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LEARNING, ADAPTIVE EXPECTATIONS, AND TECHNOLOGY SHOCKS

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Abstract. This study explores theoretical properties of the self-confirming equilibrium in the standard growth model. When rational expectations are replaced by adaptive expectations, we prove that the self-confirming equilibrium is the same as the steady state rational expectations equilibrium, but that the dynamics around the steady state are substantially different between the two equilibria. Learning has important effects on the volatility and persistence of macroeconomic variables. We show that, in contrast to Williams (2003), these important differences do not stem from escapes or large deviations from the self-confirming equilibrium. Overall, the learning model gives technology shocks a much more prominent role in shaping business cycles than does the rational expectations model.

I. Introduction

Adaptive learning models in macroeconomics have been used for many applications (Sargent, 2007).[1] In this paper, we focus on yet another application in the context of a standard growth model in which rational expectations are replaced by adaptive expectations. The stability of rational expectations under learning in real business cycle (RBC) models has been studied in the literature (Evans and Honkapohja, 2001; Bullard and Duffy, 2004). In a closely related paper, Williams (2003) considers a variety of standard learning rules in a RBC model and finds that the learning dynamics

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[1]To give a few examples, Lucas (1986), Marcet and Sargent (1989), and Evans and Honkapohja (2001) recommend selecting rational expectations equilibria that are stable under least squares learning; Schorfheide (2005), Primiceri (2006), Sargent, Williams, and Zha (2006b) use learning mechanisms to explain the rise and fall of American inflation; Adam, Marcet, and Nicolini (2008) shows how learning helps improve the fit of the model of asset pricing.

differ very little from those under rational expectations. As noted by Williams (2003), these learning rules do not separate agents' beliefs and their decision making. The only role for agents is forecasting by using a reduced-form model. Consequently, one would conclude that learning dynamics do not teach us anything new, as compared to the rational expectations version of the RBC model.

In this paper, we reexamine this conclusion in the standard growth model with both neutral and investment specific technologies. Following the commonly-used learning mechanism commended by Marcet and Nicolini (2003) and Sargent, Williams, and Zha (2006a), we study misspecified learning by separating agents' beliefs and their decision rules.[2] Rational expectations are simply replaced by adaptive expectations, while all decision equations under rational expectations remain intact. We show that this slight departure from rational expectations has important ramifications by answering the following substantive questions:

- Does there exist a self-confirming equilibrium (SCE) in our learning environment?
- Is the SCE unique?
- Are there escape dynamics away from the domain of attraction of the SCE?
- How different are the learning dynamics from the dynamics under rational expectations?
- Are they quantitatively important to affect the amplification and propagation mechanisms?
- What is the relative importance of investment-specific technology shocks to neutral technology shocks in shaping the business cycle dynamics?

We obtain the closed-form solutions for both rational expectations and learning models. These analytical solutions enable us to prove the existence and uniqueness of the SCE under all admissible parameterizations in our learning model. We further prove that the SCE coincides with the REE steady state, but that the learning

[2]Williams (2003) studies a different kind of misspecified learning in which agents do not know the true parameters of the production function. By assuming full depreciation of the capital stock, an i.i.d. technology process, and inelastic labor, he shows that learning leads to occasional, but recurrent, large deviations away from an SCE, called "escape dynamics." For other studies of escape dynamics, see Sargent (1999), Cho, Williams, and Sargent (2002), Kasa (2004), and Adam, Evans, and Honkapohja (2006).

dynamics are substantially different from REE dynamics. Unlike Marcet and Nicolini (2003), Williams (2003), and Sargent, Williams, and Zha (2006a), however, we show that learning dynamics are stationary and that the differences between learning dynamics and rational-expectations dynamics are *not* driven by escape dynamics.

Our results not only establish the theoretical validity of the learning model but enable us to draw practical implications. The dynamic responses of output, consumption, investment, and labor hours, following a neutral technology shock, are substantially larger in the adaptive expectations model than in the rational expectations model. Introducing learning non only amplifies the effects of the neutral shock on many aggregate variables but also helps improve the model's predictions on the labor market dynamics. In the rational expectations model, equilibrium hours tend to change too little and the equilibrium real wage tends to rise too much. In contrast, learning can amplify the response of labor and at the same time dampen the response of the real wage.

As for responses to a positive biased technology shock, the learning model tends to amplify the responses of all macroeconomic variables, including the real wage, as compared to the rational expectations model. Moreover, find that in the model with learning, the contributions of the biased technology shock to the variances of output, consumption, and investment are much larger than those in the model with rational expectations.

Overall, our results indicate that the growth model with adaptive expectations is capable of giving technology shocks a much more important role in explaining business cycles than is the rational expectations model.

II. The Model

In this section, we describe the standard growth model with both neutral and biased technologies. The economy is populated by a continuum of infinitely lived and identical households. The representative household is endowed with a unit of time. The household derives utility from consumption and leisure, with the utility function

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln C_t - \xi \frac{L_t^{1+\eta}}{1+\eta} \right\}, \tag{1}$$

where C_t denotes consumption, L_t denotes labor hours, $\beta \in (0,1)$ denotes the subjective discount factor, and E_0 denotes an expectation at the initial time 0.

The economy is also populated by a continuum of identical, perfectly competitive firms. The representative firm has access to a constant returns to scale technology represented by the production function

$$Y_t = K_{t-1}^{1-\alpha} \left(Z_t L_t\right)^{\alpha}, \tag{2}$$

where Y_t denotes output, K_{t-1} denotes capital input, and L_t denotes labor input. The term Z_t denotes the neutral technological change and follows the stochastic process

$$Z_t = Z_{t-1}\lambda_z \nu_t, \tag{3}$$

where λ_z is the trend component and ν_t is the stationary component that follows the AR(1) process

$$\ln \nu_t = \rho_\nu \ln \nu_{t-1} + \varepsilon_{\nu t}. \tag{4}$$

The persistence parameter $\rho_\nu \in (0,1)$ and the shock $\varepsilon_{\nu t}$ is an i.i.d. normal random variable with mean zero and variance σ_ν^2.

The economy has an initial stock of capital denoted by K_{-1}. Capital stock evolves over time according to the law of motion

$$K_t = (1-\delta)K_{t-1} + Q_t I_t, \tag{5}$$

where K_t denotes the period-t capital stock, I_t denotes investment, Q_t denotes the relative price of investment goods, and the parameter $\delta \in (0,1)$ denotes the capital depreciation rate. Following Greenwood, Hercowitz, and Krusell (1997), we assume that Q_t represents the (inverse of) investment specific technological change. Specifically, we assume that Q_t follows the stochastic process

$$Q_t = Q_{t-1}\lambda_q \mu_t, \tag{6}$$

where λ_q is the trend component and μ_t is the stationary component that follows the AR(1) process

$$\ln \mu_t = \rho_\mu \ln \mu_{t-1} + \varepsilon_{\mu t}. \tag{7}$$

The persistence parameter $\rho_\mu \in (0,1)$ and the shock $\varepsilon_{\mu t}$ is an i.i.d. normal random variable with mean zero and variance σ_μ^2.

The aggregate resource constraint is given by

$$C_t + I_t = Y_t. \tag{8}$$

III. Equilibrium Allocation and Balanced Growth

Since the model economy has perfect competition and no externality, the First Welfare Theorem applies. Thus, the equilibrium allocations are Pareto efficient and can be found by solving a social planner's problem.

The social planner maximizes the representative household's utility (1) subject to the resource constraint (8) and the capital law of motion (5). The first order conditions imply that

$$\xi L_t^{1+\eta} = \alpha Y_t / C_t, \tag{9}$$

$$1 = \beta \mathrm{E}_t \left\{ \frac{Q_t}{Q_{t+1}} \frac{C_t}{C_{t+1}} \left[1 - \delta + Q_{t+1}(1-\alpha) \frac{Y_{t+1}}{K_t} \right] \right\}. \tag{10}$$

On the balanced growth path, C_t, I_t, and Y_t grow at the same rate of $\lambda_z \lambda_q^{(1-\alpha)/\alpha}$ while the capital stock K_t grows at a faster rate of $\lambda_z \lambda_q^{1/\alpha}$. We define the following stationary variables

$$\tilde{Y}_t = \frac{Y_t}{Z_t Q_t^{(1-\alpha)/\alpha}}, \; \tilde{C}_t = \frac{C_t}{Z_t Q_t^{(1-\alpha)/\alpha}}, \; \tilde{I}_t = \frac{I_t}{Z_t Q_t^{(1-\alpha)/\alpha}}, \; \tilde{K}_t = \frac{K_t}{Z_t Q_t^{1/\alpha}}.$$

Given these stationary variables, we can rewrite the equilibrium conditions (2), (5), (8), (9), and (10) as

$$\tilde{Y}_t \left(\frac{Z_t}{Z_{t-1}} \right)^{1-\alpha} \left(\frac{Q_t}{Q_{t-1}} \right)^{(1-\alpha)/\alpha} = \tilde{K}_{t-1}^{1-\alpha} L_t^{\alpha}, \tag{11}$$

$$\tilde{K}_t \frac{Z_t}{Z_{t-1}} \left(\frac{Q_t}{Q_{t-1}} \right)^{1/\alpha} = (1-\delta) \tilde{K}_{t-1} + \tilde{I}_t \frac{Z_t}{Z_{t-1}} \left(\frac{Q_t}{Q_{t-1}} \right)^{1/\alpha}, \tag{12}$$

$$\tilde{C}_t + \tilde{I}_t = \tilde{Y}_t, \tag{13}$$

$$\xi L_t^{1+\eta} = \alpha \tilde{Y}_t / \tilde{C}_t, \tag{14}$$

$$1 = \beta \mathrm{E}_t \left[(1-\delta) \frac{\tilde{C}_t}{\tilde{C}_{t+1}} \frac{Z_t}{Z_{t+1}} \left(\frac{Q_t}{Q_{t+1}} \right)^{1/\alpha} + (1-\alpha) \frac{\tilde{C}_t}{\tilde{C}_{t+1}} \frac{\tilde{Y}_{t+1}}{\tilde{K}_t} \right]. \tag{15}$$

It follows from the above conditions that the steady state equilibrium can be described by the following equations

$$\lambda_z^{1-\alpha}\lambda_q^{(1-\alpha)/\alpha}\tilde{Y} = \tilde{K}^{1-\alpha}L^{\alpha}, \tag{16}$$

$$i_k = \frac{\tilde{I}}{\tilde{K}} = 1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}, \tag{17}$$

$$\tilde{C} + \tilde{I} = \tilde{Y}, \tag{18}$$

$$\xi L^{1+\eta} = \alpha\tilde{Y}/\tilde{C}, \tag{19}$$

$$y_k = \frac{\tilde{Y}}{\tilde{K}} = \frac{1}{\beta(1-\alpha)}\left[1 - \frac{\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}}\right]. \tag{20}$$

The consumption-output and investment-output ratios can derived from the above steady state conditions:

$$i_y = \frac{\tilde{I}}{\tilde{Y}} = \beta(1-\alpha)\frac{\lambda_z\lambda_q^{1/\alpha} - (1-\delta)}{\lambda_z\lambda_q^{1/\alpha} - \beta(1-\delta)}, \tag{21}$$

$$c_y = \frac{\tilde{C}}{\tilde{Y}} = \frac{\lambda_z\lambda_q^{1/\alpha}(1-\beta(1-\alpha)) - \beta\alpha(1-\delta)}{\lambda_z\lambda_q^{1/\alpha} - \beta(1-\delta)}. \tag{22}$$

Log-linearizing the equilibrium conditions (11), (12), (13), (14), and (15) and rearranging the terms, we obtain the following five equations describing the production function, the law of motion for capital accumulation, the resource constraint, the optimal consumption-labor-supply decision, and the optimal investment decision:

$$\hat{y}_t - \alpha\hat{l}_t + \frac{1-\alpha}{\alpha}\hat{\mu}_t + (1-\alpha)\hat{\nu}_t = (1-\alpha)\hat{k}_{t-1}, \tag{23}$$

$$\hat{k}_t - i_k\hat{i}_t + (1-i_k)\left(\alpha^{-1}\hat{\mu}_t + \hat{\nu}_t\right) = (1-i_k)\hat{k}_{t-1}, \tag{24}$$

$$c_y\hat{c}_t + i_y\hat{i}_t = \hat{y}_t, \tag{25}$$

$$\hat{y}_t = \hat{c}_t + (1+\eta)\hat{l}_t, \tag{26}$$

$$\begin{aligned}\beta(1-\alpha)y_k\hat{k}_t - \hat{c}_t + [1-\beta(1-\alpha)y_k]\left(\frac{\rho_\mu}{\alpha}\hat{\mu}_t + \rho_\nu\hat{\nu}_t\right) = \\ [\beta(1-\alpha)c_k - 1]E_t\hat{c}_{t+1} + \beta(1-\alpha)i_kE_t\hat{i}_{t+1},\end{aligned} \tag{27}$$

where the notation $\hat{x}_t$ denotes $\ln\tilde{X}_t - \ln\tilde{X}$ for $X = C, I, Y, K$ or $\ln X_t - \ln X$ for $X = L$, i_k, c_y, i_y, and y_k are steady-state ratios defined in (17), (22), (21), (20), and

$c_k = \frac{\tilde{C}}{\tilde{K}}$ is derived as

$$\beta(1-\alpha)(c_k+1) = 1 - \frac{\alpha\beta(1-\delta)}{\lambda_z \lambda_q^{1/\alpha}}. \tag{28}$$

Definition 1. Admissible values of the deep parameters are $\beta \in (0,1)$, $\eta \geq 0$, $\alpha \in (0,1)$, $\delta \in [0,1]$, $\lambda_z \geq 1$, and $\lambda_q \geq 1$.

In the literature, dynamics are often simulated for a particular set of admissible values of the deep parameters by numerically solving the rational-expectations equilibrium system given by the above conditions. We shall show, however, that the equilibrium characterized by (23)-(27) can be solved analytically for all admissible values of the deep parameters. A key step is to derive a second-order stochastic difference equation for capital, as stated in the following proposition.

Proposition 1.

$$\hat{k}_t = \gamma_1 E_t \hat{k}_{t+1} + \gamma_2 \hat{k}_{t-1} + \gamma_\mu \hat{\mu}_t + \gamma_\nu \hat{\nu}_t, \tag{29}$$

where the coefficients γ_1, γ_2, γ_μ and γ_ν are reported in Appendix A.

Proof. See Appendix B. □

The closed-form solutions for all the other variables are reported in Appendix A. The key is to solve (29). The solution depends on how agents form expectations of the endogenous accumulation process of capital, as is shown in the next section.

IV. REE vs. SCE: Analytical Results

In this section, we derive the closed-form solutions for the rational expectations equilibrium (REE) and the Self-confirming equilibrium (SCE). For the REE solution, we have the following result.

Proposition 2. The solution to the second-order differential equation (29) under the rational expectations assumption is

$$\hat{k}_t = a\hat{k}_{t-1} + b\hat{\nu}_t + c\hat{\mu}_t, \tag{30}$$

where

$$a = \frac{1-\sqrt{1-4\gamma_1\gamma_2}}{2\gamma_1}, \qquad b = \frac{\gamma_\nu}{1-(\rho_\nu + a)\gamma_1}, \qquad c = \frac{\gamma_\mu}{1-(\rho_\mu + a)\gamma_1},$$

Furthermore, this solution is stationary and unique.

Proof. See Appendix C. □

Given the shock processes and an initial condition for capital, (30) gives the dynamic solution for capital. For a comparison with the SCE solution, this dynamic solution can be expressed as

$$\hat{k}_t = b\sum_{i=0}^{\infty} b_i \varepsilon_{\nu,t-i} + c\sum_{i=0}^{\infty} c_i \varepsilon_{\mu,t-i}, \tag{31}$$

where for all $i \geq 0$,

$$b_i = \sum_{j=0}^{i} a^{i-j}\rho_\nu^j, \quad \text{while if} \quad \rho_\nu \neq a \quad \text{then} \quad b_i = \frac{a^{i+1} - \rho_\nu^{i+1}}{a - \rho_\nu};$$

$$c_i = \sum_{j=0}^{i} a^{i-j}\rho_\mu^j, \quad \text{while if} \quad \rho_\mu \neq a \quad \text{then} \quad c_i = \frac{a^{i+1} - \rho_\mu^{i+1}}{a - \rho_\mu};$$

We now assume that agents have adaptive expectations. We follow Marcet and Nicolini (2003) and Sargent, Williams, and Zha (2006a) to replace $E_t\hat{k}_{t+1}$ by $\hat{E}_t\hat{k}_{t+1}$ such that

$$\hat{E}_t\hat{k}_{t+1} = \hat{\beta}_t.$$

Agents update their beliefs $\hat{\beta}_t$ using the following constant-gain learning (CGL) algorithm:

$$\hat{\beta}_t = \hat{\beta}_{t-1} + g(\hat{k}_{t-1} - \hat{\beta}_{t-1}), \tag{32}$$

where $0 < g << 1$ is a gain representing how fast past observations are discounted in the learning regression.

The dynamics of $\hat{k}_t$ produced by (29) under the above learning algorithm (32) follow the process

$$\hat{k}_t = \gamma_1\hat{\beta}_t + \gamma_2\hat{k}_{t-1} + \gamma_\nu\hat{\nu}_t + \gamma_\mu\hat{\mu}_t. \tag{33}$$

In self-confirming equilibrium, beliefs are not contradicted by observations along the equilibrium path (Sargent, 1999). To find an SCE is to solve a fixed-point problem. For our model, the solution to the SCE is to find the fixed point $\hat{\beta}$ that solves the orthogonality condition

$$E\left[\hat{k}_t(\hat{\beta}) - \hat{\beta}\right] = 0, \tag{34}$$

where $E(\ )$ is a mathematical unconditional expectation operator and $\hat{k}_t$ itself is a function of the belief $\hat{\beta}$ in self-confirming equilibrium such that

$$\hat{k}_t(\hat{\beta}) = \gamma_1\hat{\beta} + \gamma_2\hat{k}_{t-1}(\hat{\beta}) + \gamma_\nu\hat{\nu}_t + \gamma_\mu\hat{\mu}_t.$$

Proposition 3. As $g \to 0$, the belief sequence $\{\hat{\beta}_t\}$ in (32) converges weakly to the unique and stationary SCE given by $\hat{\beta} = 0$ for all admissible values of the deep parameters.

Proof. From (33) one can see that $\hat{k}_t$ is a function of current and past beliefs and fundamental shocks. We denote this function as $\kappa(\ )$ such that

$$\hat{k}_t = \kappa(\hat{\beta}_t, \hat{\beta}_{t-1}, \ldots, \nu_t, \nu_{t-1}, \ldots, \mu_t, \mu_{t-1}, \ldots).$$

Denote

$$\tilde{\kappa}(\hat{\beta}_t, \hat{\beta}_{t-1}, \ldots, \nu_t, \nu_{t-1}, \ldots, \mu_t, \mu_{t-1}, \ldots) = \kappa(\hat{\beta}_t, \hat{\beta}_{t-1}, \ldots, \nu_t, \nu_{t-1}, \ldots, \mu_t, \mu_{t-1}, \ldots) - \hat{\beta}_t.$$

We can then rewrite the CGL algorithm (32) as

$$\hat{\beta}_t = \hat{\beta}_{t-1} + g\tilde{\kappa}(\hat{\beta}_t, \hat{\beta}_{t-1}, \ldots, \nu_t, \nu_{t-1}, \ldots, \mu_t, \mu_{t-1}, \ldots). \tag{35}$$

To prove that (34) holds at $\hat{\beta} = 0$ and the fixed point $\hat{\beta} = 0$ is unique, we denote the left-hand-side term in (34) by

$$G(\hat{\beta}) = E\tilde{\kappa}(\hat{\beta}, \hat{\beta}, \ldots, \nu_t, \nu_{t-1}, \ldots, \mu_t, \mu_{t-1}, \ldots).$$

Under our assumptions, it follows from Kushner and Yin (1997) that as $g \to 0$, the beliefs $\hat{\beta}_t$ in (35) converges weakly to the solution of the ordinary differential equation (ODE)

$$\dot{\hat{\beta}} = G(\hat{\beta}).$$

One can further show that

$$G(\hat{\beta}) = \left(\frac{\gamma_2 + \gamma_1 - 1}{1 - \gamma_2}\right)\hat{\beta}.$$

Since $\gamma_1 > 0$, $\gamma_2 > 0$, and $\gamma_2 + \gamma_1 < 1$, the ODE has a unique fixed point at $\hat{\beta} = 0$. The ODE is stable since $(\gamma_2 + \gamma_1 - 1)/(1 - \gamma_2) < 0$. □

As one can see from Proposition 3, the SCE is exactly the same as the rational expectations steady state. Since an SCE is a limit of adaptive dynamics, it is important to characterize adaptive dynamics and to study whether they are different from dynamics under rational expectations. We rewrite the stochastic processes (32) and (33) as

$$\begin{bmatrix}\hat{\beta}_t\\ \hat{k}_t\end{bmatrix} = \begin{bmatrix}1-g & g\\ (1-g)\gamma_1 & \gamma_2+g\gamma_1\end{bmatrix}\begin{bmatrix}\hat{\beta}_{t-1}\\ \hat{k}_{t-1}\end{bmatrix} + \begin{bmatrix}0 & 0\\ \gamma_\nu & \gamma_\mu\end{bmatrix}\begin{bmatrix}\hat{\nu}_t\\ \hat{\mu}_t\end{bmatrix}. \tag{36}$$

Given the initial belief $\hat{\beta}_{-1}$, the initial capital stock $\hat{k}_{-1}$, and the shock processes, the bivariate autoregressive process (36) determines the belief and capital dynamics jointly; then, (23)-(26) in Section III, or (A1)-(A4) in Appendix A, determine the dynamics of investment, labor, output, and consumption.

Proposition 4. The learning dynamics, described by (23)-(26) and (36) for $g \in (0,1)$, are stationary for all admissible values of the deep parameters.

Proof. Given (A1)-(A4) in Appendix A that characterize the dynamics of investment, labor, output, and consumption as a function of $\hat{k}_t$, it suffices to show that (36) is a stationary process. The two characteristic roots of the 2×2 coefficient matrix of $\hat{\beta}_{t-1}$ and $\hat{k}_{t-1}$ on the right-hand side of (36) are

$$\lambda_1 = \frac{(1-g+\gamma_2+g\gamma_1)-\sqrt{(1-g+\gamma_2+g\gamma_1)^2-4(1-g)\gamma_2}}{2},$$

$$\lambda_2 = \frac{(1-g+\gamma_2+g\gamma_1)+\sqrt{(1-g+\gamma_2+g\gamma_1)^2-4(1-g)\gamma_2}}{2}.$$

Since $\gamma_1 > 0$, $\gamma_2 > 0$, and $\gamma_1+\gamma_2 < 1$ for all admissible values of the deep parameters, it follows that both λ_1 and λ_2 are real numbers and for any $g \in (0,1)$, $0 < \lambda_1 < \lambda_2 < 1$. Hence, the adaptive process for $\{\hat{\beta}_t, \hat{k}_t\}$, given by (36), is stationary. □

Proposition 4 implies that the learning dynamics studied in this paper remain in the domain of attraction of the SCE (the rational expectations steady state) and there are no escape dynamics.

To assess how different the learning dynamics differ from dynamics under rational expectations, we derive the belief and capital dynamics under the CGL as

$$(1-\lambda_1 L)(1-\lambda_2 L)\,\hat{\beta}_t = g\left(\gamma_\nu \hat{\nu}_{t-1} + \gamma_\mu \hat{\mu}_{t-1}\right), \tag{37}$$

$$\left(1-\lambda_1 L\right)\left(1-\lambda_2 L\right)\hat{k}_t=\left[1-\left(1-g\right)L\right]\left(\gamma_\nu\hat{\nu}_t+\gamma_\mu\hat{\mu}_t\right), \tag{38}$$

where L is the lag operator. It follows from (37) that

$$\hat{\beta}_t=g\sum_{i=1}^{\infty}\sum_{j=1}^{i}\frac{\lambda_1^j-\lambda_2^j}{\lambda_1-\lambda_2}\left(\gamma_\nu\rho_\nu^{i-j}\varepsilon_{\nu,t-i}+\gamma_\mu\rho_\mu^{i-j}\varepsilon_{\mu,t-i}\right). \tag{39}$$

If $\rho_\nu\neq\lambda_1$ or λ_2, $\rho_\mu\neq\lambda_1$ or λ_2, and $\rho_\xi\neq\lambda_1$ or λ_2, then (39) simplifies to

$$\begin{aligned}\hat{\beta}_t=&g\gamma_\nu\sum_{i=1}^{\infty}\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\frac{\lambda_1^i-\rho_\nu^i}{\lambda_1-\rho_\nu}-\frac{\lambda_2}{\lambda_1-\lambda_2}\frac{\lambda_2^i-\rho_\nu^i}{\lambda_2-\rho_\nu}\right)\varepsilon_{\nu,t-i}\\&+g\gamma_\mu\sum_{i=1}^{\infty}\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\frac{\lambda_1^i-\rho_\mu^i}{\lambda_1-\rho_\mu}-\frac{\lambda_2}{\lambda_1-\lambda_2}\frac{\lambda_2^i-\rho_\mu^i}{\lambda_2-\rho_\mu}\right)\varepsilon_{\mu,t-i}.\end{aligned} \tag{40}$$

On the other hand, we can compute the rational expectations from (31) as

$$\begin{aligned}E_t\hat{k}_{t+1}=&bb_1\varepsilon_{\nu,t}+cc_1\varepsilon_{\mu,t}+dd_1\varepsilon_{\xi,t}\\&+b\sum_{i=1}^{\infty}b_{i+1}\varepsilon_{\nu,t-i}+c\sum_{i=1}^{\infty}c_{i+1}\varepsilon_{\mu,t-i}.\end{aligned} \tag{41}$$

A comparison of (40) and (41) shows that, although the SCE is the same as the steady state REE, the dynamics of beliefs $\hat{\beta}_t$ is both qualitatively and quantitatively different from those of expectations $E_t\hat{k}_{t+1}$. These differences lead to important differences in dynamics of other macroeconomic variables, as shown in the next section.

V. REE vs. SCE: Transmission of Technology Shocks

We now analyze the transmission mechanisms of the model under both rational and adaptive expectations. We simulate the model using one set of parameter values, but our quantitative results hold for a wide range of values. The model parameters include β, the subjective discount factor; α, the labor share of income; δ, the capital depreciation rate; η, the inverse Frisch elasticity of labor supply; λ_z and λ_q, the average growth rate of the neutral and biased technologies; ξ, the weight parameter in the preferences for leisure; ρ_ν, ρ_μ, σ_ν, and σ_μ, the parameters controlling the shock processes, and g, the constant gain in the learning process.

V.1. **Parameterization.** The set of parameter values we use for simulations are summarized in Table 1. We set $\alpha = 0.7$, corresponding to a labor income share of 70%. We set $\lambda_q = 1.008$ such that, in our quarterly model, the investment-specific technology grows at an annual rate of 3.2%, as suggested by Greenwood, Hercowitz, and Krusell (1997). We set $\lambda_z = 1.0016$ such that, given our value of λ_q and α, real per capita GDP grows at an annual rate of 2% on the path of balanced growth.[3] We set $\delta = 0.025$ (corresponding to an annual depreciation rate of 10%), a value used in most business cycle studies. We set $\beta = 0.9952$ such that the annual return to capital (net of depreciation) is 7.2%. We set $\eta = 2$, corresponding to a Frisch elasticity of 0.5(Pencavel, 1986). We set $\xi = 3.17$ such that, given the values of other parameters, the steady-state hours worked are 30% of the time endowment (which is normalized to 1). For the parameters in the shock processes, we set $\rho_\nu = \rho_\mu = 0.1$ and $\sigma_\nu = \sigma_\mu = 0.2$. Finally, we set the gain $g = 0.1$ in the learning process.

V.2. **Amplification effects.** To understand the role of adaptive expectations in transmitting the two types of technology shocks, we compute impulse responses to both shocks. Figure 1 displays the impulse responses of aggregate variables following a neutral technology shock. The solid line represents the responses under rational expectations and the dashed line represents the responses under adaptive expectations.

The responses of aggregate variables to a positive neutral technology shock in the rational expectations model should be familiar to a student of real business cycle studies. As the solid line in the figure shows, output rises on impact and continues to rise until reaching the new level. Consumption, investment, hours, the real wage, and the real interest rate all co-move with output. In the impact period, consumption responds less and investment responds more than does output. These patterns of responses are consistent with the stylized facts about business cycles. A well documented difficulty facing the standard RBC model with rational expectations lies in the labor market dynamics (Christiano and Eichenbaum, 1992). The RBC model typically fails to generate the observed large responses of labor hours and small responses of the real wage following a neutral technology shock. To understand this feature of the model, note that the shock raises the demand for labor at any given real wage, shifts the labor demand schedule out and creating a substitution effect.

[3]The average growth rate for output in the model is given by $\lambda_z \lambda_q^{(1-\alpha)/\alpha}$.

Thus, holding the labor supply schedule unchanged, the substitution effect drives up both hours and the real wage. In the mean time, since the shock raises future productivity (as it increases the trend growth rate temporarily but the level of technology permanently), the associated wealth effect raises current consumption, which shifts the labor supply curve up. The wealth effect thus tends to cancel out the substitution effect on hours, rendering the responses of equilibrium hours small; meanwhile, the wealth effect reinforces the substitution effect on the real wage, pushing up the equilibrium wage sharply, as is evident in the solid lines of Figure 1. The lower the Frisch elasticity of labor supply, the greater the rise in the real wage.[4]

Introducing learning helps alleviate some of the problems for the RBC model, especially for the labor market variables. The dashed lines in Figure 1 display the dynamic responses of the aggregate variables to a positive neutral technology shock. We first note that the shock raises the demand for labor and, as in the rational expectations model, the substitution effect works to increase the labor hours and the real wage. Unlike the rational expectations model, however, the wealth effect is dampened because agents form expectations of the future productivity and income based on past observations. Because of this dampening effect, consumption responds less to the shock and thus the labor supply curve does not shift much. What is interesting is that, following the neutral technology shock, the wealth effect is dampened sufficiently enough for the labor supply to rise sharply but for the real wage to rise only modestly. The sharp rise in labor, along with the positive productivity shock, leads to a sharp rise in output; since consumption does not rise much for the lack of a wealth effect, the sharp rise in output leads to a sharp increase in investment and in the real interest rate.

Compared to the rational expectations model, the response of hours under adaptive expectations is substantially amplified but that of the real wage is considerably dampened (the dashed lines in Figure 1). Furthermore, the responses of output, consumption, investment, and the real interest rate are all amplified.

Although the effects of neutral technology shocks are well studied, the literature on the effects of investment-specific shocks is scarce, with the notable exceptions of Greenwood, Hercowitz, and Krusell (2000), Krusell, Ohanian, Ríos-Rull, and Violante

[4]Note that even under the assumption of indivisible labor so that the aggregate labor supply elasticity is arbitrarily large (Hansen, 1985; Rogerson, 1988), the real wage still rises sharply.

(2000), and Fisher (2006). In particular, to our best knowledge, there has been no published study that examines the effects of biased technology shocks (such as the investment-specific shocks) in the context of adaptive expectations.

In Figure 2, we plot the impulse responses following a positive biased technology shock for both the rational expectations and adaptive expectations models. In both models, the shock leads to a rise in output, investment, hours, and the real interest rate, but a decline in consumption and the real wage.

For the rational expectations model, as the biased shock raises the efficiency of investment, investment goods today become cheaper and current consumption becomes more expensive. This type of shock, unlike the neutral technology shock, shifts resources from consumption to investment. Consequently, investment rises and consumption declines. The decline in consumption generates a negative wealth effect on labor supply, which shifts the labor supply curve out so that the real wage declines somewhat and hours rise. The rise in labor hours helps produce more output. Meanwhile, as capital becomes more productive following the shock, the real interest rate increases.

In the learning model, because agents do not (rationally) foresee the permanent shift in the future level of investment technology, they respond to the positive shock as though it had only a temporary effect. Consequently, the demand for current investment rises sharply, leading to a rise in investment and a fall in consumption in a magnitude more than that in the rational expectations model. The sharp decline in consumption amplifies the decline in the real wage and the rise in hours. The amplified increase in hours in turn leads to a sharp rise in output and thus in the real interest rate.

In summary, following a positive biased technology shock, the responses of all the aggregate variables are substantially amplified in the learning model, as compared to those under rational expectations. Overall, relaxing the assumption of perfect rationality helps give a larger role to both neutral and biased technology shocks in shaping business cycles.

V.3. **Relative importance of biased vs. neutral technology shocks.** In the previous section, we see that introducing leaning amplify the responses of macroeconomic variables, such as hours, output, and investment, to both types of technology shocks. In this section, we examine whether learning changes the relative importance of neutral to biased technology shocks.

In Table 2, we report the variance decomposition results for various forecasting horizons. For each variable, we first compute the contribution of the biased technology shock, relative to the neutral technology shock, in forecasting variances for both rational and adaptive expectations models. We then compute the ratio of such a contribution for the learning model to that for the rational expectations model. This ratio is reported in Table 2. If the ratio is larger than one, it means that the relative contribution of the biased technology becomes more important under the learning model than under the rational expectations model. At the one-quarter forecasting horizon, for instance, the biased technology shock accounts for only 1.55% of the output variance in the rational expectations model, while it accounts for 42.32% in the learning model (these values are not reported in the table). Thus, the contribution of the biased technology shock for the one-quarter ahead output variance in the learning model is 27.28 times that in the rational expectations model (this ratio is reported in the table).

As one can see from the table, introducing learning substantially magnifies the contribution of the biased technology shock, relative to the neutral technology shock, for output, consumption, and investment. Interestingly, the relative contribution of the biased technology shock for hours is the same across the two models.[5]

VI. Conclusion

We have studied a standard growth model with adaptive expectations in which beliefs are separate from decision rules. We have proven that there exists a unique, stable SCE in our learning model and that the SCE turns out to be the same as the steady state REE. In contrast to the existing literature, however, we have shown that the learning model can generate substantially different dynamics around the steady state than does the rational expectations model. These differences are not associated with escape dynamics.

[5]This result is in part due to the stationarity in the variable hours. Technology shocks have permanent effects on nonstationary variables like output and investment and thus are likely to change the relative impact on these variables over time and across models. These cross-time and cross-model changes are likely to be absent for a stationary variable.

Why are our theoretical results important? It is known that technology shocks in a standard growth model do not generate enough fluctuations in key macroeconomic variables such as hours and output. We have shown that introducing misspecified learning in the growth model can substantially amplify the responses of the macroeconomic variables and thus has potential to give technology shocks a much more prominent role in shaping business cycles than does the rational expectations growth model.

Appendix A. Analytical Solution

The coefficients in (29) in Proposition 1 are defined as:

$$\gamma_1 = \frac{\beta(1-\alpha)(c_k+1)+\eta}{\beta(1-\alpha)(c_k+1)\left[1+(1+\eta)c_k\right]+\eta(1-\alpha i_k)+(1-\alpha+\eta)},$$

$$\gamma_2 = \frac{(1-\alpha)(1+\eta)y_k+(1-\alpha+\eta)(1-i_k)}{\beta(1-\alpha)(c_k+1)\left[1+(1+\eta)c_k\right]+\eta(1-\alpha i_k)+(1-\alpha+\eta)},$$

$$\gamma_\nu = \frac{\rho_\nu\left\{\left[\beta(1-\alpha)(c_k+1)-1\right]\left[(1+\eta)c_k+1\right]+(1+\eta)(1-\alpha i_k)\right\}+\alpha\eta y_k-(1-\alpha+\eta)(c_k+1)}{\beta(1-\alpha)(c_k+1)\left[1+(1+\eta)c_k\right]+\eta(1-\alpha i_k)+(1-\alpha+\eta)},$$

$$\gamma_\mu = \frac{\rho_\mu\left\{\left[\beta(1-\alpha)(c_k+1)-1\right]\left[(1+\eta)c_k+1\right]+(1+\eta)(1-\alpha i_k)\right\}+\alpha\eta y_k-(1-\alpha+\eta)(c_k+1)}{\alpha\left[\beta(1-\alpha)(c_k+1)\left[1+(1+\eta)c_k\right]+\eta(1-\alpha i_k)+(1-\alpha+\eta)\right]}.$$

Other steady state ratios have been derived in Section III. One can verify that, for all admissible values of the deep parameters, that is, for any $\beta \in (0,1)$, $\eta \geq 0$, $\alpha \in (0,1)$, $\delta \in [0,1]$, $\lambda_z \geq 1$, and $\lambda_q \geq 1$, all the steady-state ratios are well-defined and positive, and so are γ_1 and γ_2.

The closed-form solutions for investment, hours, output, and consumption are derived as the the following system of equations under either RE or AE:

$$\hat{i}_t = k_i\hat{k}_t+(1-k_i)\hat{k}_{t-1}+(k_i-1)\hat{\nu}_t+\left(\frac{k_i-1}{\alpha}\right)\hat{\mu}_t, \tag{A1}$$

$$\begin{aligned}\hat{l}_t =& \frac{1}{\left[(1+\eta)c_k+\alpha i_k\right]}\hat{k}_t-\frac{\left[1-\alpha i_k\right]}{\left[(1+\eta)c_k+\alpha i_k\right]}\hat{k}_{t-1}\\ &+\frac{(1-\alpha i_k)}{\left[(1+\eta)c_k+\alpha i_k\right]}\hat{\nu}_t+\frac{(\alpha^{-1}-i_k)}{\left[(1+\eta)c_k+\alpha i_k\right]}\hat{\mu}_t,\end{aligned} \tag{A2}$$

$$\begin{aligned}\hat{y}_t =& \frac{\alpha}{\left[(1+\eta)c_k+\alpha i_k\right]}\hat{k}_t+\frac{\left[(1-\alpha)(1+\eta)c_k-\alpha(1-i_k)\right]}{\left[(1+\eta)c_k+\alpha i_k\right]}\hat{k}_{t-1}\\ &+\frac{\alpha\,(1-i_k)-(1-\alpha)(1+\eta)c_k}{\left[(1+\eta)c_k+\alpha i_k\right]}\hat{\nu}_t\\ &+\frac{(1-i_k)-\alpha^{-1}(1-\alpha)(1+\eta)c_k}{\left[(1+\eta)c_k+\alpha i_k\right]}\hat{\mu}_t,\end{aligned} \tag{A3}$$

$$
\begin{aligned}
\hat{c}_t =& \frac{\alpha - 1 - \eta}{[(1+\eta)c_k + \alpha i_k]}\hat{k}_t + \frac{[(1-\alpha)(1+\eta)y_k + (1+\eta-\alpha)(1-i_k)]}{[(1+\eta)c_k + \alpha i_k]}\hat{k}_{t-1} \\
&+ \frac{\alpha\,(1+\eta i_k) - (1+\eta)\,[1+(1-\alpha)c_k]}{[(1+\eta)c_k + \alpha i_k]}\hat{\nu}_t \\
&+ \frac{(1+\eta i_k) - \alpha^{-1}(1+\eta)\,[1+(1-\alpha)c_k]}{[(1+\eta)c_k + \alpha i_k]}\hat{\mu}_t.
\end{aligned}
\tag{A4}
$$

It is clear how an equilibrium can be solved. Once the solution for capital is obtained, as shown in Section IV, Equation (A1) can be used to solve for investment, (A2) for labor, (A3) for output, and (A4) for consumption.

We derive the closed-form solution for labor under RE as

$$
\begin{aligned}
\hat{l}_t =& b_l\varepsilon_{\nu t} + c_l\varepsilon_{\mu t} \\
&+ \sum_{i=1}^{\infty}[a_l b b_{i-1} + b_l\rho_\nu^i]\varepsilon_{\nu,t-i} + \sum_{i=1}^{\infty}[a_l c c_{i-1} + c_l\rho_\mu^i]\varepsilon_{\mu,t-i},
\end{aligned}
\tag{A5}
$$

where

$$
\begin{aligned}
a_l &= \frac{a - 1 + \alpha i_k}{(1+\eta)c_k + \alpha i_k}, \\
b_l &= \frac{b + 1 - \alpha i_k}{(1+\eta)c_k + \alpha i_k}, \\
c_l &= \frac{c + \alpha^{-1} - i_k}{(1+\eta)c_k + \alpha i_k}.
\end{aligned}
$$

Similarly, one can derive the closed-form solutions for investment, output, and consumption under RE.

One can also derive the closed-form solutions for these variables under AE.

Appendix B. Proof of Proposition 1

By successive substitutions in (23)-(27), one can derive (29). Specific steps are described below.

We begin by first deriving the following two relations from (25) and (26):

$$
\hat{y}_t = \hat{i}_t - (1+\eta)c_i\hat{l}_t, \tag{A6}
$$

$$
\hat{c}_t = \hat{i}_t - (1+\eta)y_i\hat{l}_t. \tag{A7}
$$

Substituting (A6) into (23), we get:

$$
\hat{i}_t = [(1+\eta)c_i + \alpha]\,\hat{l}_t + (1-\alpha)\hat{k}_{t-1} - (1-\alpha)\hat{\nu}_t - \left(\frac{1-\alpha}{\alpha}\right)\hat{\mu}_t. \tag{A8}
$$

Substituting (A8) into (24) yields

$$\hat{k}_t = [(1+\eta)c_k + \alpha i_k]\,\hat{l}_t + (1-\alpha i_k)\hat{k}_{t-1} - (1-\alpha i_k)\hat{\nu}_t - \left(\frac{1-\alpha i_k}{\alpha}\right)\hat{\mu}_t. \tag{A9}$$

Substituting (A7) and (A8) into (27) yields

$$\begin{aligned}\left[1 - \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}} + \eta\right] E_t\hat{l}_{t+1} - \left[1 - \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}}\right]\hat{k}_t \\ = (1-\alpha+\eta)\hat{l}_t - (1-\alpha)\hat{k}_{t-1} \\ + (1-\alpha)\hat{\nu}_t + \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}} E_t\hat{\nu}_{t+1} \\ + \frac{1-\alpha}{\alpha}\hat{\mu}_t + \frac{\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}} E_t\hat{\mu}_{t+1}.\end{aligned} \tag{A10}$$

Rewrite (A9) as

$$\begin{aligned}\hat{l}_t =& \frac{1}{(1+\eta)c_k + \alpha i_k}\hat{k}_t \\ &- \frac{1-\alpha i_k}{(1+\eta)c_k + \alpha i_k}\hat{k}_{t-1} \\ &+ \frac{1-\alpha i_k}{(1+\eta)c_k + \alpha i_k}\hat{\nu}_t \\ &+ \frac{\{1-\alpha i_k\}\left(\frac{1}{\alpha}\right)}{(1+\eta)c_k + \alpha i_k}\hat{\mu}_t.\end{aligned} \tag{A11}$$

It follows that

$$\begin{aligned}\left[1 - \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}} + \eta\right] E_t\hat{l}_{t+1} =& \frac{\left[1 - \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}} + \eta\right]}{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left(\frac{1-c_y}{i_y}\right)\left[\frac{c_y(1+\eta)}{1-c_y} + \alpha\right]} E_t\hat{k}_{t+1} \\ &- \frac{\left[1 - \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}} + \eta\right]\{1-\alpha i_k\}}{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left(\frac{1-c_y}{i_y}\right)\left[\frac{c_y(1+\eta)}{1-c_y} + \alpha\right]}\hat{k}_t \\ &+ \frac{\left[1 - \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}} + \eta\right]\left\{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left[\frac{(1-c_y)(1-\alpha)}{i_y}\right] + \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right\}}{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left(\frac{1-c_y}{i_y}\right)\left[\frac{c_y(1+\eta)}{1-c_y} + \alpha\right]} E_t\hat{\nu}_{t+1} \\ &+ \frac{\left[1 - \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}} + \eta\right]\left\{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left[\frac{(1-c_y)(1-\alpha)}{i_y}\right] + \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right\}\left(\frac{1}{\alpha}\right)}{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left(\frac{1-c_y}{i_y}\right)\left[\frac{c_y(1+\eta)}{1-c_y} + \alpha\right]} E_t\hat{\mu}_{t+1}.\end{aligned} \tag{A12}$$

Substituting (A11) and (A12) into (A10), and rearranging, we get

$$\chi_{k,1}E_t\hat{k}_{t+1} + \chi_{k,0}\hat{k}_t + \chi_{k,-1}\hat{k}_{t-1} + \chi_{\nu,1}E_t\hat{\nu}_{t+1} + \chi_{\nu,0}\hat{\nu}_t + \chi_{\mu,1}E_t\hat{\mu}_{t+1} + \chi_{\mu,0}\hat{\mu}_t = 0,$$

where

$$\chi_{k,1} = \frac{\left[1 - \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}} + \eta\right]}{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left(\frac{1-c_y}{i_y}\right)\left[\frac{c_y(1+\eta)}{1-c_y} + \alpha\right]}$$

$$\chi_{k,0} = -\frac{\left[1 - \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}} + \eta\right]\{1-\alpha i_k\} + (1-\alpha+\eta)}{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left(\frac{1-c_y}{i_y}\right)\left[\frac{c_y(1+\eta)}{1-c_y} + \alpha\right]} - \left[1 - \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}}\right]$$

$$\chi_{k,-1} = \frac{(1-\alpha+\eta)\{1-\alpha i_k\}}{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left(\frac{1-c_y}{i_y}\right)\left[\frac{c_y(1+\eta)}{1-c_y} + \alpha\right]} + (1-\alpha)$$

$$\chi_{\nu,1} = \frac{\left[1 - \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}} + \eta\right]\left\{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left[\frac{(1-c_y)(1-\alpha)}{i_y}\right] + \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right\}}{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left(\frac{1-c_y}{i_y}\right)\left[\frac{c_y(1+\eta)}{1-c_y} + \alpha\right]} - \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}}$$

$$\chi_{\nu,0} = -\frac{(1-\alpha+\eta)\left\{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left[\frac{(1-c_y)(1-\alpha)}{i_y}\right] + \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right\}}{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left(\frac{1-c_y}{i_y}\right)\left[\frac{c_y(1+\eta)}{1-c_y} + \alpha\right]} - (1-\alpha)$$

$$\chi_{\mu,1} = \frac{\left[1 - \frac{\alpha\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}} + \eta\right]\left\{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left[\frac{(1-c_y)(1-\alpha)}{i_y}\right] + \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right\}\left(\frac{1}{\alpha}\right)}{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left(\frac{1-c_y}{i_y}\right)\left[\frac{c_y(1+\eta)}{1-c_y} + \alpha\right]} - \frac{\beta(1-\delta)}{\lambda_z\lambda_q^{1/\alpha}}$$

$$\chi_{\mu,0} = -\frac{(1-\alpha+\eta)\left\{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left[\frac{(1-c_y)(1-\alpha)}{i_y}\right] + \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right\}\left(\frac{1}{\alpha}\right)}{\left(1 - \frac{1-\delta}{\lambda_z\lambda_q^{1/\alpha}}\right)\left(\frac{1-c_y}{i_y}\right)\left[\frac{c_y(1+\eta)}{1-c_y} + \alpha\right]} - \frac{1-\alpha}{\alpha}.$$

Further simplifying, we get

$$\hat{k}_t = \chi^k_{k,1}E_t\hat{k}_{t+1} + \chi^k_{k,-1}\hat{k}_{t-1} + \chi^k_{\nu,1}E_t\hat{\nu}_{t+1} + \chi^k_{\nu,0}\hat{\nu}_t + \chi^k_{\mu,1}E_t\hat{\mu}_{t+1} + \chi^k_{\mu,0}\hat{\mu}_t,$$

where

$$\chi^k_{k,1} = \frac{\beta(1-\alpha)(c_k+1)+\eta}{\beta(1-\alpha)(c_k+1)\left[1+(1+\eta)c_k\right] + \eta(1-\alpha i_k) + (1-\alpha+\eta)}$$

$$\chi^k_{k,-1} = \frac{(1-\alpha)(1+\eta)y_k + (1-\alpha+\eta)(1-i_k)}{\beta(1-\alpha)(c_k+1)\left[1+(1+\eta)c_k\right] + \eta(1-\alpha i_k) + (1-\alpha+\eta)}$$

$$\chi^k_{\nu,1} = \frac{\left[\beta(1-\alpha)(c_k+1)-1\right]\left[(1+\eta)c_k+1\right] + (1+\eta)(1-\alpha i_k)}{\beta(1-\alpha)(c_k+1)\left[1+(1+\eta)c_k\right] + \eta(1-\alpha i_k) + (1-\alpha+\eta)}$$

$$\chi^k_{\nu,0} = -\frac{(1-\alpha+\eta)(c_k+1) - \alpha\eta y_k}{\beta(1-\alpha)(c_k+1)\left[1+(1+\eta)c_k\right] + \eta(1-\alpha i_k) + (1-\alpha+\eta)}$$

$$\chi^k_{\mu,1} = \frac{[\beta(1-\alpha)(c_k+1)-1]\,[(1+\eta)c_k+1]+(1+\eta)(1-\alpha i_k)}{\beta(1-\alpha)(c_k+1)\,[1+(1+\eta)c_k]+\eta(1-\alpha i_k)+(1-\alpha+\eta)}\left(\frac{1}{\alpha}\right)$$

$$\chi^k_{\mu,0} = -\frac{(1-\alpha+\eta)(c_k+1)-\alpha\eta y_k}{\beta(1-\alpha)(c_k+1)\,[1+(1+\eta)c_k]+\eta(1-\alpha i_k)+(1-\alpha+\eta)}\left(\frac{1}{\alpha}\right).$$

Simplifying further, we have

$$\hat{k}_t = \chi^k_{k,1}E_t\hat{k}_{t+1} + \chi^k_{k,-1}\hat{k}_{t-1} + \left(\rho_\nu\chi^k_{\nu,1}+\chi^k_{\nu,0}\right)\hat{\nu}_t + \left(\rho_\mu\chi^k_{\mu,1}+\chi^k_{\mu,0}\right)\hat{\mu}_t,$$

which gives the results in Appendix A.

Appendix C. Proof of Proposition 2

Because (29) is a second-order differential equation, there are only two solutions. We will show, next, that one solution is stationary and the other explosive. Thus, there is a unique stationary solution.

The coefficient a in (30) takes on one of the following two values:

$$a_1 = \frac{1-\sqrt{1-4\gamma_1\gamma_2}}{2\gamma_1}, \qquad a_2 = \frac{1+\sqrt{1-4\gamma_1\gamma_2}}{2\gamma_1}.$$

We can verify that $\gamma_1 > 0$ and $\gamma_2 > 0$ for all admissible values of the deep parameters. We can further show that $\gamma_1+\gamma_2 < 1$ if and only if $\beta(1-\delta) < \lambda_z\lambda_q^{1/\alpha}$, which holds too for all admissible values of the deep parameters.

Since $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_1+\gamma_2 < 1$, we have $\gamma_1 \in (0,1)$, $\gamma_2 \in (0,1)$, and $4\gamma_1\gamma_2 < 1$. It follows that a_1 and a_2 are real numbers. Knowing the above ranges for γ_1 and γ_2, we can in fact show that $a_1 \in (0,1)$ and $a_2 > 1$. We can then verify that $(\rho_\nu + a_1)\gamma_1 < 1$ and $(\rho_\mu + a_1)\gamma_1 < 1$, and so the solution prescribed by $a = a_1$ above corresponds to a (unique) stationary rational expectations equilibrium.[6] Given the initial condition $\hat{k}_{-1}$ and the driving processes, (30) completely pins down capital, and then (A1), (A2), (A3), and (A4) determine investment, labor, output, and consumption, respectively. From now on, whenever we mention REE, we refer to this stationary REE, where we also write a_1 simply as a.

[6]We can also show that, provided $\rho_\nu \neq a_1$ and $\rho_\mu \neq a_1$, the solution prescribed by $a = a_2$ above corresponds to an explosive path.

Table 1. Parameter values

Preference	$\beta = 0.9952$	$\xi = 3.17$	$\eta = 2.0$
Labor share	$\alpha = 0.7$		
Capital Depreciation	$\delta = 0.025$		
Neutral Technology	$\lambda_z = 1.0016$	$\rho_\nu = 0$	$\sigma_\nu = 0.2$
Biased Technology	$\lambda_q = 1.008$	$\rho_\mu = 0$	$\sigma_\mu = 0.2$
Learning Gain	$g = 0.1$		

Table 2. Relative contribution of biased technology shocks: learning vs. rational expectations

Forecast Horizon	Output	Consumption	Investment	Hours
1 quarter	27.28	2.50	1.67	1.00
4 quarters	14.02	3.86	1.70	1.00
8 quarters	7.78	6.35	1.75	1.00
16 quarters	3.94	12.25	1.86	1.00
24 quarters	2.71	11.27	1.96	1.00

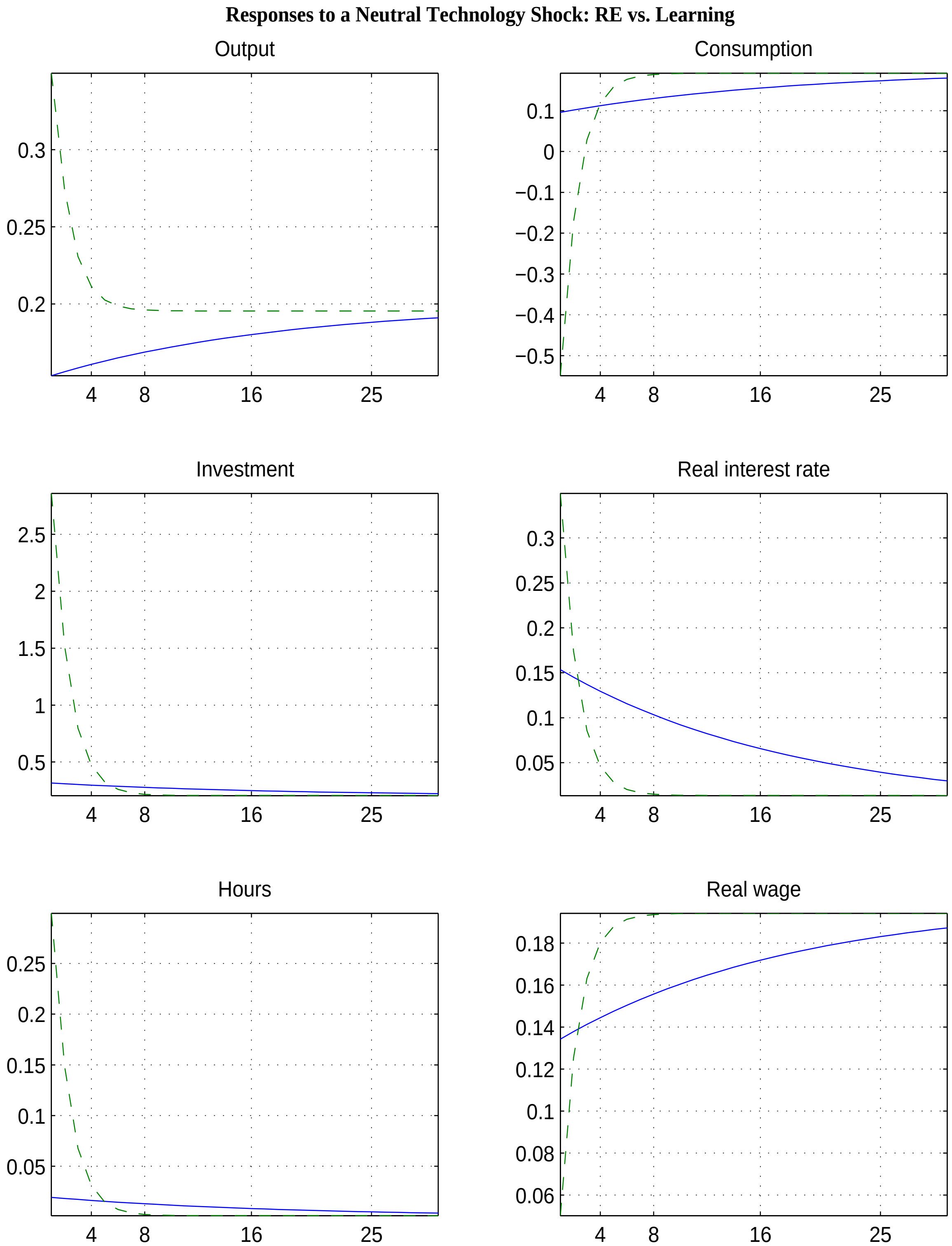

FIGURE 1. Impulse responses to the neutral technology shock. The solid line represents the responses from the model with rational expectations. The dashed line represents the responses from the model with adaptive expectations.

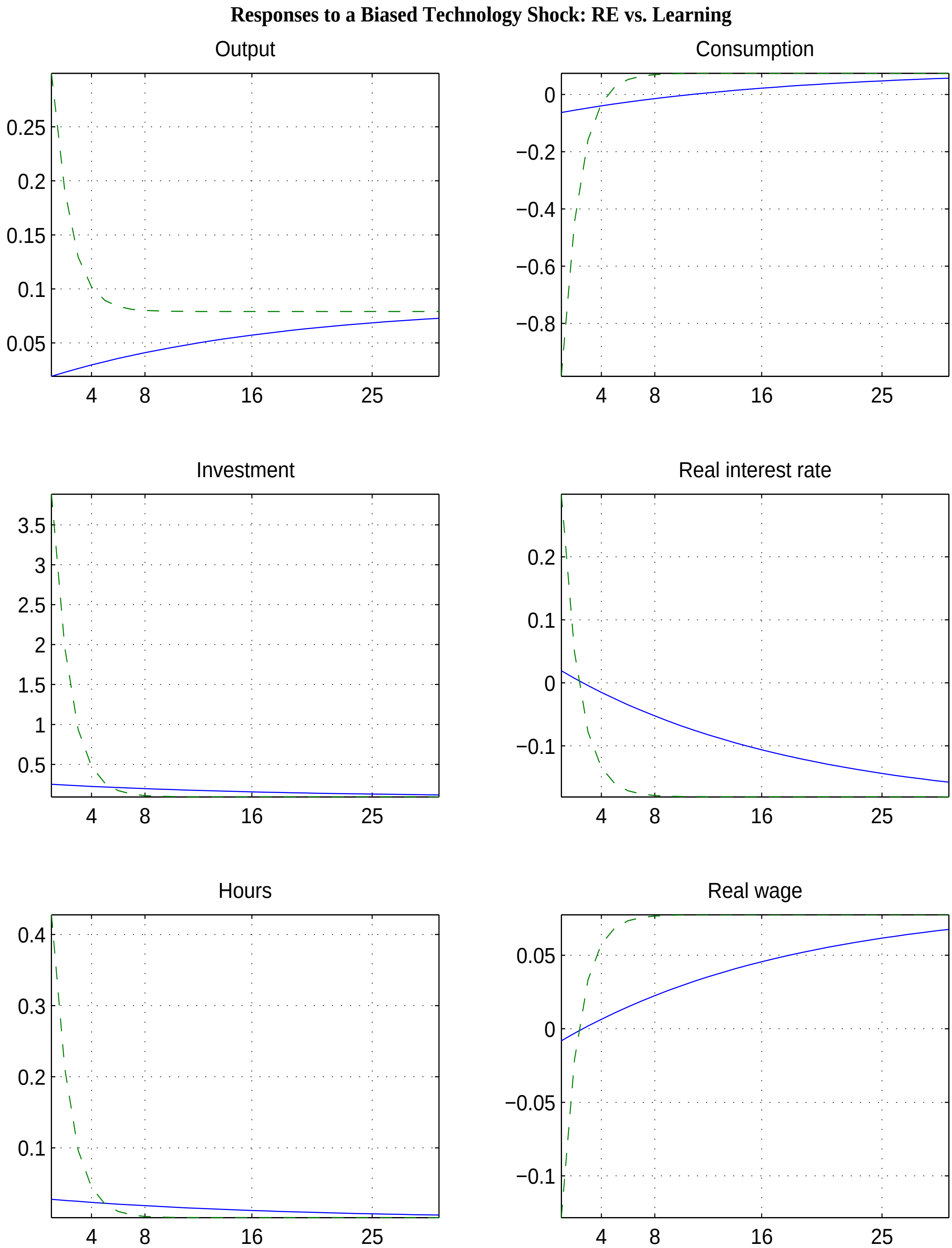

FIGURE 2. Impulse responses to the biased technology shock. The solid line represents the responses from the model with rational expectations. The dashed line represents the responses from the model with adaptive expectations.

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