

# Adverse Selection in Competitive Search Equilibrium

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## **Abstract**

We extend the notion of competitive search equilibrium to an environment with adverse selection. Uninformed principals post contracts to attract informed agents. Agents observe the contracts and apply for one, trading off the probability of matching with a principal against the terms of trade offered by the contract. We characterize equilibria as the solution to a constrained optimization problem and show that in equilibrium principals offers separating contracts to attract different types of agents. We then present a set of examples to illustrate the usefulness of our model.

# 1 Introduction

This paper studies equilibrium and efficiency in an economy with adverse selection, where uninformed principals interact with informed agents. We use a version of the solution concept known as competitive search equilibrium, extended to include private information. More precisely, we consider a static environment, where a large number of homogeneous principals interact with a large number of heterogeneous agents. The type of an agent is private information. Principals compete by posting contracts at some cost, where a contract specifies the action profiles for the matched principal and agent, and hence payoffs, if a principal and agent happen to match.

Agents observe the posted contracts and direct their search towards the most attractive ones. Matching is limited by search frictions and the restriction that each principal can match with at most one agent. For example, if fewer principals offer a particular contract than the number of agents who wish to obtain it, each agent is matched only probabilistically. For each type of contract posted, principals and agents form rational expectations about market tightness—the ratio of (the measure of) principals posting that contract to (the measure of) agents that direct their search to that contract—as well as the composition of agents searching for the contract.

Part of the contribution of this paper is technical: we develop a canonical extension to the competitive search model (Montgomery, 1991; Peters, 1991; Moen, 1997; Shimer, 1996; Acemoglu and Shimer, 1999; Burdett, Shi and Wright, 2001; Mortensen and Wright, 2002) that allows for ex-ante heterogeneous agents with private information about their type. We prove that, under mild assumptions, including a weak version of a single-crossing condition, there exists an equilibrium where principals offer separating contracts: each contract posted attracts only one type of agent, and different types of agents direct their search towards different types of contracts. The expected utility of each type of agent is uniquely determined in equilibrium. Moreover, the set of competitive search equilibria is easily found by sequentially solving a constrained optimization problem for each type of agent.

After formulating our definition of equilibrium and discussing existence, uniqueness, and so on, we present a series of examples and applications to illustrate the usefulness of the approach. We also use these examples to show how some well known results in either contract theory or search theory can change when we combine elements of both in one model. And we use the examples to explore the role of our assumptions and discuss what happens when some of them are relaxed

The first example is a modified version of the classic signaling model (Spence, 1973). Firms want to hire workers who are heterogeneous both in their productivity and their

cost of signalling. More productive workers find it cheaper to provide a useless signal. In equilibrium, firms separate workers by making more productive workers provide a bigger signal. Hence, in equilibrium, the terms of the posted contracts are distorted, but we find that the market tightness associated to any contract is not distorted. Although equilibrium always exists, a Pareto improvement is feasible if there are sufficiently few unproductive agents or if the difference in signalling costs is small relative to the difference in productivity levels. We also consider a version of this example where signalling is cheaper for less productive workers. In this case there exists an equilibrium where workers are pooled and nobody signals.

Our second example is a modified version of the Rothschild and Stiglitz (1976) insurance model.<sup>1</sup> Agents are risk-averse and subject to an idiosyncratic shock. For example, in a labor market context, the shock determines their productivity, and workers differ in the probability of a good shock. Firms are risk neutral and can only verify ex post the realization of a worker's productivity. In equilibrium, firms separate workers by partially insuring them against the productivity shock. In particular, they offer full insurance only to the worse types. We also show that, even if a pooling contract does not Pareto dominate the equilibrium, a partial pooling allocation—only pooling some types of workers—may be Pareto superior.

This example shows something that might not have been obvious ex ante: competitive search resolves the famous nonexistence problem in Rothschild and Stiglitz (1976). Thus, when there are relatively few low productivity workers, equilibrium may not exist in the original Rothschild-Stiglitz model, because any separating contract is less profitable than a pooling contract that cross-subsidizes low productivity workers. In our model, such a pooling contract is infeasible, regardless of the composition of the worker pool. To understand why, suppose a principal posts a contract that attracts all types of workers. Because a firm can match with at most one worker, the more workers who try to obtain a contract, the less likely each worker is to match. This drives some workers away from the contract. Critically, the most desirable workers are the first to leave, because their outside option—trying to obtain a separating contract—is more attractive. This means that only undesirable workers would be attracted by such a deviation, which makes it unprofitable.

In the first two examples, our use of competitive search equilibrium affects the contracts that are offered in equilibrium but asymmetric information does not affect the equilibrium search frictions. Our third example reverses this: asymmetric information makes it harder for some agents to find a trading partner but does not affect the terms of trade conditional on finding a partner. We study a stylized model of asset trade. Agents want to sell a heterogeneous object, say apples that could be good or bad, to principals for a different homogeneous object, say bananas. Principals value apples (in units of bananas) more than

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<sup>1</sup>This example builds on a discussion between Daron Acemoglu and one of the authors.

agents do, so there are gains from trade. However, bad apples are valued less than good apples—they are lemons.

We show that in equilibrium there are fewer contracts posted to attract agents holding good apples than the number of agents holding good apples. This implies that, even in the absence of search frictions, by which we mean that everyone on the short side of the market will match, good-apple holders can only trade probabilistically. The probability of a meeting, rather than the terms of trade within a meeting, screens out the agents holding bad apples. In this example again, the competitive search equilibrium is Pareto dominated by a pooling allocation when there are enough few agents holding the bad asset. We also find that if there are no gains from trade in bad apples, adverse selection can shut down the market for good apples as well, an extreme version of the lemons problem.

Our paper is related to a growing literature exploring search models with private information. In particular, Faig and Jerez (2005), Guerrieri (2008), and Moen and Rosén (2006) propose different extensions of competitive search models with one-side private information. However, in all these papers agents are homogeneous, *ex ante*, and all heterogeneity is match-specific. Inderst and Müller (1999) is an exception that extends the standard notion of competitive search to an environment with *ex ante* heterogeneous agents. That model is a special case of our first example, and the analysis is closely related. Inderst and Wambach (2001) explore a version of the Rothschild and Stiglitz (1976) model with a finite number of principals and agents and a capacity constraint for each agent that is related to our second example. Those papers however do not develop a general framework for analyzing competitive search with adverse selection in a wide variety of applications.

Many papers have studied related economies with adverse selection but without search. Driven by the nonexistence issue in Rothschild and Stiglitz (1976), Miyazaki (1977), Wilson (1977), and Riley (1979) propose alternative notions of equilibrium that offer possible resolutions. In contrast to these papers, we generally find that there is no cross-subsidization in equilibrium. Prescott and Townsend (1984) attempt to study adverse selection in competitive economies, concluding pessimistically that “there do seem to be fundamental problems for the operation of competitive markets for economies or situations which suffer from adverse selection.” (p. 44) More recently, Bisin and Gottardi (2006) propose a notion of Walrasian equilibrium with competitive markets, but agents are restricted to trade only incentive-compatible contracts. They also find that there always exists a separating equilibrium. Although our notion of equilibrium is more strategic, their equilibrium allocation has some features that are similar. In particular, the incentive-compatibility condition that they impose on the set of admissible trades is analogous to a condition that arises endogenously in our model.

The rest of the paper is organized as follows. In Section 2, we develop the general environment, define competitive search equilibrium, and discuss the critical assumptions. In Section 3, we show how to find competitive search equilibria by solving a relatively simple constrained optimization problem. We prove that a separating equilibrium always exists and show that the equilibrium vector of agents' payoffs is unique. In Section 4, we define the class of incentive-feasible allocations, and discuss whether equilibrium outcomes are efficient within this class. In Section 5, we explore the examples and applications discussed above; in each case we characterize equilibria and discuss efficiency. In Section 6, we explore examples where we relax some of the key assumptions. Section 7 concludes.

## 2 Model

We consider a static model. There is a measure 1 of agents, a fraction  $\pi_i > 0$  of whom are of type  $i \in \{1, 2, \dots, I\}$ . The type is the agent's private information. There is also a large measure of ex ante homogeneous principals. Principals and agents have a single opportunity to match bilaterally. Let  $\mathbb{Y} \subset \mathbb{R}^n$  denote the space of feasible action profiles for principals and agents that are matched, and assume  $\mathbb{Y}$  is compact and nonempty. A typical action profile  $y \in \mathbb{Y}$  may specify actions by the principal, actions by the agent, and transfers between them, among other possibilities. A type  $i$  agent matched with a principal gets payoff  $u_i(y)$  if they undertake the action profile  $y \in \mathbb{Y}$ , while unmatched agents get a payoff normalized to zero. Assume  $u_i : \mathbb{Y} \mapsto \mathbb{R}$  is continuous and bounded for all  $i$ .

A principal may post a contract, at cost  $k$ , which gives him an opportunity to match with an agent. A principal who matches with a type  $i$  agent gets a payoff  $v_i(y) - k$  if they undertake the action profile  $y \in \mathbb{Y}$ . A principal who does not post a contract gets payoff 0, while one who posts a contract but fails to match gets  $-k$ . Assume  $v_i : \mathbb{Y} \mapsto \mathbb{R}$  is continuous and bounded for all  $i$ . We use the revelation principle to assume without loss of generality that the contracts are revelation mechanisms. More precisely, a contract is a vector of action profiles,  $C \equiv \{y_1, y_2, \dots, y_I\} \in \mathbb{Y}^I$ , specifying that if a principal and agent match, the latter (truthfully) announces her type  $i$ , and they implement  $y_i$ . A contract  $C \equiv \{y_1, y_2, \dots, y_I\}$  is incentive compatible if  $u_i(y_i) \geq u_i(y_j)$  for all  $i$ .<sup>2</sup> Let  $\mathbb{C} \subset \mathbb{Y}^I$  denote the set of incentive compatible contracts. Principals only post incentive compatible contracts.

We turn now to the matching process. Given (some) principals post a  $C \in \mathbb{C}$ , all agents observe the set of posted contracts and direct their search to the most attractive ones. Let  $\Theta(C)$  denote the ratio of principals offering contract  $C$  to agents who direct their search

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<sup>2</sup>Note that we are not concerned with moral hazard in this paper. An action profile  $y \in \mathbb{Y}$  can be implemented by any principal and agent  $i$ .

towards that contract,  $\Theta : \mathbb{C} \mapsto [0, \infty]$ . Let  $p_i(C)$  denote the share of these agents whose type is  $i$ , with  $P(C) \equiv \{p_1(C), p_2(C), \dots, p_I(C)\} \in \Delta^I$ , the  $I$ -dimensional unit simplex. That is,  $P(C)$  satisfies  $p_i(C) \geq 0$  for all  $i$  and  $\sum_i p_i(C) = 1$  and so  $P : \mathbb{C} \mapsto \Delta^I$ . These objects are determined endogenously in equilibrium.

A type  $i$  agent seeking contract  $C$  matches with a principal with probability  $\mu(\Theta(C))$ , independent of her type, where  $\mu : [0, \infty] \mapsto [0, 1]$  is nondecreasing. A principal offering contract  $C$  matches with a type  $i$  agent with probability  $\eta(\Theta(C))p_i(C)$ , where  $\eta : [0, \infty] \mapsto [0, 1]$  is nonincreasing. Note that  $\mu(\theta) = \theta\eta(\theta)$  for all  $\theta$  since the left hand side is the matching probability of an agent and the right hand side is the matching probability of a principal times the principal-agent ratio. Together with the monotonicity of  $\mu$  and  $\eta$ , this implies both functions are continuous. It is convenient to let  $\bar{\eta} \equiv \eta(0) > 0$  denote the highest probability that a principal can match with an agent, obtained when the principal-agent ratio for a contract is 0. Similarly let  $\bar{\mu} \equiv \mu(\infty) > 0$  denote the highest probability that an agent can match with a principal.

We summarize the setup by writing the expected utilities of principals and agents. The expected utility of a principal who posts  $C = \{y_1, y_2, \dots, y_I\} \in \mathbb{C}$  is

$$\eta(\Theta(C)) \sum_{i=1}^I p_i(C) v_i(y_i) - k.$$

The expected utility of a type  $i$  agent who seeks contract  $C = \{y_1, y_2, \dots, y_I\} \in \mathbb{C}$  and reports type  $j$  is

$$\mu(\Theta(C)) u_i(y_j).$$

Incentive compatibility says that the agent is willing to report truthfully,  $u_i(y_i) \geq u_i(y_j)$ .

We are now in a position to propose a generalized notion of competitive search equilibrium for environments with ex ante heterogeneous agents and asymmetric information.

**Definition 1** *A Competitive Search Equilibrium is a vector  $\bar{U} = \{\bar{U}_1, \dots, \bar{U}_I\} \in \mathbb{R}_+^I$ , a measure  $\lambda$  on  $\mathbb{C}$  with support  $\bar{\mathbb{C}}$ , a function  $\Theta(C) : \mathbb{C} \mapsto [0, \infty]$ , and a function  $P(C) : \mathbb{C} \mapsto \Delta^I$  satisfying*

(i) *principals' **profit maximization** and **free-entry**: for any incentive-compatible contract  $C = \{y_1, \dots, y_I\} \in \mathbb{C}$ ,*

$$\eta(\Theta(C)) \sum_{i=1}^I p_i(C) v_i(y_i) \leq k,$$

*with equality if  $C \in \bar{\mathbb{C}}$ ;*

(ii) *agents' optimal search*: for any  $C = \{y_1, \dots, y_I\} \in \mathbb{C}$  and  $i$  such that  $\Theta(C) < \infty$  and  $p_i(C) > 0$ ,

$$\bar{U}_i = \mu(\Theta(C))u_i(y_i),$$

where

$$\bar{U}_i = \max_{C' \in \mathbb{C}} \mu(\Theta(C'))u_i(y'_i),$$

and  $C' = \{y'_1, \dots, y'_I\}$ ;

(iii) *market clearing* :

$$\int \frac{p_i(C)}{\Theta(C)} d\lambda(\{C\}) \leq \pi_i \text{ for any } i,$$

with equality if  $\bar{U}_i > 0$ .

In our equilibrium, principals and agents take as given the expected utility of all types of agents,  $\bar{U} = \{\bar{U}_1, \dots, \bar{U}_I\}$ . Notice that  $\bar{U}_i \geq 0$  for all  $i$ , as all agents can choose not to participate and obtain their outside option 0. Moreover, principals and agents know the market tightness and the distribution of agents' types associated with each contract,  $\Theta$  and  $P$ . Given this, principals post contracts to maximize their expected profits, but by free entry profits are driven to zero. Agents search optimally for contracts. If a contract  $C$  offers type  $i$  agents less than  $\bar{U}_i$ , they will not direct their search towards that contract. Also, in equilibrium, no incentive compatible  $C$  offers type  $i$  agents more than  $\bar{U}_i$ , because then more type  $i$  agents would apply, reducing  $\Theta(C)$  and  $\bar{U}_i$ . To attract type  $i$  agents, a contract must offer exactly  $\bar{U}_i$ . Finally, the market clearing condition guarantees that agents of any type  $i$  direct their search towards some contract, unless they are indifferent between matching and their outside option,  $\bar{U}_i = 0$ .

Let us define

$$\bar{\mathbb{Y}}_i \equiv \{y \in \mathbb{Y} \mid \bar{\eta}v_i(y) \geq k \text{ and } u_i(y) > 0\},$$

the set of action profiles for type  $i$  agents that deliver positive utility to the agent while permitting the principal to make non-negative profits if the principal-agent ratio is equal to zero. Also define

$$\bar{\mathbb{Y}} \equiv \bigcup_i \bar{\mathbb{Y}}_i.$$

For much of the analysis, we make several assumptions that we now present and discuss.

**Assumption A1** *Monotonicity*: for all  $y \in \bar{\mathbb{Y}}$ ,

$$v_1(y) \leq v_2(y) \leq \dots \leq v_I(y).$$

This says that, for any fixed action profile, principals weakly prefer higher types (which is virtually a normalization, and could be relaxed at the cost of notation).

**Assumption A2** *Sorting: for all  $y \in \bar{\mathbb{Y}}$ ,  $\varepsilon > 0$ , and  $i$ , there exists a  $y' \in B_\varepsilon(y)$  such that*

$$u_j(y') > u_j(y) \text{ for all } j \geq i \text{ and } u_j(y') < u_j(y) \text{ for all } j < i,$$

where  $B_\varepsilon(y) \equiv \{y' \in \mathbb{Y} | d(y, y') < \varepsilon\}$  and  $d(y, y')$  is the Euclidean distance between the two points, so  $B_\varepsilon$  is a ball of radius  $\varepsilon$ .

This is an important assumption, guaranteeing that it is possible for principals to design contracts that attract desirable but not undesirable agents. This is a generalized version of a standard single-crossing condition. Thus, assume that  $y \equiv (y_1, y_2)$  has two dimensions, that we can ignore boundaries in the action profiles, e.g. because  $\mathbb{Y} = \mathbb{R}^2$ , and that agents' utility functions are differentiable. Then A2 holds if the marginal rate of substitution between  $y_1$  and  $y_2$  is higher for higher types,

$$\frac{\partial u_i(y_1, y_2) / \partial y_1}{\partial u_i(y_1, y_2) / \partial y_2}$$

is increasing in  $i$ . In Section 6.1 we consider an example where this fails, and show this can substantially change the nature of equilibrium.

**Assumption A3** *Gain-from-trade: for all  $i$ ,  $\bar{\mathbb{Y}}_i$  is non-empty.*

This assumption ensures that all agents are offered a contract in equilibrium. In Section 6.2 we explore an example where this fails for one type,  $\bar{\mathbb{Y}}_1 = \emptyset$ , and show that this can actually cause the market to shut down despite the presence of gains from trade.

**Assumption A4** *Local non-satiation: For all  $i$ ,  $y \in \bar{\mathbb{Y}}_i$ , and  $\varepsilon > 0$ , there exists a  $y' \in B_\varepsilon(y)$  such that  $v_i(y') > v_i(y)$ .*

This is a mild assumption, immediately satisfied in any example where the action profile  $y$  allows transfers. Our main results go through even without this assumption if we restrict attention to strictly monotonic matching functions.

It is useful to note that A3 and A4 together imply that there are points with strict gains from trade,  $y \in \mathbb{Y}$  with  $\bar{\eta}v_i(y) > k$  and  $u_i(y) > 0$ . Take any  $y \in \bar{\mathbb{Y}}_i$ . Fix  $\varepsilon > 0$  such that for all  $y' \in B_\varepsilon(y)$ ,  $u_i(y') > 0$ . Then A4 ensures that there is some  $y' \in B_\varepsilon(y)$  with  $v_i(y') > v_i(y)$  and hence  $\bar{\eta}v_i(y') > \bar{\eta}v_i(y) \geq k$  and  $u_i(y') > 0$ .

### 3 Characterization

As a first step, consider the following optimization problem for any type  $i$ :

$$\begin{aligned} & \max_{\theta \in [0, \infty], y \in \mathbb{Y}} \mu(\theta) u_i(y) && \text{(P-}i\text{)} \\ & \text{s.t. } \eta(\theta) v_i(y) \geq k, \\ & \text{and } \mu(\theta) u_j(y) \leq \bar{U}_j \text{ for all } j < i. \end{aligned}$$

We say that three vectors  $(\bar{U}_1, \dots, \bar{U}_I)$ ,  $(\theta_1, \dots, \theta_I)$ , and  $(y_1, \dots, y_I)$  solve problem (P) if for all  $i$ ,  $(\theta_i, y_i)$  solves problem (P- $i$ ) given  $(\bar{U}_1, \dots, \bar{U}_{i-1})$  and  $\bar{U}_i = \mu(\theta_i) u_i(y_i)$ . Note this implies that  $\bar{U}_i$  is the maximized value of problem (P- $i$ ).

**Lemma 1** *Assume A1-A4. There exist vectors  $(\bar{U}_1, \dots, \bar{U}_I)$ ,  $(\theta_1, \dots, \theta_I)$ , and  $(y_1, \dots, y_I)$  that solve problem (P). At a solution,*

$$\begin{aligned} & \bar{U}_i > 0 \text{ for all } i, \\ & \eta(\theta_i) v_i(y_i) = k \text{ for all } i, \\ & \mu(\theta_i) u_j(y_i) \leq \bar{U}_j \text{ for all } i, j. \end{aligned}$$

**Proof.** In the first step, we prove that there exists a solution to (P) and that  $\bar{U}_i > 0$  for all  $i$ . The second and third steps establish the rest.

**Step 1** The objective function in (P-1) is continuous in  $(\theta, y)$  because  $\mu$  and  $u_1$  are continuous. The set of  $(\theta, y)$  satisfying the constraint  $\eta(\theta) v_1(y) \geq k$  is closed because  $\eta$  and  $v_1$  are continuous. Since  $[0, \infty] \times \mathbb{Y}$  is compact, the constraint set is compact, and A3 ensures it is nonempty, containing at least  $(0, y)$  for any  $y \in \bar{\mathbb{Y}}_1$ . Then (P-1) has a solution, with maximized value  $\bar{U}_1$ . For any maximizer  $(\theta_1, y_1)$ ,  $\bar{U}_1 = \mu(\theta_1) u_1(y_1)$ .

Moreover,  $\bar{U}_1 > 0$ . Recall that A3 and A4 jointly imply that there exist points  $y \in \mathbb{Y}$  satisfying  $\bar{\eta} v_1(y) > k$  and  $u_1(y) > 0$ . Fix such a  $y$  and set  $\theta > 0$  such that  $\eta(\theta) v_1(y) \geq k$ . By construction,  $\mu(\theta) u_1(y) > 0$ , so positive values of the objective function are attainable. The maximized value may be higher still.

We now proceed by induction. Fix  $i > 1$  and assume that we have solved problem (P- $j$ ) for all  $j < i$ , in the process defining  $(\bar{U}_j, \theta_j, y_j)$  with  $\bar{U}_j > 0$ . We now solve problem (P- $i$ ) and thus define  $(\bar{U}_i, \theta_i, y_i)$ . The objective function is continuous in  $(\theta, y)$  because  $\mu$  and  $u_i$  are continuous functions. The set of  $(\theta, y)$  satisfying the constraints is compact because  $\eta$ ,  $\mu$ ,  $v_i$ , and  $u_j$  for any  $j < i$ , are all continuous functions, and  $y \in \mathbb{Y}$  which is compact by

assumption, and A3 ensures it is nonempty, containing  $(0, y)$  for any  $y \in \bar{\mathbb{Y}}_i$ . Hence (P- $i$ ) has a solution, with maximized value  $\bar{U}_i$ . For any maximizer  $(\theta_i, y_i)$ ,  $\bar{U}_i = \mu(\theta_i)u_i(y_i)$ .

Also,  $\bar{U}_i > 0$ . Recall that A3 and A4 imply that there exists  $y \in \mathbb{Y}$  satisfying  $\bar{\eta}v_i(y) > k$  and  $u_i(y) > 0$ . Fix such a  $y$  and take any  $\theta > 0$  small enough such that  $\eta(\theta)v_i(y) \geq k$  and  $\mu(\theta)u_j(y) \leq \bar{U}_j$  for all  $j < i$ . By construction,  $\mu(\theta)u_i(y) > 0$ , so positive values of the objective function are attainable. This completes the induction step and proves that there exist vectors  $(\bar{U}_1, \dots, \bar{U}_I)$ ,  $(\theta_1, \dots, \theta_I)$ , and  $(y_1, \dots, y_I)$  which solve problem (P).

**Step 2** Suppose by way of contradiction that there exists  $i$  such that  $(\theta_i, y_i)$  solves (P- $i$ ), but  $\eta(\theta_i)v_i(y_i) > k$ . This together with the fact, shown in the previous step, that  $\bar{U}_i = \mu(\theta_i)u_i(y_i) > 0$ , implies that  $y_i \in \bar{\mathbb{Y}}_i$ . Fix  $\varepsilon > 0$  such that for all  $y' \in B_\varepsilon(y_i)$ ,  $\eta(\theta_i)v_i(y') > k$  and  $u_i(y') > 0$ , so  $B_\varepsilon(y_i) \subset \bar{\mathbb{Y}}_i$ . Let  $h \in \{1, \dots, I\}$  be the largest minimizer of  $\bar{U}_j/u_j(y_i)$  with  $u_j(y_i) > 0$ . This implies in particular that

$$\frac{\bar{U}_h}{u_h(y_i)} \leq \frac{\bar{U}_i}{u_i(y_i)} = \mu(\theta_i).$$

The first inequality holds by the choice of  $h$ , while the second equality follows because  $(\bar{U}_i, \theta_i, y_i)$  is part of a solution to problem (P). If  $h < i$ , the assumption that  $h$  is the largest minimizer of  $\bar{U}_j/u_j(y_i)$  implies the first inequality is strict; however, this would violate one of the constraints in problem (P- $i$ ),  $\bar{U}_h \geq \mu(\theta_i)u_h(y_i)$ , so  $h \geq i$ .

Let  $\theta' \leq \theta_i$  solve  $\mu(\theta')u_h(y_i) = \bar{U}_h$ . Given that  $y_i \in \bar{\mathbb{Y}}_i \subset \bar{\mathbb{Y}}$ , assumption A2 ensures there exists a  $y' \in B_\varepsilon(y_i)$  such that

$$\begin{aligned} u_j(y') &> u_j(y_i) \text{ for all } j \geq h, \\ u_j(y') &< u_j(y_i) \text{ for all } j < h. \end{aligned}$$

Then the pair  $(\theta', y')$  satisfies the constraints of problem (P- $h$ ):

1.  $\eta(\theta')v_h(y') \geq \eta(\theta_i)v_i(y') > k$ , where  $\eta(\theta') \geq \eta(\theta_i)$  because  $\theta' \leq \theta_i$  and  $\eta$  is nondecreasing;  $v_h(y') \geq v_i(y')$  since  $y' \in \bar{\mathbb{Y}}_i \subset \bar{\mathbb{Y}}$  and so assumption A1 applies; and  $\eta(\theta_i)v_i(y') > k$  from the original choice of  $\varepsilon$ .
2.  $\mu(\theta')u_j(y') < \mu(\theta')u_j(y_i) = (\bar{U}_h/u_h(y_i))u_j(y_i) \leq \bar{U}_j$  for all  $j < h$ , where the first inequality comes from the choice of  $y'$ , the second equality comes the definition of  $\theta'$ , and third inequality from the choice of  $h$ :  $\bar{U}_h/u_h(y_i) \leq \bar{U}_j/u_j(y_i)$  for all  $j$ .

Moreover, the pair  $(\theta', y')$  achieves a higher value of the objective function in problem (P- $h$ )

than does  $(\theta_h, y_h)$ , given that

$$\mu(\theta')u_h(y') > \mu(\theta')u_h(y_i) = \bar{U}_h.$$

This implies that  $(\theta_h, y_h)$  does not solve (P- $h$ ), a contradiction. This proves that  $\eta(\theta_i)v_i(y_i) = k$  for all  $i$ .

**Step 3** Fix  $i$  and suppose by way of contradiction that there exists  $j > i$  such that  $\mu(\theta_i)u_j(y_i) > \bar{U}_j$ . Let  $h$  be the smallest such  $j$ . First notice that  $y_i \in \bar{Y}_i$  given that the pair  $(\theta_i, y_i)$  solves (P- $i$ ) so that  $\mu(\theta_i)u_i(y_i) > 0$  and  $\eta(\theta_i)v_i(y_i) \geq k$ . Moreover, the pair  $(\theta_i, y_i)$  satisfies the constraints of problem (P- $h$ ) since

1.  $\eta(\theta_i)v_h(y_i) \geq \eta(\theta_i)v_i(y_i) \geq k$ , where the first inequality holds by A1 because  $h > i$  and  $y_i \in \bar{Y}_i \subset \bar{Y}$ , and the second holds because  $(\theta_i, y_i)$  solves (P- $i$ );
2.  $\mu(\theta_i)u_l(y_i) \leq \bar{U}_l$  for all  $l < h$ , which holds for
  - (a)  $l < i$  because  $(\theta_i, y_i)$  satisfy the constraints of (P- $i$ ),
  - (b)  $l = i$  because  $\bar{U}_i = \mu(\theta_i)u_i(y_i)$  by problem (P),
  - (c)  $i < l < h$  by the choice of  $h$  as the smallest violation of  $\mu(\theta_i)u_j(y_i) > \bar{U}_j$ .

Since by assumption,  $\mu(\theta_i)u_h(y_i) > \bar{U}_h = \mu(\theta_h)u_h(y_h)$ ,  $(\theta_h, y_h)$  does not solve problem (P- $h$ ), a contradiction. This proves that  $\mu(\theta_i)u_j(y_i) \leq \bar{U}_j$  for all  $i$  and  $j$ . ■

The next proposition shows that a solution to problem (P) can be used to construct an equilibrium.

**Proposition 1** *Assume A1-A4. Consider three vectors  $(\bar{U}_1, \dots, \bar{U}_I)$ ,  $(\theta_1, \dots, \theta_I)$ ,  $(y_1, \dots, y_I)$  that solve problem (P). Then, there exists a Competitive Search Equilibrium  $\{\bar{U}, \lambda, \bar{C}, \Theta, P\}$  with  $\bar{C} = \{C_1, \dots, C_I\}$ , where  $C_i = (y_i, \dots, y_i)$ ,  $\Theta(C_i) = \theta_i$ , and  $p_i(C_i) = 1$ .*

**Proof.** We proceed by construction.

- For  $i \in \{1, \dots, I\}$  and  $C_i = (y_i, \dots, y_i)$ , let  $\Theta(C_i) = \theta_i$ . Otherwise, for any incentive compatible contract  $C' = \{y'_1, \dots, y'_I\} \in \mathbb{C}$ , let  $J(C') = \{j | u_j(y'_j) > 0\}$  denote the types that attain positive utility from the contract. If  $J(C') \neq \emptyset$  and  $\min_{j \in J(C')} \{\bar{U}_j / u_j(y'_j)\} < \bar{\mu}$  then

$$\mu(\Theta(C')) = \min_{j \in J(C')} \frac{\bar{U}_j}{u_j(y'_j)}.$$

If this equation has multiple solutions for  $\Theta(C')$ , as may happen if  $\mu$  is not strictly increasing, pick one, e.g. the smallest such value for  $\Theta(C')$ . Otherwise, if  $J(C') = \emptyset$  or  $\min_{j \in J(C')} \{\bar{U}_j/u_j(y'_j)\} \geq \bar{\mu}$ , then  $\Theta(C') = \infty$ .

- For  $i \in \{1, \dots, I\}$  and  $C_i = (y_i, \dots, y_i)$ , let  $p_i(C_i) = 1$  and so  $p_j(C_i) = 0$  for  $j \neq i$ . Otherwise, define  $P(C')$  such that  $p_h(C') > 0$  only if  $h \in \arg \min_{j \in J(C')} \{\bar{U}_j/u_j(y'_j)\}$ . If there are multiple elements of the arg min, pick one such value for  $P(C')$ , e.g.  $p_h(C') = 1$  if  $h$  is the smallest element of the arg min. If  $J(C') = \emptyset$ , again choose  $P(C')$  arbitrarily, e.g. set  $p_1(C') = 1$ .
- The set of posted contracts is  $\bar{\mathbb{C}} = \{C_1, \dots, C_I\}$ , where  $C_i = (y_i, \dots, y_i)$ .
- The vector of expected utilities is  $\bar{U} = \{\bar{U}_1, \dots, \bar{U}_I\}$ .
- $\lambda$  is such that, for any  $i$ ,  $\lambda(\{C_i\}) = \pi_i/\Theta(C_i)$  if  $\Theta(C_i) < \infty$  and  $\lambda(\{C_i\}) = 0$  if  $\Theta(C_i) = \infty$ .

Next we check that the proposed equilibrium satisfies the three conditions in Definition 1.

**Condition (i)** By construction, profit maximization and free entry hold for any posted contract. For any  $i$ ,  $(\theta_i, y_i)$  solve problem (P- $i$ ) and in particular satisfy the constraint  $\eta(\theta_i)v_i(y_i) \geq k$ . Lemma 1 implies that the constraint is binding.

Now consider an arbitrary incentive compatible contract; we show that principals' profit maximization and free-entry condition is satisfied. Suppose, to the contrary, that there exists an incentive compatible contract  $C' = (y'_1, \dots, y'_I) \in \mathbb{C}$  with  $\eta(\Theta(C')) \sum_i p_i(C')v_i(y'_i) > k$ . Note that this implies  $\eta(\Theta(C')) > 0$  and so  $\Theta(C') < \infty$ . In particular there exists some type  $j$  with  $p_j(C') > 0$  and  $\eta(\Theta(C'))v_j(y'_j) > k$ . Since  $p_j(C') > 0$  and  $\Theta(C') < \infty$ , our construction of  $\Theta(C')$  implies  $\bar{U}_j = \mu(\Theta(C'))u_j(y'_j)$ . Then for all  $h$ ,

$$\bar{U}_h \geq \mu(\Theta(C'))u_h(y'_h) \geq \mu(\Theta(C'))u_h(y'_j),$$

where the first inequality follows from the construction of  $\Theta$  and the second from the requirement that  $C'$  is incentive compatible. Hence  $(\Theta(C'), y'_j)$  satisfies the constraints of problem (P- $j$ ). Since  $\eta(\Theta(C'))v_j(y'_j) > k$ , Lemma 1 implies that there exists some pair  $(\theta'', y''_j)$  that satisfies the constraints of problem (P- $j$ ) but delivers a higher value of the objective function,  $\mu(\theta'')u_j(y''_j) > \bar{U}_j$ . This implies we have not solved (P- $j$ ), a contradiction.

**Condition (ii)** By construction, the equilibrium functions  $\Theta$  and  $P$  ensure that  $\bar{U}_i \geq \mu(\Theta(C'))u_i(y'_i)$  for all contracts  $C' = \{y'_1, \dots, y'_I\}$ , with equality if  $\Theta(C') < \infty$  and  $p_i(C') > 0$ .

Moreover, problem (P) ensures  $\bar{U}_i = \mu(\theta_i)u_i(y_i)$  where  $\theta_i = \Theta(C_i)$  and  $C_i = \{y_i, \dots, y_i\}$  is the equilibrium contract offered to type  $i$  agents.

**Condition (iii)** The market clearing condition is satisfied by the construction of  $\lambda$ . ■

The next proposition shows that any equilibrium can be characterized using problem (P).

**Proposition 2** *Assume A1-A4. If  $\{\bar{U}, \lambda, \bar{C}, \Theta, P\}$  is a competitive search equilibrium, then for any  $C \equiv \{y_1, y_2, \dots, y_I\} \in \bar{C}$  with  $p_i(C) > 0$ ,  $(\Theta(C), y_i)$  solves problem (P- $i$ ) and  $\bar{U}_i = \mu(\Theta(C))u_i(y_i)$ .*

**Proof.** We prove this in four steps. First, we prove that the constraint  $\eta(\Theta(C))v_i(y_i) \geq k$  in problem (P- $i$ ) is satisfied. Second, we prove that the constraint  $\mu(\Theta(C))u_j(y_j) \leq \bar{U}_j$  in problem (P- $i$ ) is satisfied for all  $j$ . Third, we prove that  $(\Theta(C), y_i)$  solves problem (P- $i$ ). Finally, we prove that such a pair  $(\Theta(C), y_i)$  actually delivers  $\bar{U}_i$  to type  $i$ .

**Step 1** We start by proving that the constraint  $\eta(\Theta(C))v_i(y_i) \geq k$  is satisfied in problem (P- $i$ ). To find a contradiction, suppose that there is a  $C \in \bar{C}$  with  $p_i(C) > 0$  for some  $i$  and  $\eta(\Theta(C))v_i(y_i) < k$ . The first part of the definition of equilibrium implies  $\eta(\Theta(C))\sum_j p_j(C)v_j(y_j) = k$ , and so there is a type  $h$  with  $p_h(C) > 0$  and  $\eta(\Theta(C))v_h(y_h) > k$ . This implies in particular that  $\Theta(C) < \infty$ . Since  $\eta(\Theta(C)) \leq \bar{\eta}$ ,  $\bar{\eta}v_h(y_h) > k$  as well. Moreover, because  $p_h(C) > 0$ , optimal search implies  $\mu(\Theta(C))u_h(y_h) = \bar{U}_h$ . Since  $\bar{U}_h > 0$ ,  $u_h(y_h) > 0$  as well. It follows that  $y_h \in \bar{Y}_h \subset \bar{Y}$ .

Next, fix  $\varepsilon > 0$  such that for all  $y \in B_\varepsilon(y_h)$ ,  $\eta(\Theta(C))v_h(y) > k$ . Then assumption A2 together with  $y_h \in \bar{Y}$  guarantees that there exists  $y' \in B_\varepsilon(y_h)$  such that

$$\begin{aligned} u_j(y') &> u_j(y_h) \text{ for all } j \geq h, \\ u_j(y') &< u_j(y_h) \text{ for all } j < h. \end{aligned}$$

Notice that  $y' \in \bar{Y}_h$  as well, given that  $u_h(y') > u_h(y_h) > 0$  and  $\bar{\eta}v_h(y') \geq \eta(\Theta(C))v_h(y') > k$ .

Next consider the contract  $C' = \{y', \dots, y'\}$ . Note that

$$\mu(\Theta(C'))u_h(y') \leq \bar{U}_h = \mu(\Theta(C))u_h(y_h) < \mu(\Theta(C))u_h(y'),$$

where the first inequality uses the optimal search condition for contract  $C'$ , the second equality holds by optimal search because  $\Theta(C) < \infty$  and  $p_h(C) > 0$ , and the last inequality holds by the construction of  $y'$ . Since  $\mu$  is nondecreasing, this implies  $\Theta(C') < \Theta(C)$ .

Next observe that for all  $j < h$ ,

$$\mu(\Theta(C'))u_j(y') < \mu(\Theta(C))u_j(y_h) \leq \mu(\Theta(C))u_j(u_j) \leq \bar{U}_j,$$

where the first inequality uses  $\Theta(C') < \Theta(C)$  and the construction of  $y'$ , the second uses incentive compatibility of  $C$ , and the third inequality follows from the optimal search condition for contract  $C$ . Thus  $p_j(C') = 0$  for all  $j < h$ .

Finally, the profits from posting contract  $C'$  are

$$\eta(\Theta(C')) \sum_j p_j(C')v_j(y') \geq \eta(\Theta(C'))v_h(y') \geq \eta(\Theta(C))v_h(y') > k.$$

The first inequality follows because  $p_j(C') = 0$  if  $j < h$  and  $v_h(y')$  is increasing in  $h$  by assumption A1 together with  $y' \in \bar{\mathbb{Y}}$ . The second follows because  $\Theta(C') < \Theta(C)$  and  $\eta$  is non-increasing. The last inequality uses the construction of  $\varepsilon$ . Offering the contract  $C'$  is therefore strictly profitable, a contradiction.

**Step 2** We next prove that the constraint  $\mu(\Theta(C))u_j(y_i) \leq \bar{U}_j$  is satisfied for all  $j$ . Optimal search implies  $\mu(\Theta(C))u_j(y_j) \leq \bar{U}_j$  while incentive compatibility implies  $u_j(y_i) \leq u_j(y_j)$ .

**Step 3** Next we prove that  $(\Theta(C), y_i)$  solves problem (P- $i$ ). We have proved it satisfies the constraints in this problem, so to find a contradiction, suppose there exists  $(\theta', y')$  with  $\eta(\theta')v_i(y') \geq k$ ,  $\mu(\theta')u_j(y') \leq \bar{U}_j$  for all  $j < i$ , and  $\mu(\theta')u_i(y') > \bar{U}_i$ . Note that  $\eta(\infty) = 0$  implies  $\theta' < \infty$ . Also, the last two inequalities imply

$$\frac{\bar{U}_i}{u_i(y')} < \mu(\theta') \leq \frac{\bar{U}_j}{u_j(y')}$$

for all  $j < i$ . Notice that  $y' \in \bar{\mathbb{Y}}_i$ , because  $\mu(\theta')u_i(y') > \bar{U}_i > 0$  and  $\bar{\eta}v_i(y') \geq \eta(\theta')v_i(y') \geq k$ . Fix  $\varepsilon > 0$  such that for all  $y \in B_\varepsilon(y')$  and for all  $j < i$ ,

$$\frac{\bar{U}_i}{u_i(y)} < \min \left\{ \mu(\theta'), \frac{\bar{U}_j}{u_j(y)} \right\}.$$

We now use assumption A4. Given that  $y' \in \bar{\mathbb{Y}}_i$  we can choose  $y'' \in B_\varepsilon(y')$  such that  $v_i(y'') > v_i(y')$ . Notice that  $y'' \in \bar{\mathbb{Y}}_i$ : monotonicity of  $\eta$  implies  $\bar{\eta}v_i(y'') > \eta(\theta')v_i(y') \geq k$  and by construction  $\mu(\theta')u_i(y'') > \bar{U}_i > 0$ .

Now consider the contract  $C'' = \{y'', \dots, y''\}$ . Optimal search implies

$$\mu(\Theta(C'')) \leq \frac{\bar{U}_i}{u_i(y'')}.$$

Since by construction  $\bar{U}_i/u_i(y'') < \mu(\theta')$ , the pair of inequalities and monotonicity of  $\mu$  imply  $\Theta(C'') < \theta'$ . Moreover, using  $\bar{U}_i/u_i(y'') < \bar{U}_j/u_j(y'')$ , optimal search implies  $p_j(C'') = 0$  for all  $j < i$ . Then assumption A1 together with  $y'' \in \bar{Y}_i$  and monotonicity of  $\eta$  imply that the profits of a principal posting  $C''$  are

$$\eta(\Theta(C'')) \sum_j p_j(C'') v_j(y'') \geq \eta(\theta') v_i(y'') > k,$$

where the first inequality holds because  $\eta$  is decreasing,  $\Theta(C'') < \theta'$ , and this contract attracts type  $i$  or better agents; and the last inequality holds by construction. Then posting the contract  $C''$  is strictly profitable, a contradiction, completing the step.

**Step 4** Finally, we need to prove that for any  $C = \{y_1, \dots, y_I\} \in \bar{\mathbb{C}}$  with  $p_i(C) > 0$ , the pair  $(\Theta(C), y_i)$  that solves problem (P- $i$ ) actually delivers  $\bar{U}_i$  to type  $i$ . Principals' profit maximization implies that if  $C \in \bar{\mathbb{C}}$ ,  $\eta(\Theta(C)) > 0$  and hence  $\Theta(C) < \infty$ . Agents' optimal search then implies  $\bar{U}_i = \mu(\Theta(C)) u_i(y_i)$ , completing the proof. ■

The next proposition is basically a corollary of the preceding results.

**Proposition 3** *Assume A1-A4 hold. Then competitive search equilibrium exists. Moreover, the equilibrium vector  $\bar{U}$  is unique.*

**Proof.** Lemma 1 shows that, under A1-A4, there exists a solution for problem (P- $i$ ) for all  $i$ . Moreover, Proposition 1 shows that, under the same assumptions, if the vectors  $(\bar{U}_1, \dots, \bar{U}_I)$ ,  $(\theta_1, \dots, \theta_I)$ , and  $(y_1, \dots, y_I)$  solve problem (P), there exists a Competitive Search Equilibrium  $\{\bar{U}, \lambda, \bar{\mathbb{C}}, \Theta, P\}$  with the same  $\bar{U}$ ,  $\bar{\mathbb{C}} = \{C_1, \dots, C_I\}$ , where  $C_i = (y_i, \dots, y_i)$ ,  $\Theta(C_i) = \theta_i$ , and  $p_i(C_i) = 1$ . This proves the first claim of the corollary. Moreover, Proposition 2 shows that if a Competitive Search Equilibrium  $\{\bar{U}, \lambda, \bar{\mathbb{C}}, \Theta, P\}$  exists, then  $\bar{U}_i$  is the maximum value of the objective of problem (P- $i$ ) for all  $i$ , and Lemma 1 shows also that there exists a unique maximum value  $\bar{U}_i$  for the objective of problem (P- $i$ ) for all  $i$ . The second claim of the corollary immediately follows. ■

## 4 Incentive Feasible Allocations

In this section we set the stage for studying the efficiency properties of equilibrium. In particular, we define an incentive feasible allocation, which is necessary to look for Pareto improvements. An allocation is a vector of expected utilities for the different types of agents, a set of posted contracts, and the associated market tightness and composition of agents applying for the posted contracts. It is incentive feasible whenever: (1) each posted contract offers the maximal expected utility to agents who direct their search for that contract and no more to those who do not; (2) the economy's resource constraint is satisfied; and (3) markets clear.

**Definition 2** *An allocation is a vector  $\bar{U}$  of expected utilities for the agents, a measure  $\lambda$  over the set of incentive-compatible contracts  $\mathbb{C}$ , with support  $\bar{\mathbb{C}}$ , a function  $\tilde{\Theta} : \bar{\mathbb{C}} \mapsto [0, \infty]$ , and a function  $\tilde{P} : \bar{\mathbb{C}} \mapsto \Delta^I$ .*

**Definition 3** *An allocation  $\{\bar{U}, \lambda, \bar{\mathbb{C}}, \tilde{\Theta}, \tilde{P}\}$  is incentive feasible if*

1. *for any  $C \in \bar{\mathbb{C}}$  and  $i \in \{1, \dots, I\}$  such that  $\tilde{p}_i(C) > 0$  and  $\tilde{\Theta}(C) < \infty$ ,*

$$\bar{U}_i = \mu(\tilde{\Theta}(C))u_i(y_i),$$

*and*

$$\bar{U}_i \equiv \max_{C' \in \bar{\mathbb{C}}} \mu(\tilde{\Theta}(C'))u_i(y'_i)$$

*where  $C' = \{y'_1, \dots, y'_I\}$ ;*

- 2.

$$\int \left( \eta(\tilde{\Theta}(C)) \sum_{i=1}^I \tilde{p}_i v_i(y_i) - k \right) d\lambda(C) = 0;$$

3. *for all  $i \in \{1, \dots, I\}$ ,*

$$\int \frac{\tilde{p}_i(C)}{\tilde{\Theta}(C)} d\lambda(C) \leq \pi_i,$$

*with equality if  $\bar{U}_i > 0$ .*

## 5 Examples

### 5.1 Schooling

Our first example shows that in competitive search equilibrium principals can separate agents by distorting directly the terms of the posted contracts in a version of the classic signaling

model.<sup>3</sup> Agents here can be thought of as workers that are heterogeneous both in terms of their productivity and cost of producing a signal, say schooling. Principals can be thought of as firms that are willing to pay more to hire more productive workers, but cannot observe productivity, only schooling. We show that if the cost of schooling is lower for more productive workers, firms will use schooling as a costly signal to separate workers.

An action profile here consists of two elements,  $y = \{t, x\}$ , where  $t$  denotes a transfer from the firm to the worker and  $x \geq 0$  denotes a costly signal that the worker must send. The payoff of a matched worker of type  $i$  who undertakes action  $\{t, x\}$  is

$$u_i(t, x) = t - \frac{x}{a_i},$$

where higher values of  $a_i$  imply that the signal  $x$  is less costly. The payoff of a firm matched with a type  $i$  agent who takes action  $\{t, x\}$  is

$$v_i(t, x) = b_i - t,$$

where  $b_i$  is the productivity of the worker.

Assume  $\mu$  is strictly concave and continuously differentiable. Also assume that  $I = 2$  and, without loss of generality, that type 2 workers are more productive than type 1 workers:  $b_2 > b_1$ . We restrict the set of feasible action profiles to  $\mathbb{Y} = \{(t, x) | t \in [0, b_2] \text{ and } x \in [0, b_2 \max\{a_1, a_2\}]\}$ . The firm would never profit from offering a transfer higher than  $b_2$  and the worker would never accept a negative transfer. Moreover, a worker would never provide a signal higher than  $b_2 \max\{a_1, a_2\}$ , given that the transfer is bounded by  $b_2$ . With these restrictions, the space of action profiles that provide positive utility to a type  $i$  worker and nonnegative profit to a firm when the firms/workers ratio is zero are

$$\bar{\mathbb{Y}}_i = \{(t, x) | x/a_i < t \leq b_i - k/\bar{\eta}\}.$$

The fact that  $b_2 > b_1$  immediately implies that assumption A1 is satisfied. Assumption A3 is satisfied as long as  $\bar{\eta}b_1 > k$ , which we assume in what follows. Assumption A4 holds because  $\{t, x\} \in \bar{\mathbb{Y}}_i$  implies  $t > 0$  and so can be reduced to raise  $v_i(t, x)$ .

The critical assumption is A2. Consider points  $\{t, 0\} \in \bar{\mathbb{Y}}$ . There are nearby points  $(t', x')$  with  $u_1(t', x') < u_1(t, 0)$  and  $u_2(t', x') > u_2(t, 0)$  if and only if  $a_2 > a_1$ , so the more productive worker finds it less costly to signal her type. At any other point  $\{t, x\} \in \bar{\mathbb{Y}}$ , assumption A2 holds for all values of  $a_1$  and  $a_2$ . We therefore impose  $a_2 > a_1$  in what follows and discuss the nature of equilibrium if this assumption is violated in Section 4.

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<sup>3</sup>This example is the same as Inderst and Müller (1999), where  $a_i = b_i$ .

Using the results in Section 3, we can characterize a competitive search equilibrium using vectors  $(\bar{U}_1, \bar{U}_2)$ ,  $(t_1, x_1, t_2, x_2)$ , and  $(\theta_1, \theta_2)$  that solve problem (P). In this example, problem (P-i) is

$$\begin{aligned} \bar{U}_i &= \max_{\theta \in [0, \infty], (t, x) \in \mathbb{Y}} \mu(\theta) \left( t - \frac{x}{a_i} \right) \\ \text{s.t. } &\eta(\theta)(b_i - t) \geq k, \\ \text{and } &\mu(\theta) \left( t - \frac{x}{a_j} \right) \leq \bar{U}_j \text{ for } j \leq i. \end{aligned}$$

We claim the following result:

**Result 1** *There exists a unique competitive search equilibrium with  $\mu'(\theta_i)b_i = k$ , for  $i = 1, 2$ , and so  $\theta_1 < \theta_2$ ;  $x_1 = 0$ ,*

$$x_2 = \frac{a_1}{\mu(\theta_2)} \left[ \left( \frac{\mu(\theta_2)}{\mu'(\theta_2)} - \theta_2 \right) - \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) \right] k > 0;$$

and

$$t_i = \left( \frac{1}{\mu'(\theta_i)} - \frac{\theta_i}{\mu(\theta_i)} \right) k$$

for  $i = 1, 2$ . Moreover,

$$\bar{U}_1 = \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) k \text{ and } \bar{U}_2 = \left[ \frac{a_1}{a_2} \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) + \left( 1 - \frac{a_1}{a_2} \right) \left( \frac{\mu(\theta_2)}{\mu'(\theta_2)} - \theta_2 \right) \right] k.$$

We prove this result by first solving problem (P-1), finding  $\bar{U}_1$  and then using it to solve problem (P-2).

Consider problem (P-1). Using  $\eta(\theta) = \mu(\theta)/\theta$ , we write it as

$$\bar{U}_1 = \max_{\theta \in [0, \infty], (t, x) \in \mathbb{Y}} \mu(\theta) \left( t - \frac{x}{a_1} \right) \text{ s.t. } \frac{\mu(\theta)}{\theta}(b_1 - t) \geq k.$$

Proposition 1 implies that the constraint is binding so that we can eliminate  $t$  and reduce the problem to

$$\bar{U}_1 = \max_{\theta \in [0, \infty], x \in [0, b_2 a_2]} \mu(\theta) \left( b_1 - \frac{x}{a_1} \right) - \theta k.$$

It is easy to see that at the optimum  $x = 0$  and  $\theta = \theta_1$ , where  $\theta_1$  solves the necessary and sufficient first order condition,  $\mu'(\theta_1)b_1 = k$ . Using this to eliminate  $b_1$  from the objective function delivers the expression for  $\bar{U}_1$ . Also use the constraint to solve for  $t_1$ .

Next, solve problem (P-2) using the solution for  $\bar{U}_1$ , that is,

$$\begin{aligned}\bar{U}_2 &= \max_{\theta \in [0, \infty], (t, x) \in \mathbb{Y}} \mu(\theta) \left( t - \frac{x}{a_2} \right) \\ \text{s.t. } & \frac{\mu(\theta)}{\theta} (b_2 - t) \geq k, \\ \text{and } & \mu(\theta) \left( t - \frac{x}{a_1} \right) \leq \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) k.\end{aligned}$$

Again using Proposition 1, the first constraint binds and so we can eliminate  $t$ . It is easy to verify that the second constraint must be binding as well, so we can use it to eliminate  $x$  and then check that at the solution  $x \geq 0$ . The problem reduces to

$$\bar{U}_2 = \max_{\theta \in [0, \infty]} \left[ \frac{a_1}{a_2} \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) k + \left( 1 - \frac{a_1}{a_2} \right) (\mu(\theta)b_2 - \theta k) \right].$$

Then, at the solution  $\theta = \theta_2$ , where  $\theta_2$  solves the necessary and sufficient first order condition,  $\mu'(\theta_2)b_2 = k$ . Note that concavity of  $\mu$  implies  $\theta_1 < \theta_2$ . Substituting into the objective function gives the expression for  $\bar{U}_2$ . Finally, use the constraints to compute  $t_2$  and  $x_2$ . Again, concavity of  $\mu$  ensures that  $\mu(\theta)/\mu'(\theta) - \theta$  is increasing in  $\theta$ , which in turn implies  $x_2 > 0$ .

In equilibrium, some firms post contracts to attract only type-1 workers and others post contracts to attract only type-2 workers. In order to separate the two types of workers, firms have to offer distorted contracts to workers of type 2, requiring a costly signal, while the contracts offered to the type-1 workers are undistorted. Two features of this example merit mention. First, the market tightness margin is efficient for both types of contracts, unaffected by private information or costly signaling. This is because the worker bears the full cost of the distortion. Second, without additional assumptions, we cannot rule out the possibility that the transfer to type-2 workers is lower than the transfer to type-1 workers. This is in addition to the costly signal. Their reward for applying to firms offering type 2 contracts is only a higher probability of trade,  $\theta_2^* > \theta_1^*$ .

We now propose an incentive feasible allocation that, depending on parameters, may Pareto dominate the equilibrium. Consider an allocation where the contracts are pooling, that is, treat the two types of workers identically,  $t_1 = t_2 = t$  and  $x_1 = x_2 = 0$ . Thus, consider an allocation with  $\bar{C} = \{C\}$ , where  $C = ((t, 0), (t, 0))$ . Moreover, assume  $\tilde{\Theta}(C) = \theta^*$ , with  $\theta^*$  solving  $\mu'(\theta^*)(\pi_1 b_1 + \pi_2 b_2) = k$ ,  $\tilde{p}_i(C) = \pi_i$ , and  $\lambda(\{C\}) = 1/\theta^*$ . Note that this defines  $\theta_1 < \theta^* < \theta_2$ , an intermediate level of market tightness. Finally, choose  $t$  such that firms

make zero profit, that is,

$$t = \left( \frac{1}{\mu'(\theta^*)} - \frac{\theta^*}{\mu(\theta^*)} \right) k$$

The contract is incentive compatible since the types are treated identically. Moreover, all workers are attracted to the posted contracts and get the same expected utility, so condition (1) of feasibility is satisfied. Also, the resource constraint and the market clearing condition are satisfied by the choice of  $t$  and  $\lambda$ . Hence, it constitutes an incentive feasible allocation.

The expected utility of (both types of) workers is now

$$\bar{U} = \left( \frac{\mu(\theta^*)}{\mu'(\theta^*)} - \theta^* \right) k.$$

Compare this with the equilibrium. Since  $\theta^* > \theta_1$ , trivially  $\bar{U} > \bar{U}_1$ . On the other hand,  $\bar{U} \geq \bar{U}_2$  if and only if

$$\frac{\mu(\theta^*)}{\mu'(\theta^*)} - \theta^* \geq \frac{a_1}{a_2} \left( \frac{\mu(\theta_1)}{\mu'(\theta_1)} - \theta_1 \right) + \left( 1 - \frac{a_1}{a_2} \right) \left( \frac{\mu(\theta_2)}{\mu'(\theta_2)} - \theta_2 \right).$$

This holds if  $a_1/a_2$  is sufficiently close to 1 (screening is very costly) or if  $\pi_1$  is sufficiently close to zero (there are a few type-1 workers). The reason is that in equilibrium, firms who want to attract type 2 need to screen out type 1 agents. If a firm failed to do so, he would be swamped by type 1 workers. This may not be socially optimal, however. If there are few type 1 workers, or screening is costly, it is socially optimal to subsidize type 1 workers and eliminate costly screening.

## 5.2 Insurance

The next example is closer to the original Rothschild and Stiglitz (1976) environment, where risk neutral principals offer insurance contracts to risk averse agents who are heterogeneous in their probability of experiencing a loss. This example illustrates several features of our environment. For one thing, we do not require that utility is quasi-linear. Also, we do not require that there are search frictions, but instead we can have everyone on the short side of the market match. And, even when a pooling contract does not Pareto dominate the equilibrium, a partial pooling allocation, where only some types of agents are pooled together, may be Pareto superior.

To be concrete, we again imagine worker-firm matches, where the productivity of a match is initially unknown. Some workers (agents) are more likely to be productive than others, but firms (principals) can only verify the ex post realization of productivity, not a worker's type. More precisely, a type  $i$  worker produces one unit of output with probability  $p_i$  and 0

otherwise. A contract specifies the worker's consumption conditional on whether the pair can produce. Workers are risk averse and firms risk neutral. In the absence of adverse selection, the marginal utility of consumption would be equalized across states. By incompletely insuring workers against the risk of being unproductive, a firm can keep undesirable workers from directing their search toward a particular contract (which may provide an explanation for why firms do not insure workers against layoff risk).

Now an action profile consists of a pair of consumption levels,  $y = \{c_e, c_u\}$ . The payoff of a matched type  $i$  worker who undertakes action profile  $\{c_e, c_u\}$  is

$$u_i(c_e, c_u) = p_i U(c_e) + (1 - p_i)U(c_u),$$

where  $p_1 < p_2 < \dots < p_I < 1$  and  $U : [\underline{c}, \infty) \rightarrow \mathbb{R}$  is increasing and strictly concave with  $\lim_{c \rightarrow \underline{c}} U(c) = -\infty$  for some  $\underline{c} < 0$  and  $U(0) = 0$ . The payoff of a firm matched with a type  $i$  worker who undertakes action profile  $\{c_e, c_u\}$  is

$$v_i(c_e, c_u) = p_i(1 - c_e) - (1 - p_i)c_u.$$

To ensure that A1 is satisfied, we restrict the set of feasible action profiles to  $\mathbb{Y} = \{(c_e, c_u) | c_u + 1 \geq c_e \geq \underline{c} \text{ and } c_u \geq \underline{c}\}$ . Finally, assume  $\bar{\eta}p_1 > k$ , which guarantees that there are gains from trade for all types, so A3 is satisfied; for example, set  $c_e = c_u = p_i - k/\bar{\eta} > 0$ .

The assumption that  $\lim_{c \rightarrow \underline{c}} U(c) = -\infty$  ensures that action profiles of the form  $\{c_e, \underline{c}\}$  yield negative utility for all types and so are not in  $\bar{\mathbb{Y}}$ . To verify A2, consider an incremental increase in  $c_e$  to  $c_e + dc_e$  and an incremental reduction in  $c_u$  to  $c_u - dc_u$  for some  $dc_e > 0$  and  $dc_u > 0$ . For a type- $i$  worker, this raises utility by approximately  $p_i u'(c_e)dc_e - (1 - p_i)u'(c_u)dc_u$ , which is positive if and only if

$$\frac{dc_e}{dc_u} > \frac{1 - p_i}{p_i} \frac{u'(c_u)}{u'(c_e)}.$$

Since  $(1 - p_i)/p_i$  is decreasing in  $i$ , an appropriate choice of  $dc_e/dc_u$  yields an increase in utility if and only if  $j \geq i$ , which verifies A2. Finally, since a reduction in  $c_u$  raises  $v_i(y)$  and is feasible, A4 is satisfied. Our propositions therefore apply.

To illustrate that search frictions are not essential for our results, assume that the number of matches is determined by the short side of the market,  $\mu(\theta) = \min\{\theta, 1\}$ . This assumption allows us to focus on the risk of layoffs. Then we have:

**Result 2** *There exists a competitive search equilibrium with  $\theta_i = 1$  for all  $i$ ;  $c_{e,1} = c_{u,1} =$*

$p_1 - k$ ; and for  $i > 1$ ,  $c_{e,i} > c_{e,i-1}$  and  $c_{u,i} < c_{u,i-1}$  solve

$$p_i(1 - c_{e,i}) - (1 - p_i)c_{u,i} = k$$

and  $p_{i-1}U(c_{e,i}) + (1 - p_{i-1})U(c_{u,i}) = p_{i-1}U(c_{e,i-1}) + (1 - p_{i-1})U(c_{u,i-1})$ .

**Proof.** We start by solving problem P-1:

$$\bar{U}_1 = \max_{\theta \in [0, \infty], (c_e, c_u) \in \mathbb{Y}} \min\{\theta, 1\} (p_1 U(c_e) + (1 - p_1) U(c_u))$$

s.c.  $\min\{1, \theta^{-1}\} (p_1(1 - c_e) - (1 - p_1)c_u) \geq k$ .

At the solution,  $c_{u,1} = c_{e,1} = c_1$  and so this reduces to

$$\bar{U}_1 = \max_{\theta \in [0, \infty], c \geq c} \min\{\theta, 1\} U(c)$$

s.c.  $\min\{1, \theta^{-1}\} (p_1 - c) \geq k$ .

Eliminate consumption  $c$  by substituting the binding constraint into the objective function:

$$\bar{U}_1 = \max_{\theta \in [0, \infty]} \min\{\theta, 1\} U(p_1 - \max\{1, \theta\}k).$$

The solution is attained at  $\theta_1 = 1$ , delivering  $\bar{U}_1 = U(p_1 - k)$ . Hence, the equilibrium consumption of type-1 workers  $c_{e,1} = c_{e,2} = p_1 - k$ .

Turn next to a typical problem P- $i$ :

$$\bar{U}_i = \max_{\theta \in [0, \infty], (c_e, c_u) \in \mathbb{Y}} \min\{\theta, 1\} (p_i U(c_e) + (1 - p_i) U(c_u))$$

s.c.  $\min\{1, \theta^{-1}\} (p_i(1 - c_e) - (1 - p_i)c_u) \geq k$

and  $\min\{\theta, 1\} (p_j U(c_e) + (1 - p_j) U(c_u)) \leq \bar{U}_j$  for all  $j < i$ .

We claim first that the solution to this problem again ensures  $\theta_i = 1$ . If  $\theta_i > 1$ , reducing  $\theta_i$  to 1 relaxes the first constraint without otherwise affecting the solution to the problem. If  $\theta_i < 1$ , consider the following variation: raise  $\theta_i$  to 1 and increase  $c_e$  and reduce  $c_u$  while keeping both  $\theta(p_i U(c_e) + (1 - p_i) U(c_u))$  and  $p_i c_e + (1 - p_i)c_u$  unchanged. That is, the perturbed consumption levels  $c_e > c_{e,i}$  and  $c_u < c_{u,i}$  are defined by

$$\theta_i (p_i U(c_{e,i}) + (1 - p_i) U(c_{u,i})) = p_i U(c_e) + (1 - p_i) U(c_u)$$

and  $p_i c_{e,i} + (1 - p_i)c_{u,i} = p_i c_e + (1 - p_i)c_u$ .

By construction, this does not affect the value of the objective function nor the first constraint. Suppose it fails to relax one of the remaining constraints for a given  $j < i$ , then it must be that

$$\theta_i(p_j U(c_{e,i}) + (1 - p_j)U(c_{u,i})) \leq p_j U(c_e) + (1 - p_j)U(c_u).$$

Multiply this together with  $p_i U(c_e) + (1 - p_i)U(c_u) = \theta_i(p_i U(c_{e,i}) + (1 - p_i)U(c_{u,i}))$  and simplify to obtain

$$(p_i - p_j)(U(c_{e,i})U(c_u) - U(c_e)U(c_{u,i})) \geq 0.$$

But since  $p_i > p_j$ ,  $c_{e,i} < c_e$ , and  $c_u < c_{u,i}$ , this is a contradiction. Thus the perturbation relaxes each of these constraints. We may therefore without loss of generality focus on the problem with  $\theta_i = 1$ :

$$\begin{aligned} \bar{U}_i &= \max_{(c_e, c_u) \in \mathbb{Y}} (p_i U(c_e) + (1 - p_i)U(c_u)) \\ \text{s.c. } &p_i(1 - c_e) - (1 - p_i)c_u \geq k \\ &\text{and } p_j U(c_e) + (1 - p_j)U(c_u) \leq \bar{U}_j \text{ for all } j < i. \end{aligned}$$

Next we claim that the solution to this problem has  $c_{e,i} > c_{e,i-1}$  and  $c_{u,i} < c_{u,i-1}$ . To prove this, note that setting  $c_e = c_{e,i-1}$  and  $c_u = c_{u,i-1}$  is feasible from the construction of problem P- $(i - 1)$ :

$$\begin{aligned} \bar{U}_{i-1} &= p_{i-1}U(c_{e,i-1}) + (1 - p_{i-1})U(c_{u,i-1}), \\ p_{i-1}(1 - c_{e,i-1}) - (1 - p_{i-1})c_{u,i-1} &= k \\ \text{and } p_j U(c_{e,i-1}) + (1 - p_j)U(c_{u,i-1}) &\leq \bar{U}_j \text{ for all } j < i - 1. \end{aligned}$$

Moreover, it is not optimal because it implies that the first constraint in problem P- $i$  is slack:  $p_i(1 - c_{e,i-1}) - (1 - p_i)c_{u,i-1} > p_{i-1}(1 - c_{e,i-1}) - (1 - p_{i-1})c_{u,i-1} = k$ . Relative to this, setting  $c_{e,i} \leq c_{e,i-1}$  and  $c_{u,i} \leq c_{u,i-1}$ , with one inequality strict, reduces the value of the objective function and so is not optimal. Setting  $c_{e,i} \geq c_{e,i-1}$  and  $c_{u,i} \geq c_{u,i-1}$  with one inequality strict would violate the constraint for  $i - 1$  and so is not feasible. Setting  $c_{e,i} \leq c_{e,i-1}$  and  $c_{u,i} \geq c_{u,i-1}$  with one inequality strict may be feasible, but only when it reduces the value of the objective function. That is, we use the requirements that the deviation does not attract

type  $i - 1$  workers and that it delivers higher utility to type  $i$  workers:

$$\begin{aligned} p_{i-1}U(c_{e,i-1}) + (1 - p_{i-1})U(c_{u,i-1}) &\geq p_{i-1}U(c_{e,i}) + (1 - p_{i-1})U(c_{u,i}) \\ p_iU(c_{e,i-1}) + (1 - p_i)U(c_{u,i-1}) &\leq p_iU(c_{e,i}) + (1 - p_i)U(c_{u,i}). \end{aligned}$$

Multiply these together and simplify algebraically to obtain  $(p_i - p_{i-1})(U(c_{e,i})U(c_{u,i-1}) - U(c_{e,i-1})U(c_{u,i})) \geq 0$ , a contradiction. This proves that  $c_{e,i} > c_{e,i-1}$  and  $c_{u,i} < c_{u,i-1}$ .

Finally, we claim that for all  $i > 1$ , the equilibrium consumption specified by type- $i$  contracts satisfies two binding constraints, firms earn zero profits and type  $i - 1$  workers are indifferent about applying for type- $i$  contracts,

$$\begin{aligned} p_i(1 - c_{e,i}) - (1 - p_i)c_{u,i} &= k \\ \text{and } p_{i-1}U(c_{e,i}) + (1 - p_{i-1})U(c_{u,i}) &= \bar{U}_{i-1}. \end{aligned}$$

In particular, these equations will typically exhibit two solutions; we look for the unique solution with  $c_{e,i} > p_1 - k > c_{u,i}$ . It is straightforward to prove that these equations must bind for  $i = 2$ . For any  $i > 2$ , suppose we have established this result for type  $i - 1$  and proceed by induction. In particular, we know that

$$\begin{aligned} p_{i-1}U(c_{e,i-1}) + (1 - p_{i-1})U(c_{u,i-1}) &= \bar{U}_{i-1}, \\ p_{i-2}U(c_{e,i-1}) + (1 - p_{i-2})U(c_{u,i-1}) &= \bar{U}_{i-2}, \end{aligned}$$

The first equation comes from the construction of  $\bar{U}_{i-1}$ , the second from the assumption that type  $i - 2$  workers are indifferent about applying for type  $i - 1$  contracts. Suppose that for some  $j < i - 2$

$$p_jU(c_{e,i}) + (1 - p_j)U(c_{u,i}) = \bar{U}_j.$$

Substituting for  $\bar{U}_j$  using the optimal search condition for  $j < i - 2$  gives

$$p_jU(c_{e,i}) + (1 - p_j)U(c_{u,i}) \geq p_jU(c_{e,i-1}) + (1 - p_j)U(c_{u,i-1}).$$

Combining this last inequality with the constraint for type  $i - 1$ ,

$$p_{i-1}U(c_{e,i}) + (1 - p_{i-1})U(c_{u,i}) \leq \bar{U}_{i-1},$$

we obtain  $(p_{i-1} - p_j)(U(c_{e,i-1})U(c_{u,i}) - U(c_{e,i})U(c_{u,i-1})) \geq 0$ . Since  $p_{i-1} > p_j$  for all  $j < i - 2$ , and given that  $c_{e,i} > c_{e,i-1}$ ,  $c_{u,i} < c_{u,i-1}$ , we obtain a contradiction. Instead, the constraint

for type  $i - 1$  must bind and the remaining constraints are slack, completing the proof. ■

An interesting feature of a competitive search equilibrium is that  $c_{u,i}$  may be negative, so a worker is worse off in a bad match than without a match at all. If one thinks of a bad match as a layoff, an optimal contract may give a worker lower utility if she is laid off than if she never gets a job.

We now show the equilibrium need not be efficient. Observe that a worker with  $p_i$  close to 1 suffers little from the distortions introduced by adverse selection. At the extreme, if  $p_I = 1$ , setting  $c_{u,I} = \underline{c}$  excludes all other workers without distorting the type  $I$  contract. More generally, adverse selection has the biggest impact on the utility of workers with an intermediate value of  $p_i$ . A Pareto improvement may therefore require only partial pooling.

To be concrete, suppose  $p_1 = 1/4$ ,  $p_2 = 1/2$ , and  $p_3 = 3/4$  and there are equal numbers of type 1 and type 3 workers, so half of all matches are productive. Also assume  $U(c) = \log(1 + c)$  and  $k = 1/8$ . Then in a competitive search equilibrium,  $c_{e,1} = c_{u,1} = 0.125$  and  $\bar{U}_1 = U(0.125)$ ;  $c_{e,2} = 0.786$ ,  $c_{u,2} = -0.036$ , and  $\bar{U}_2 = U(0.312)$ ; and  $c_{e,3} = 0.858$ ,  $c_{u,3} = -0.073$ , and  $\bar{U}_3 = U(0.561)$ . Pooling all three types, the best incentive-feasible allocation sets  $c_e = c_u = 3/8$  and  $\bar{U}_i = U(3/8)$ , since half of all matches are productive. This reduces the utility of a type-3 agent.

Instead consider an allocation that pools type 1 and type 2 workers. If there are sufficiently few type 1 workers, it is feasible to set consumption at  $c_e = c_u > 0.312$ , delivering utility greater than  $\bar{U}_2$  to both type 1 and 2. For example, suppose  $\pi_1 = \pi_3 = 0.1$  and  $\pi_2 = 0.8$ . Then the utility of type 1 and 2 rises to  $U(25/72) = U(0.347)$ . By raising the utility of type 2, it is easier to exclude them from type 3 contracts, reducing the requisite inefficiency of those contracts. This raises the utility of those workers, in this case to  $U(0.573)$ .

This example is similar to the model of Rothschild and Stiglitz (1976, p. 630), where they “consider an individual who will have income of size  $W$  if he is lucky enough to avoid accident. In the event an accident occurs, his income will be only  $W - d$ . The individual can insure against this accident by paying to an insurance company a premium  $\alpha_1$  in return for which he will be paid  $\hat{\alpha}_2$  if an accident occurs. Without insurance his income in the two states, ‘accident,’ ‘no accident,’ was  $(W, W - d)$ ; with insurance it is now  $(W - \alpha_1, W - d + \alpha_2)$  where  $\alpha_2 = \hat{\alpha}_2 - \alpha_1$ .” We can always normalize the utility of an uninsured individual to zero and then express the utility of an individual who anticipates an accident with probability  $p_i$  as

$$u_i(\alpha_1, \alpha_2) = p_i U(W - \alpha_1) + (1 - p_i) U(W - d + \alpha_2) - \kappa_i,$$

where  $\kappa_i \equiv p_i U(W) + (1 - p_i) U(W - d)$ . Setting  $W = d = 1$  and defining  $c_e = 1 - \alpha_1$  and  $c_u = \alpha_2$ , this is equivalent to our example, except for a level shift in the utility function.

Still, our characterization result above carries through to this environment as long as there are gains from trade for each type,  $U(p_i - k) > p_i U(1) + (1 - p_i)U(0)$  for all  $i$ .

Rothschild and Stiglitz (1976) prove that in any equilibrium, principals who attract type  $i$  agents,  $i > 1$ , offer incomplete insurance so as to deter type  $i - 1$  agents. Under some conditions, however, such an equilibrium might not exist. Starting from this configuration of contracts, a principal may consider deviating by offering a pooling contract that attracts multiple types of agents. This is profitable if the least cost separating contract is Pareto inefficient.

Such a deviation is never profitable in our environment. In Rothschild and Stiglitz (1976), a deviating principal can attract and serve all the agents in the economy, or at least a representative cross-section. This is not the case in our model, where a principal cannot serve all the agents who are potentially attracted to a contract. Instead, they are rationed through the endogenous movement in market tightness  $\theta$ . Whether such a deviation is profitable depends on which agents are most willing to accept a decline in market tightness. In this model, high type agents will quickly give up on the pooling contract if it is too crowded with low type agents. Low type agents, who have a lower outside option,  $\bar{U}_{i-1} < \bar{U}_i$ , are more persistent. A principal who tries to offer a pooling contract will end up with a long queue of type-1 agents, the worst possible outcome.

### 5.3 Asset Markets

A feature of the previous examples is that market tightness is not distorted:  $\theta_i$  is at the constrained efficient level for any  $i$ , maximizing  $\mu(\theta)b_i - \theta k$ . We now consider an example where principals may use tightness to screen out undesirable types. Although all of our results hold more generally, to stress the point, we assume  $\mu(\theta) = \min\{\theta, 1\}$ , so matching is determined by the short side of the market and  $\bar{\eta} = 1$ . In this case, without any private information,  $\theta$  would typically be equal to 1.

Consider an asset market where sellers (agents) have private information about the value of their asset. Although buyers (principals) always value an asset more than the seller does, some sellers' assets are more valuable than others. Market tightness, or probabilistic trading, is a useful screening device since sellers who hold a more valuable asset are more willing to accept a low probability of trade at a given price. Thus, this example shows how an illiquid market may serve as a useful screening device when asset holders have private information about asset values.

We assume that each type- $i$  buyer is endowed with one indivisible object, say an apple, of type  $i$ , with value  $a_i^A > 0$  for the buyer and  $a_i^P > 0$  for the seller. Each seller is endowed with

a different indivisible object, say a banana, which has value  $b$  for both buyers and sellers. The action profile in a contract for type  $i$  buyers consists of a pair  $\{\alpha_i, \beta_i\}$ , where  $\alpha_i$  is the probability that the buyer gives the seller the apple and  $\beta_i$  is the probability that the seller gives the buyer the banana. The payoff of a matched type  $i$  buyer who reports to be a type  $j$  buyer is

$$u_i(\alpha_j, \beta_j) = \beta_j b - \alpha_j a_i^A,$$

while the payoff of a seller matched with a type  $i$  buyer who reports truthfully is

$$v_i(\alpha_i, \beta_i) = \alpha_i a_i^P - \beta_i b.$$

We set  $I = 2$  and impose a number of restrictions on payoffs. First, both sellers and buyers prefer type 2 apples and both types of buyers like apples:

$$a_2^A > a_1^A \geq 0 \text{ and } a_2^P > a_1^P \geq 0.$$

Second, there would be are gains from trade, including the cost of posting, if the seller is sure to trade. Also, bananas are valuable enough that sellers will not give up their entire endowment for an apple:

$$a_i^A + k < a_i^P < b + k \text{ for } i = 1, 2.$$

The available action profiles are  $\mathbb{Y} = [0, 1] \times [0, 1]$ , with  $\bar{\mathbb{Y}}_i = \{(\alpha, \beta) | \alpha a_i^A < \beta b \leq \alpha a_i^P - k\}$ . Using these restrictions, we verify our four assumptions. As a preliminary step, note that  $(\alpha, \beta) \in \bar{\mathbb{Y}}_i$  implies  $\alpha > k/(a_i^P - a_i^A) > 0$  and  $\beta > k a_i^A / b(a_i^P - a_i^A) > 0$ , so in any equilibrium contract, trades are bounded away from zero. If  $\alpha$  were too close to 0, sellers would be unwilling to post contracts. But then  $\beta$  must be large enough for buyers to be willing to apply for contracts.

Since  $\alpha > 0$  whenever  $(\alpha, \beta) \in \bar{\mathbb{Y}}_i$ , the restriction  $a_1^P < a_2^P$  ensures A1 holds. The restriction  $a_i^A + k < a_i^P < b + k$  guarantees that  $(\alpha, \beta) = (1, (a_i^P - k)/b) \in \bar{\mathbb{Y}}_i$  for all  $i$ , and so A3 holds. A4 holds because for any  $(\alpha, \beta) \in \bar{\mathbb{Y}}_i$ , a movement to  $(\alpha, \beta - \varepsilon)$  with  $\varepsilon > 0$  is feasible and raises the seller's utility.

The important assumption is again A2, here guaranteed by the restriction that  $a_1^A < a_2^A$ . Fix  $(\alpha, \beta) \in \bar{\mathbb{Y}}$  and  $\gamma \in (a_1^A/b, a_2^A/b)$ . For arbitrary  $\delta > 0$ , consider the action profile  $(\alpha', \beta') = (\alpha - \delta, \beta - \gamma\delta)$ . Such an action profile is feasible for sufficiently small  $\delta$  because  $(\alpha, \beta) \in \bar{\mathbb{Y}}$  guarantees that  $\alpha > 0$  and that  $\beta > 0$ . By construction,

$$u_2(\alpha', \beta') - u_2(\alpha, \beta) = \delta(a_2^A - \gamma b) > 0$$

and

$$u_1(\alpha', \beta') - u_1(\alpha, \beta) = \delta(a_1^A - \gamma b) < 0.$$

Now for fixed  $\varepsilon > 0$ , choose  $\delta \leq \varepsilon/\sqrt{1 + \gamma^2}$ . This ensures  $(\alpha', \beta') \in B_\varepsilon(\alpha, \beta)$  and so assumption A2 holds.

**Result 3** *There exists a unique competitive search equilibrium with  $\alpha_i = 1$ ,  $\beta_i = (a_i^P - k)/b$ ,  $\theta_1 = 1$ ,  $\bar{U}_1 = a_1^P - a_1^A - k$ ,*

$$\theta_2 = \frac{a_1^P - a_1^A - k}{a_2^P - a_1^A - k}, \quad \text{and } \bar{U}_2 = \frac{(a_2^P - a_2^A - k)(a_1^P - a_1^A - k)}{a_2^P - a_1^A - k}.$$

**Proof.** We characterize the competitive search equilibrium using problem (P). Write problem (P-1) as

$$\begin{aligned} \bar{U}_1 &= \max_{\theta \in [0, \infty], (\alpha, \beta) \in \mathbb{Y}} \min\{\theta, 1\}(\beta b - \alpha a_1^A) \\ \text{s.t. } &\min\{1, \theta^{-1}\}(\alpha a_1^P - \beta b) \geq k. \end{aligned}$$

Proposition 1 ensures that the constraint is binding, so that we can use it to eliminate  $\beta$  and rewrite the problem as

$$\begin{aligned} \bar{U}_1 &= \max_{\theta \in [0, \infty], \alpha \in [0, 1]} \min\{\theta, 1\} \alpha (a_1^P - a_1^A) - \theta k \\ \text{s.t. } &\alpha a_1^P - k \max\{\theta, 1\} \in [0, b], \end{aligned}$$

where the constraint ensures  $\beta \in [0, 1]$ . Temporarily ignoring this constraint, the solution is to set  $\alpha = 1$ . Then the assumption that  $a_1^P > a_1^A + k$  implies it is optimal to set  $\theta = 1$  as well. And the assumption that  $b + k > a_1^P$  ensures that the constraint is satisfied. We conclude from this analysis that  $\bar{U}_1 = a_1^P - a_1^A - k$ .

Next we turn to problem (P-2):

$$\begin{aligned} \bar{U}_2 &= \max_{\theta \in [0, \infty], (\alpha, \beta) \in \mathbb{Y}} \min\{\theta, 1\}(\beta b - \alpha a_2^A) \\ \text{s.t. } &\min\{1, \theta^{-1}\}(\alpha a_2^P - \beta b) \geq k \\ &\min\{\theta, 1\}(\beta b - \alpha a_1^A) \leq a_1^P - a_1^A - k. \end{aligned}$$

One can again prove that both constraints are binding. Eliminating  $b$  from the problem

using the first constraint gives

$$\begin{aligned}\bar{U}_2 &= \max_{\theta \in [0, \infty], \alpha \in [0, 1]} \min\{\theta, 1\} \alpha (a_2^P - a_2^A) - \theta k \\ \text{s.t. } &\alpha a_2^P - k \max\{\theta, 1\} \in [0, b] \\ &\min\{\theta, 1\} \alpha (a_2^P - a_1^A) - \theta k = a_1^P - a_1^A - k.\end{aligned}$$

Use the last constraint to eliminate  $\alpha$ :

$$\begin{aligned}\bar{U}_2 &= \max_{\theta \in [0, \infty]} \frac{a_1^P - a_1^A - (1 - \theta)k}{a_2^P - a_1^A} (a_2^P - a_2^A) - \theta k \\ \text{s.t. } &\frac{a_1^P - a_1^A - (1 - \theta)k}{\min\{\theta, 1\} (a_2^P - a_1^A)} a_2^P - k \max\{\theta, 1\} \in [0, b] \\ \text{and } &\frac{a_1^P - a_1^A - (1 - \theta)k}{\min\{\theta, 1\} (a_2^P - a_1^A)} \in [0, 1].\end{aligned}$$

We include the constraints to remember that  $\alpha$  and  $\beta$  are probabilities, lying between 0 and 1. Since by assumption  $a_1^A < a_2^A < a_2^P$ , the objective function is decreasing in  $\theta$ . We thus set  $\theta$  equal to the smallest value consistent with the two constraints, that is

$$\theta_2 = \frac{a_1^P - a_1^A - k}{a_2^P - a_1^A - k} < 1.$$

This implies  $\alpha_2 = 1$ , so the second constraint binds; the first constraint reduces to  $a_2^P - k \in [0, b]$ , which we have assumed holds. The value of the program is

$$\bar{U}_2 = \frac{(a_2^P - a_2^A - k)(a_1^P - a_1^A - k)}{a_2^P - a_1^A - k}.$$

In the absence of private information, we would have  $\theta_2 = 1$  and  $\bar{U}_2 = a_2^P - a_2^A - k$ . Relative to this benchmark, sellers post too few contracts designed to attract type 2 buyers, so that many of them fail to match. Since type 2 hold better apples than type 1 buyers, they are more willing to accept a lower matching probability in return for more bananas when they do match. Note that the obvious alternative, setting  $\theta_2 = 1$  but rationing through the probability of exchange,  $\alpha_2 < 1$ , is more costly because it involves creating more contracts at cost  $k$  per contract. Reducing the meeting rate is a more cost-effective rationing mechanism than directly rationing trades in meetings.

We again ask whether there is a feasible allocation that Pareto dominates the equilibrium. Consider the allocation in which only a pooling contract is posted, with  $\alpha_1 = \alpha_2 = 1$  and

$\beta_1 = \beta_2 = \beta$ . That is,  $\bar{Y} = \{C\}$ , where  $C = ((1, \beta), (1, \beta))$ . Moreover,  $\tilde{\Theta}(C) = 1$ ,  $\tilde{p}_i(C) = \pi_i$ , and  $\lambda(\{C\}) = 1$ . Finally, set  $\beta = (\pi_1 a_1^P + \pi_2 a_2^P - k)/b$ . The choice of  $\beta$  ensures that the resource constraint holds and the choice of  $\lambda$  ensures that markets clear. All the buyers apply to the same contract so they receive trivially their maximum possible utility. Hence, the allocation is incentive feasible. The expected payoff for type  $i$  buyers is

$$\bar{U}_i = \pi_1 a_1^P + \pi_2 a_2^P - a_i^A - k,$$

for  $i = 1, 2$ . Since  $a_1^P < a_2^P$ , type-1 buyers are always better off than in equilibrium. Type 2 buyers are better off if and only if

$$\pi_1 a_1^P + \pi_2 a_2^P - a_2^A - k > \frac{(a_2^P - a_2^A - k)(a_1^P - a_1^A - k)}{a_2^P - a_1^A - k}.$$

Since  $\pi_2 = 1 - \pi_1$ , this reduces to

$$\pi_1 < \frac{a_2^P - a_2^A - k}{a_2^P - a_1^A - k}.$$

By assumption both the numerator and denominator are positive, but the numerator is smaller because  $a_2^A > a_1^A$ . Thus type 2 buyers prefer the pooling allocation only if there is sufficiently little cross subsidization, so  $\pi_1$  is small. The cost of cross subsidizing type 1 buyers then does not offset the benefit, the increased efficiency of trade.

## 6 Relaxing Assumptions

We have imposed four assumptions in our analysis of equilibrium. As discussed above, A1 and A4 are not crucial, but A2 and A3 play a substantive role in the analysis. We now explore variations on our examples where these assumptions are relaxed. If the sorting condition A2 fails, there may be no means of screening the good agents, and so a natural competitive search equilibrium will involve pooling. We show that other less efficient equilibria may exist where the screening technology is wastefully used. Assumption A3 ensures that all agents are offered a contract. If it fails, the impossibility of offering a contract to some agents may eliminate the gains from trade with other more desirable agents.

### 6.1 Pooling

Consider the same example described in Section 5.1, but assume that signaling is cheaper for the less productive workers,  $a_1 \geq a_2$  while  $b_1 < b_2$ . As we show below, this violates

assumption A2. Firms would like to screen out low productivity workers, but the only available screening technology works against them. If a firm posts a contract with the desire to attract only the more productive workers, the less productive ones would report to be more productive. We show that in this case there is a class of equilibria in which firms attract both types of workers and may prescribe the less productive workers to send a costly signal with different level of intensity. The most efficient (and natural) of these equilibria is the one where firms offer the same action profiles to all the workers and do not use the screening technology at all. Notice that, in this case, to characterize the equilibrium we cannot use the analysis in Section 3, which relies on assumption A2, but we need to go back to the primitive definition 1 of a competitive search equilibrium.

As in Section 5.1, the terms of trade prescribed to type  $i$  by some contract  $C \in \mathbb{C}$  consists of two elements,  $y_i = (t_i, x_i)$ , where  $t_i$  denotes a transfer from the firm to a type- $i$  worker and  $x_i \geq 0$  denotes a costly signal that a type- $i$  worker must send. The payoff of a matched worker of type  $i$  who reports type  $j$  is  $u_i(y_j) = t_j - x_j/a_i$ , and the payoff of a firm matched with a type  $i$  worker who reports his type truthfully is  $v_i(y_i) = b_i - t_i$ . Assume that  $I = 2$  and that type 2 workers are the more productive ones, that is,  $b_2 > b_1$ , while  $a_1 \geq a_2$ , so type 1 workers find screening less costly. Also, assume  $\mathbb{Y} = \{(t, x) | t \in [0, b_2] \text{ and } x \in [0, b_2 a_1]\}$ , so both the firms' and workers' payoffs are continuous and bounded.

Following our analysis in Section 5.1, it is straightforward to verify assumptions A1, A3, and A4. However, assumption A2 is violated. Fix  $t$  and set  $x = 0$ . For any nearby contract  $(t', x')$ ,

$$u_1(t', x') - u_1(t, 0) = t' - t - x'/a_1 \geq t' - t - x'/a_2 = u_2(t', x') - u_2(t, 0),$$

since  $x' \geq 0$ . It follows that there is no such value of  $(t', x')$  with  $u_1(t', x') < u_1(t, 0)$  and  $u_2(t', x') > u_2(t, 0)$ .

We claim that there exists a class of equilibria indexed by  $x_1 \in [0, a_1(b_2 - b_1)(1 - \pi_1)/\pi_1]$ . In equilibrium, all firms post the same contract  $C = \{(t + x_1/a_1, x_1), (t, 0)\}$ , where  $t$  is chosen to ensure that firms make zero profits. Moreover, given that all the posted contracts are the same, all types of workers look for the same contracts, and  $p_i(C) = \pi_i$ .

A firm might consider offering a contract that attracts only one type of worker. If he tries to attract only type 1 workers, he will lose the benefit of cross-subsidization and thus is unable to attract them while earning positive profits. If he tries to attract type 2 workers, he will be unable to devise a contract that will exclude type-1 workers, again making such a deviation unprofitable.

**Result 4** *Suppose  $a_1 \geq a_2$ . For any  $x_1 \in [0, a_1(b_2 - b_1)(1 - \pi_1)/\pi_1]$ , there exists a Competitive*

Search Equilibrium where  $\bar{\mathbb{C}} = \{C\}$  with  $C = \{(t + x_1/a_1, x_1), (t, 0)\}$  and

$$t = \pi_1 \left( b_1 - \frac{x_1}{a_1} \right) + \pi_2 b_2 - \frac{\theta}{\mu(\theta)} k,$$

where  $\theta$  solves

$$\mu'(\theta) \left( \pi_1 \left( b_1 - \frac{x_1}{a_1} \right) + \pi_2 b_2 \right) = k.$$

Moreover, the expected utility of both types of workers is

$$\bar{U} = \mu(\theta)t.$$

**Proof.** Fix  $x_1 \leq a_1(b_2 - b_1)(1 - \pi_1)/\pi_1$ . Our proof proceeds by constructing an equilibrium. Assume that  $\bar{\mathbb{C}} = \{C\}$  where  $C = \{(t + x_1/a_1, x_1), (t, 0)\}$ ,  $t$  and  $\theta$  are defined above, and  $\bar{U}_1 = \bar{U}_2 = \mu(\theta)t$  and  $\lambda(\{C\}) = 1/\theta$ . Moreover,  $\Theta(C) = \theta$  and  $p_i(C) = \pi_i$  for  $i = 1, 2$ . For any other incentive compatible contract  $C' = \{(t'_1, x'_1), (t'_2, x'_2)\} \in \bar{\mathbb{C}}$ ,  $C' \neq C$ , suppose  $\Theta(C')$  solves

$$\bar{U}_1 = \mu(\Theta(C')) \left( t'_1 - \frac{x'_1}{a_1} \right)$$

if this defines  $\Theta(C') < \infty$ ; if  $\bar{U} \geq \bar{\mu}(t'_1 - x'_1/a_1)$ ,  $\Theta(C') = \infty$ . Finally, suppose  $p_1(C') = 1$  and  $p_2(C') = 0$  for all such contracts.

By construction, profit maximization and free entry hold for the posted contract  $C$ . In particular,  $t$  is chosen so that firms break even.

For any other incentive-compatible contract  $C' \neq C$ , workers' optimal search holds by construction for type-1 workers. For type-2 workers, we need to verify that

$$\bar{U}_2 \geq \mu(\Theta(C')) \left( t'_2 - \frac{x'_2}{a_2} \right).$$

To prove this, note that  $t'_1 - x'_1/a_1 \geq t'_2 - x'_2/a_1 \geq t'_2 - x'_2/a_2$ , where the first inequality comes from incentive compatibility of  $C'$  and the second from the assumption  $a_1 \geq a_2$  and the feasibility restriction  $x'_2 \geq 0$ . This implies that

$$\mu(\Theta(C')) \left( t'_2 - \frac{x'_2}{a_2} \right) \leq \mu(\Theta(C')) \left( t'_1 - \frac{x'_1}{a_1} \right) = \bar{U}_1,$$

which establishes the desired inequality since  $\bar{U}_1 = \bar{U}_2$ .

Next, the firm's profit maximization and free entry condition for an incentive-compatible contract  $C' \neq C$  reduces to

$$\eta(\Theta(C'))(b_1 - t'_1) \leq k.$$

Since  $\eta(\infty) = 0$ , this is obviously satisfied for contracts with  $\Theta(C') = \infty$ . Otherwise, use the construction of  $\Theta(C')$  to eliminate  $t'_1$  from this requirement. We need to show that

$$\mu(\Theta(C')) \left( b_1 - \frac{x'_1}{a_1} \right) - \Theta(C')k \leq \bar{U}_1.$$

An upper bound on the left hand side is obtained by setting  $x'_1 = 0$  and choosing  $\Theta(C')$  to maximize the left hand side,  $\mu'(\Theta(C'))b_1 = k$ . The restriction  $x_1 \leq a_1(b_2 - b_1)(1 - \pi_1)/\pi_1$  implies that  $b_1 \leq \pi_1 \left( b_1 - \frac{x_1}{a_1} \right) + \pi_2 b_2$ , from which it follows that  $\Theta(C') \leq \theta$ . That is,

$$\mu(\Theta(C')) \left( b_1 - \frac{x'_1}{a_1} \right) - \Theta(C')k \leq \left( \frac{\mu(\Theta(C'))}{\mu'(\Theta(C'))} - \Theta(C') \right) k \leq \left( \frac{\mu(\theta)}{\mu'(\theta)} - \theta \right) k = \bar{U}_1,$$

where the first inequality uses the preceding discussion, the second inequality holds because  $\Theta(C') \leq \theta$ , and the third holds from the construction of  $\bar{U}_1$ .

Finally, the market clearing condition holds by construction. ■

This result shows that there exists a class of Competitive Search Equilibria parameterized by  $x_1$ , the extent to which low productivity workers are forced to use the costly signal. In this example, firms would like to be able to screen the more productive type 2 workers, but the assumption  $a_1 \leq a_2$  implies that they can use the costly signal only to attract less productive type 1 workers. Still, there are equilibria where the signal is used because firms fear that if they did not require signalling from type 1 workers, they would be stuck exclusively with that type of worker.

The costly signal is socially wasteful, so these equilibria can be Pareto ranked. In particular, the “pooling” equilibrium characterized by  $x_1 = 0$  is Pareto optimal, at least within this class. In such an equilibrium, all the firms offer the same contract that prescribes the same action profile to all the workers:  $x = 0$  and a transfer which ensures that the firms break even.

Next, we show that when  $a_1$  is strictly larger than  $a_2$ , then any Competitive Search Equilibrium where all the firms post the same contract falls in the class of equilibria characterized in Result 4.

**Result 5** *Suppose  $a_1 > a_2$ . If there exists a Competitive Search Equilibrium with  $\bar{C} = \{C\}$  where  $C = ((t_1, x_1), (t_2, x_2))$ ,  $p_i(C) = \pi_i$ , and  $\Theta(C) < \infty$ , then  $x_2 = 0$ , and  $t_1 - x_1/a_1 = t_2$ .*

**Proof.** Throughout this proof, we suppose there is an equilibrium characterized by the single incentive compatible contract  $C = ((t_1, x_1), (t_2, x_2))$ . In the first step we prove that  $x_2 = 0$  and in the second that  $t_1 - x_1/a_1 = t_2$ .

**Step 1** Suppose  $x_2 > 0$ . Given that  $p_i(C) = \pi_i$  and  $\Theta(C) < \infty$ , optimal search requires that the expected utility of type  $i$  workers satisfies

$$\bar{U}_i = \mu(\Theta(C)) \left( t_i - \frac{x_i}{a_i} \right),$$

for  $i = 1, 2$ . Next, consider the contract  $C' = ((t_2 - x_2/a_2, 0), (t_2 - x_2/a_2, 0))$ . Optimal search of type 2 workers requires that

$$\mu(\Theta(C')) \left( t_2 - \frac{x_2}{a_2} \right) \leq \bar{U}_2 = \mu(\Theta(C)) \left( t_2 - \frac{x_2}{a_2} \right),$$

which implies that  $\Theta(C') \leq \Theta(C)$ . Moreover, notice that

$$t_1 - \frac{x_1}{a_1} \geq t_2 - \frac{x_2}{a_1} > t_2 - \frac{x_2}{a_2},$$

where the first inequality follows from incentive compatibility of  $C$ , and the second from the fact that  $a_1 > a_2$  and  $x_2 > 0$ . This, together with  $\Theta(C') \leq \Theta(C)$ , implies that

$$\bar{U}_1 = \mu(\Theta(C)) \left( t_1 - \frac{x_1}{a_1} \right) > \mu(\Theta(C')) \left( t_2 - \frac{x_2}{a_2} \right).$$

It follows that type 1 workers will never search for  $C'$ , that is,  $p_1(C') = 0$ . Hence, given that  $\Theta(C') \leq \Theta(C) < \infty$ , it must be that  $p_2(C') = 1$  and  $\Theta(C') = \Theta(C)$ . Then, the expected profits for a firm posting  $C'$  are

$$\eta(\Theta(C')) \left( b_2 - t_2 + \frac{x_2}{a_2} \right) > \eta(\Theta(C)) (\pi_1(b_1 - t_1) + \pi_2(b_2 - t_2)) = k.$$

The first inequality follows from  $b_1 < b_2$ ; the fact shown above that  $t_1 > t_2 - x_2/a_2 + x_1/a_1 \geq t_2 - x_2/a_2$ ; and  $\Theta(C') = \Theta(C)$ . The second equality follows from the fact that in the proposed equilibrium firms post  $C$  and break even. Hence, contract  $C'$  represents a profitable deviation, a contradiction.

**Step 2** We now prove that  $t_1 - x_1/a_1 = t_2$ . Notice that incentive compatibility of  $C$  and the result from the previous step that  $x_2 = 0$  imply that  $t_1 - x_1/a_1 \geq t_2$ . To derive a contradiction, suppose that  $t_1 - x_1/a_1 > t_2$ . Consider a contract  $C' = ((t_2 + x_1/a_1, x_2), (t_2, 0))$ . Then

$$\mu(\Theta(C')) t_2 \leq \bar{U}_2 = \mu(\Theta(C)) t_2,$$

where the first inequality follows from optimal search of type-2 workers for  $C'$  and the second equality from optimal search of the same workers for  $C$  together with the assumption that  $p_2(C) = \pi_2 > 0$ . This implies that  $\Theta(C') \leq \Theta(C)$ . Hence,

$$\bar{U}_1 = \mu(\Theta(C)) \left( t_1 - \frac{x_1}{a_1} \right) > \mu(\Theta(C')) t_2,$$

where the first equality follows from optimal search of type 1 workers for  $C$  and the assumption that  $p_1(C) = \pi_1 > 0$  and the second inequality comes from  $\Theta(C') \leq \Theta(C)$  and the assumption that  $t_1 - x_1/a_1 > t_2$ . Hence, it must be that  $p_1(C') = 0$ , and, given that  $\Theta(C') \leq \Theta(C) < \infty$ ,  $p_2(C') = 1$  and  $\Theta(C') = \Theta(C)$ . It follows that the expected profits for a firm offering  $C'$  are

$$\eta(\Theta(C'))(b_2 - t_2) \geq \eta(\Theta(C))(\pi_1(b_1 - t_1) + \pi_2(b_2 - t_2)) = k,$$

given that  $\Theta(C') = \Theta(C)$  and  $b_2 - t_2 > b_1 - (t_2 + x_1/a_1) > b_1 - t_1$ , where the last inequality follows from assumption. This shows that  $C'$  represents a profitable deviation, a contradiction. ■

## 6.2 No Trade

Now, consider the same example described in Section 5.3, with the only difference that there are no gains from trade for type 1 buyers, that is, assumption A3 does not hold. More specifically, we continue to assume that

$$a_2^A > a_1^A \geq 0 \text{ and } a_2^P > a_1^P \geq 0.$$

For type 2 buyers, we assume that there are gains from trade, including the cost of posting, if the seller is sure to trade with a buyer. We also assume that bananas are valuable enough that sellers will not give up their entire endowment for an apple:

$$a_2^A + k < a_2^P < b + k.$$

For type 1 buyers, however, there are no gains from trade:

$$a_1^P \leq a_1^A + k < b + k.$$

For convenience of notation, we express the matching function in the general form  $\mu(\theta)$  with  $\bar{\eta} = 1$  and note that the results apply in the special case where  $\mu(\theta) = \min\{1, \theta\}$  that we

analyzed before.

First, we prove that in any Competitive Search Equilibrium it must be that  $\bar{U}_1 = \bar{U}_2 = 0$ . It is worth noting that if we solve problem (P) in this case, we obtain these maximized values.

**Result 6** *In any Competitive Search Equilibrium  $\bar{U}_1 = \bar{U}_2 = 0$ .*

**Proof.** In the first step we show that  $\bar{U}_1 = 0$  and in the second step that  $\bar{U}_2 = 0$ .

**Step 1** Suppose that  $\bar{U}_1 > 0$ . Then there must be some contract  $C = ((\alpha_1, \beta_1), (\alpha_2, \beta_2))$  with  $\Theta(C) < \infty$ ,  $p_1(C) > 0$ , and

$$\bar{U}_1 = \mu(\Theta(C))(\beta_1 b - \alpha_1 a_1^A).$$

We first prove that  $p_2(C) > 0$  as well. Suppose  $p_1(C) = 1$ . Then, the zero profit condition implies that

$$\mu(\Theta(C))(\alpha_1 a_1^P - \beta_1 b) - \Theta(C)k = 0.$$

Summing together the two conditions above we obtain

$$\bar{U}_1 = \mu(\Theta(C))\alpha_1(a_1^P - a_1^A) - \Theta(C)k \leq 0,$$

where the last inequality comes straight from the assumption that  $a_1^P \leq a_1^A + k$ ,  $\mu(\Theta(C)) \leq \Theta(C)$ , and  $\alpha_1 \leq 1$ . This contradicts  $\bar{U}_1 > 0$ , and hence it must be that  $p_2(C) > 0$ .

Next we prove that  $\alpha_2 > 0$  and  $\beta_2 > 0$ . First suppose  $\alpha_2 = 0$ . Optimal search and profit maximization imply that

$$\begin{aligned} \bar{U}_1 &= \mu(\Theta(C))(\beta_1 b - \alpha_1 a_1^A), \\ \bar{U}_2 &= \mu(\Theta(C))\beta_2 b, \\ \mu(\Theta(C))[(\alpha_1 a_1^P - \beta_1 b)p_1(C) - \beta_2 b(1 - p_1(C))] - \Theta(C)k &= 0. \end{aligned}$$

By summing up the last condition with a weighted average of the first two, with weights respectively  $p_1(C)$  and  $1 - p_1(C)$ , we obtain

$$p_1(C)\bar{U}_1 + (1 - p_1(C))\bar{U}_2 = \mu(\Theta(C))\alpha_1 p_1(C)(a_1^P - a_1^A) - \Theta(C)k \leq 0,$$

where the last inequality comes again from the assumption that  $a_1^P \leq a_1^A + k$ . But since  $\bar{U}_1 > 0$  and  $\bar{U}_2 \geq 0$ , this is a contradiction, which proves  $\alpha_2 > 0$ . Now suppose  $\beta_2 = 0$ . Then  $\bar{U}_2 = \mu(\Theta(C))(-\alpha_2 a_2^A) < 0$ , a contradiction.

To summarize what we have proved so far, take any contract  $C = ((\alpha_1, \beta_1), (\alpha_2, \beta_2))$  that is offered in equilibrium with  $\Theta(C) < \infty$ , and that attracts type 1 buyers. Then it also attracts type 2 buyers and it has  $\alpha_2 > 0$  and  $\beta_2 > 0$ . It must also involve some cross-subsidization. That is,

$$\eta(\theta)(\alpha_1 a_1^P - \beta_1 b) < k < \eta(\theta)(\alpha_2 a_2^P - \beta_2 b)$$

The first inequality follows from the assumption that  $a_1^P \leq a_1^A + k$  together with  $\bar{U}_1 > 0$  which implies  $\beta_1 b > \alpha_1 a_1^A$ , while the second follows because sellers must break even on average.

We now find a profitable deviation. Consider a contract  $C' = ((\alpha', \beta'), (\alpha', \beta'))$  satisfying three conditions:

$$\begin{aligned} \beta_2 b - \alpha_2 a_2^A &< \beta' b - \alpha' a_2^A \\ \beta_2 b - \alpha_2 a_1^A &> \beta' b - \alpha' a_1^A \\ (\alpha_1 a_1^P - \beta_1 b)p_1(C) + (\alpha_2 a_2^P - \beta_2 b)p_2(C) &< \alpha' a_2^P - \beta' b. \end{aligned}$$

The first two conditions can be expressed as  $(\beta_2 - \beta')b/a_1^A > \alpha_2 - \alpha' > (\beta_2 - \beta')b/a_2^A$ . Since  $a_1^A < a_2^A$ , this implies in particular that  $\alpha' < \alpha_2$  and  $\beta' < \beta_2$ , and so it is feasible. The last inequality holds for any  $(\alpha', \beta')$  in a neighborhood of  $(\alpha_2, \beta_2)$  since  $\alpha_2 a_2^P - \beta_2 b > \alpha_1 a_1^P - \beta_1 b$ .

We now show that such a contract will only attract type 2 buyers and so represents a profitable deviation. First,

$$\mu(\Theta(C'))(\beta' b - \alpha' a_2^A) \leq \bar{U}_2 = \mu(\Theta(C))(\beta_2 b - \alpha_2 a_2^A) < \mu(\Theta(C))(\beta' b - \alpha' a_2^A),$$

where the first inequality uses optimal search by type 2 buyers, the second equality uses the level of  $\bar{U}_2$ , and the third uses the assumption that  $\beta_2 b - \alpha_2 a_2^A < \beta' b - \alpha' a_2^A$ . This implies  $\Theta(C') < \Theta(C)$ .

Moreover,  $\beta_1 b - \alpha_1 a_1^A \geq \beta_2 b - \alpha_2 a_1^A > \beta' b - \alpha' a_1^A$ , where the first inequality comes from incentive compatibility of  $C$  and the second from the construction of  $C'$ . This condition, together with  $\Theta(C) > \Theta(C)'$ , implies

$$\bar{U}_1 = \mu(\Theta(C))(\beta_1 b - \alpha_1 a_1^A) > \mu(\Theta(C'))(\beta' b - \alpha' a_1^A).$$

Optimal search by type 1 buyers implies  $p_1(C') = 0$ .

Finally, the expected profit of a seller offering  $C'$  is

$$\eta(\Theta(C'))(\alpha' a_2^P - \beta' b) - k > \eta(\Theta(C))((\alpha_1 a_1^P - \beta_1 b)p_1(C) + (\alpha_2 a_2^P - \beta_2 b)p_2(C)) - k = 0,$$

where we use  $\Theta(C') < \Theta(C)$ , the monotonicity of  $\eta$ , and the choice of  $\alpha'$  and  $\beta'$ . Hence, offering  $C'$  represents a profitable deviation, a contradiction that proves  $\bar{U}_1 = 0$ .

**Step 2** Suppose there is a contract  $C$  with  $p_2(C) > 0$  and  $\Theta(C) < \infty$ . Optimal search for  $C$  implies

$$0 = \bar{U}_1 \geq \mu(\Theta(C))(\beta_1 b - \alpha_1 a_1^A).$$

Notice that  $\beta_1 b - \alpha_1 a_1^A \geq \beta_2 b - \alpha_2 a_1^A \geq \beta_2 b - \alpha_2 a_2^A$ , where the first inequality comes from incentive compatibility of  $C$  and the second one from  $a_1^A < a_2^A$ . Thus

$$\mu(\Theta(C))(\beta_1 b - \alpha_1 a_1^A) \geq \mu(\Theta(C))(\beta_2 b - \alpha_2 a_2^A) = \bar{U}_2,$$

where the last equality follows from optimal search,  $p_2(C) > 0$  and  $\Theta(C) < \infty$ . Stringing together inequalities, we obtain  $0 \geq \bar{U}_2$ . ■

We finish by proving that there exists a Competitive Search Equilibrium where no contracts are offered,  $\bar{C} = \emptyset$ , and for any incentive compatible contract  $C = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\}$ ,  $p_1(C) = 1$ . Moreover, if  $\beta_1 b > \alpha_1 a_1^A$ ,  $\Theta(C) = 0$ ; otherwise  $\Theta(C) = \infty$ .

It is straightforward to verify sellers' profit maximization and free entry condition. Since  $\eta(\infty) = 0$ , any contract with  $\beta_1 b \leq \alpha_1 a_1^A$  satisfies the condition. For a contract with  $\beta_1 b > \alpha_1 a_1^A$ ,  $\Theta(C) = 0$  and so  $\eta(\Theta(C)) = 1$ . Sellers' profit maximization and free entry condition would be violated if

$$\alpha_1 a_1^P - \beta_1 b > k.$$

Adding inequalities to eliminate  $\beta_1 b$ , we obtain  $\alpha_1(a_1^P - a_1^A) > k$ . Since  $\alpha_1 \leq 1$  and  $a_1^P \leq a_1^A + k$  by assumption, we obtain a contradiction. Thus the first part of the definition of equilibrium is satisfied.

We turn now to the second part of the definition of equilibrium. By construction, for any incentive-compatible contract with  $\beta_1 b > \alpha_1 a_1^A$ ,  $\Theta(C) = \mu(\Theta(C)) = 0$  and therefore  $\mu(\Theta(C))(\beta_i b - \alpha_i a_i^A) = 0$  for  $i = 1, 2$ , consistent with buyers' optimal search. For any other incentive compatible contract,  $\mu(\Theta(C)) = 1$  and so  $\mu(\Theta(C))u_1(y_1) = \beta_1 b - \alpha_1 a_1^A \leq 0$  by assumption. Moreover, incentive compatibility implies  $u_2(y_2) \geq u_2(y_1) = \beta_1 b - \alpha_1 a_2^A$ . Since  $a_2^A > a_1^A$ , this is again non-positive:  $\mu(\Theta(C))u_2(y_2) \leq 0$ . This proves that buyers' optimal search condition holds.

The final piece of the definition of equilibrium, market clearing, has no content because  $\bar{C} = \emptyset$  and  $\bar{U}_1 = \bar{U}_2 = 0$ .

It is worth stressing that, although there is no trade in this example, when assumption A3 fails, this need not be the case. Consider our second example, in Section 5.2, but suppose  $\bar{\eta} p_i \leq k$  if  $i < \bar{i}$ . Then one can show that in a competitive search equilibrium,  $\bar{U}_i = 0$  if  $i < \bar{i}$ ,

while otherwise a competitive search equilibrium can be found by solving problem P- $i$ . In particular, sellers offer contracts that attract each type  $i \geq \bar{i}$ , guaranteeing them a positive level of utility.

## 7 Conclusion

This paper has developed a canonical model of adverse selection in a competitive search equilibrium. Under a version of a single crossing property, we prove that there is a unique equilibrium in which principals offer separating contracts to agents. We characterize the equilibrium via the solution to a set of constrained optimization problems and illustrate the use of our framework through three examples, including versions of the Spence (1973) signalling model, the Rothschild and Stiglitz (1976) insurance model, and a simple model of asset trade.

Given the tractability of our framework, we anticipate little difficulty in extending the results to a dynamic environment with repeated rounds of contract posting and search. This is important for many applications. For example, a worker who fails to find a job today may search again the following period.

It may also be interesting to study a framework where the informed party posts contracts. In a standard competitive search model, the equilibrium allocation does not depend on who posts contracts and who searches. With asymmetric information, contract posting by informed parties may introduce multiplicity of equilibrium through the usual signaling mechanism. While the equilibrium we study seems robust to this variant of the model, other equilibria may arise.

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