$ by (19), $m_2\left(z^2\right) \in W\left(\theta_2, z_2\right)$ for all $z_2 \in \mathbf{Z}$. By the self-generation assumption $W\left(\theta_2, z_2\right) \subset B\left(W\right)\left(\theta_2, z_2\right)$, we know that $\left(\theta_2, z_2, m_2\left(z^2\right)\right) \in \text{graph} B\left(W\right)$. So we can define

$$\begin{aligned}
c_{y,2}\left(z^2\right) &= f_{c_y}\left(\theta_2, z_2, m_2\right), \\
c_{o,2}\left(z^2\right) &= f_{c_o}\left(\theta_2, z_2, m_2\right), \\
\theta_3\left(z^2\right) &= f_\theta\left(\theta_2, z_2, m_2\right), \\
q_2\left(z^2\right) &= f_q\left(\theta_2, z_2, m_2\right), \\
\lambda_2\left(z^2\right) &= f_\lambda\left(\theta_2, z_2, m_2\right).
\end{aligned}$$

We continue in this fashion and define $\left(c_{y,t}\left(z^t\right), c_{o,t}\left(z^t\right), \theta_{t+1}\left(z^t\right), q_t\left(z^t\right), \lambda_t\left(z^t\right)\right)_{t\geq 0}$ and $m_{t+1}\left(z^{t+1}\right)$, $t \geq 0$.

By the definition of the operator B and the above construction,

$$
\begin{aligned}
D_1 u^i \left(c_{y,t}^i \left(z^t\right)\right) &= \beta \sum_{z_{t+1} \in \mathbf{Z}} \pi \left(z_{t+1}|z_t\right) m_{t+1}^{ji} \left(z^{t+1}\right) + \lambda_t^{ji} \left(z^t\right), \\
m_{t+1}^{ji} \left(z^{t+1}\right) &= \left(q_{t+1}^j \left(z^{t+1}\right) + d_{t+1}^j \left(z^{t+1}\right)\right) D_1 u^i \left(c_{o,t+1}^i \left(z^{t+1}\right)\right), \\
\lambda_t^{ji} \left(z^t\right) &\geq 0, \ \lambda_t^{ji} \left(z^t\right) \theta_{t+1}^{ji} \left(z^t\right) = 0.
\end{aligned}
$$

Thus, we have the Euler equation

$$
D_1 u^i \left(c_{y,t}^i \left(z^t\right), z_t\right) = \beta \sum_{z_{t+1} \in \mathbf{Z}} \pi \left(z_{t+1}|z_t\right) \left(q_{t+1}^j \left(z^{t+1}\right) + d_{t+1}^j \left(z^{t+1}\right)\right) D_1 u^i \left(c_{o,t+1}^i \left(z^{t+1}\right)\right) + \lambda_t^{ji} \left(z^t\right).
$$

Since u^i is strictly concave, $\left(c_{y,t}^i, c_{o,t+1}^i, \theta_{t+1}^i\right)_{t \geq 0}$ maximizes agent i's utility for all i. Further, the definition of the operator B and the above construction imply that $\left(c_{y,t}^i, c_{o,t}^i, \theta_{t+1}, q_t\right)_{t \geq 0}$ satisfies the market clearing conditions in Definition 3. Thus, $\left(c_{y,t}^i, c_{o,t}^i, \theta_{t+1}, q_t\right)_{t \geq 0}$ is a SCE. This implies that $(\theta_0, z_0, m_0) \in \text{graph} V^*$. Thus, $B(W) \subset V^*$.

Proof of Theorem 8: Identical to that of Theorem 2.

Proof of Theorem 9: Identical to that of Theorem 3.

REFERENCES

D. Abreu, D. Pierce and E. Stacchetti (1990), Toward a theory of repeated games with discounting, Econometrica, 58, 1041-1063.

A. Atkeson (1991), International lending with moral hazard and risk repudiation, Econometrica, 59, 1069-1089.

Y. Balasko and K. Shell (1980), The overlapping generations model, I: The case of pure exchange without money, Journal of Economic Theory, 23, 281-306.

T. Bewley (1972), Existence of equilibria in economies with infinitely many commodities, Journal of Economic Theory

D. Bizer and K. L. Judd (1989), Taxation under uncertainty, American Economic Review, 79, 331-336.

L. E. Blume (1992), New techniques for the study of stochastic equilibrium processes, Journal of Mathematical Economics, 9, 61-70.

A. Castaneda, J. Diaz-Gimenez and J. V. Rios-Rull (2003), Accounting for wealth inequality, Journal of Political Economy, 111, 818-857.

R. Chang (1998), Credible monetary policy in an infinite horizon model: Recursive approaches, Journal of Economic Theory, 81, 431-461.

W. J. Coleman (1991), Equilibrium in a production economy with an income tax, Econometrica, 59, 1091-1104.

D. Duffie, J. Geanakoplos, A. Mas-Colell and A. McLennan (1994), Stationary Markov equilibria, Econometrica, 62, 745-781.

J. Fernandez-Villaverde and A. Tsyvinski (2004), Optimal fiscal policy in a business cycle model without commitment, manuscript, University of Pennsylvania.

J. Greenwood and G. Huffman (1995), On the existence of non-optimal equilibria in dynamic stochastic economies, Journal of Economic Theory, 65, 611-623.

H. Heaton and D. Lucas (1998), Evaluating the effects of incomplete markets on risk sharing and asset pricing, Journal of Political Economy, 104, 443-487.

A. Imrohoroglu (1989), Cost of business cycles with indivisibilities and liquidity constraints, Journal of Political Economy, 97, 1364-1383.

L. E. Jones and R. E. Manuelli (1999), Notes on the existence of equilibrium with distortions in infinite horizon economies, manuscript, University of Minnesota.

K. L. Judd, S. Yeltekin and J. Conklin (2003), Computing supergame equilibria, Econometrica, 71, 1239-1254.

T. Krebs (2004), Non-existence of recursive equilibria on compact spaces when markets are incomplete, forthcoming in Journal of Economic Theory.

F. Kubler and H. M. Polemarchakis (2004), Stationary Markov equilibria for overlapping generations, forthcoming in Economic Theory.

F. Kubler and K. Schmedders (2002), Recursive equilibria in economies with incomplete markets, Macroeconomic Dynamics, 6, 284-306.

F. Kubler and K. Schmedders (2003), Stationary equilibria in asset-pricing models with incomplete markets and collateral, Econometrica, 71, 1767-1795.

P. Krusell and A. A. Smith (1998), Income and wealth heterogeneity in the macroeconomy, Journal of Political Economy, 106, 867-896.

F. E. Kydland, and E. C. Prescott (1980), Dynamic optimal taxation, rational expectations and optimal control, Journal of Economic Dynamics and Control, 2, 79-91.

A. Marcet, and R. Marimon (1998), Recursive contracts, manuscript, Universitat Pompeu Fabra, Barcelona.

A. Marcet and K. J. Singleton (1992), Equilibrium asset prices and savings of heterogeneous agents in the presence of incomplete markets and portfolio constraints, manuscript, Universitat Pompeu Fabra, Barcelona.

J. Miao (2003), Competitive equilibria of economies with a continuum of consumers and aggregate shocks, manuscript, Boston University.

C. Phelan, and E. Stacchetti (2002), Sequential equilibria in a Ramsey tax model, Econometrica, 69, 1491-1518.

J. V. Rios-Rull (1999), Computation of equilibria in heterogeneous agent models, in R. Marimon and A. Scott, Computational Methods for the Study of Dynamic Economies: An Introduction, Oxford University Press.

M. S. Santos (2002), On non-existence of Markov equilibria for competitive-market economies, Journal of Economic Theory, 105, 73-98.