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Saving and Investing for Early Retirement: A Theoretical Analysis

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Abstract

We study optimal consumption and portfolio choice in a framework where investors save for early retirement and assume that agents can adjust their labor supply only through an irreversible choice of their retirement time. We obtain closed form solutions and analyze the joint behavior of retirement time, portfolio choice, and consumption. Investing for early retirement tends to increase savings and stock market exposure, and reduce the marginal propensity to consume out of accumulated personal wealth. Contrary to common intuition, prior to retirement an investor might find it optimal to increase the proportion of financial wealth held in stocks as she ages, even when she receives a constant income stream and the investment opportunity set is also constant. This is particularly true when the wealth of the investor increases rapidly due to strong stock market performance, as was the case in the late 1990's. We also show that the model can potentially provide a rational explanation for the paradoxical fact that some investors saving for retirement chose to increase their alloaction to stocks as the market was booming and reduce it thereafter.

JEL Codes: G0, E2

Keywords: Continuous time, Optimal Stopping, Retirement, Life Cycle Portfolio Choice, Savings, Marginal Propensity to Consume, Indivisible Labor

1 Introduction

> Two years ago, when the stock market was soaring, 401(k)'s were swelling and (..) early retirement seemed an attainable goal. All you had to do was invest that big job-hopping pay increase in a market that produced double-digit gains like clockwork, and you could start taking leisurely strolls down easy street at the ripe old age of, say, 55. (Business Week 31 December 2001)

The dramatic rise of the stock market between 1995 and 2000 significantly increased the proportion of workers opting for early retirement (Gustman and Steinmeier [2002]). The above quote from Business Week demonstrates the reasoning behind the decision to retire early: A booming stock market raises the amount of funds available for retirement and allows a larger fraction of the population to exit the workforce prematurely.

As a matter of fact, retirement savings seem to be one of the primary motivations behind investing in the stock market for most individuals. Accordingly, there is an increased need to understand the interactions between optimal retirement, portfolio choice, and savings, especially in light of the growing popularity of 401(k) retirement plans. These plans give individuals a great amount of freedom when determining how to save for retirement. This increased flexibility has also raised concerns about the rationality in agents' portfolio and savings decisions. Having a benchmark against which to check the rationality of people's choices is crucial both from a policy perspective and in order to form the basis of sound financial advice.

In this paper we develop a theoretical model to address some of the interactions between savings, portfolio choice and retirement in a utility maximizing framework. We assume that agents are faced with a constant investment opportunity set and a constant wage rate while still in the workforce. Their utility exhibits constant relative risk aversion and is nonseparable in leisure and consumption. The major point of departure from preexisting literature is that we model the labor supply choice as an optimal stopping problem: An individual can work for a fixed (non-adjustable) amount of time and earn a constant wage but is free to exit the workforce (forever) at any time she chooses. In other words, we assume that workers can work either full time or retire. As such, individuals are faced with three questions to decide: 1) how much to consume 2) how to invest the savings and 3) when to retire. The incentive to quit work comes from a discrete jump in their utility due to an increase in leisure once retired. When retired, they cannot return to the workforce.[1] We also consider two extensions of the basic framework. In the first extension we disallow the agent to choose retirement past a prespecified deadline. In a second extension we disallow her to borrow against the net present value (NPV) of her human capital. (i.e. we require financial wealth to be non-negative)

The major results that we obtain can be summarized as follows:

First, we show that the agent will enter retirement when a certain wealth threshold is reached, which we determine explicitly. In this sense, wealth plays a dual role in our model. Not only does it determine the resources available for future consumption, it also controls the "distance" to retirement.

Second, the option to retire early strengthens the incentives to save compared to the case where early retirement is not allowed. The reason is that saving not only increases consumption in the future but also brings retirement "closer". Moreover, this incentive is wealth dependent: As the individual approaches the critical wealth threshold to enter retirement, the "option" value of retiring early becomes progressively more important and the saving motive becomes stronger.

Third, the marginal propensity to consume (mpc) out of wealth declines as wealth increases and early retirement becomes more likely. The intuition is simple: An increase in wealth will bring retirement closer, therefore decreasing the length of time the individual remains in the workforce. Conversely, a decline in wealth will postpone retirement. Thus, changes in wealth are somewhat counterbalanced by the behavior of the remaining NPV of income and thus the effect of a marginal change in wealth on consumption becomes attenuated. Once again this attenuation is strongest for rich individuals who are closer to their goal of early retirement.

Fourth, the optimal portfolio is tilted more towards stocks compared to the case where early retirement is not allowed. An adverse shock in the stock market will be absorbed by postponing the retirement time. Thus, the individual is more inclined to take risks as she can always postpone her retirement time instead of cutting back her consumption in the event of a declining stock market. Moreover, in order to bring retirement closer, the most effective way is to invest the extra savings in the stock market instead of the bond market.

Fifth, the choice of portfolio over the life cycle exhibits some new and interesting patterns. We show that there exist cases where an agent might find it optimal to increase the percent of financial wealth that she invests in the stock market as she ages (in expectation). This is true *even though* her income and the investment opportunity set are constant. This happens, because wealth increases over time and hence the option of early retirement becomes more relevant ("in the money" in option pricing terminology). Accordingly, the tilting of the optimal portfolio towards stocks becomes stronger. Indeed, as we show in a calibration exercise, the model predicts that -prior to retirement- portfolio holdings could increase, especially when the stock market exhibits extraordinary returns as it did in the late 1990's when many workers experienced rapid increases in wealth, thus opting for an earlier retirement date. In fact our model suggests a possible partial rationalization for the (apparently irrational) behavior of individuals who increased their portfolios as the stock market was rising and then liquidated stock as the market collapsed.[2] We believe that this model is particularly succesful in accounting for this paradoxical fact in a plausible rational way.

This paper is related to a number of strands in the literature, which is surveyed in Ameriks and Zeldes (2001)[3]. The paper closest to ours is Bodie, Merton, and Samuelson (1992) (henceforth BMS). The major difference between BMS and this paper is the different assumption about the ability of agents to adjust their labor supply. In BMS labor can be adjusted in a continuous fashion. However, there seems to be a significant amount of evidence that labor supply is to a large extent indivisible. In many jobs workers work either full time or they are retired. Moreover, it appears that most people do not return to work after they retire, or if they do, they return to less well paying jobs or work only part time. As BMS claim in the conclusion of their paper

> Obviously, the opportunity to vary continuously one's labor without cost is a far cry from the workings of actual labor markets. A more realistic model would allow limited flexibility in varying labor and leisure. One current research objective is to analyze the retirement problem as an optimal stopping problem and to evaluate the accompanying portfolio effects.

This is precisely the direction we take here. There are at least two major directions in which our results differ from BMS. First, we show that the optimal retirement decision introduces an option-type element in the decision of the individual, that is entirely absent if labor is adjusted continuously. Second, the horizon and wealth effects on portfolio and consumption choice in our paper are fundamentally different than in BMS. For instance, the holdings of stock in BMS are a *constant multiple* of the sum of (financial) wealth and human capital. This multiple is not constant in our setup, but instead depends on wealth[4]. Third, the model presented here allows for a clear way to model retirement, which is difficult in the literature allowing for a continuous choice of labor/leisure. An important implication is that in our setup we can calibrate the parameters of the model to observed retirement decisions. In the BMS framework calibration to microeconomic data is harder, because individuals do not seem to adjust their labor supply continuously. Liu and Neiss (2002) study a framework similar to BMS, but force an important constraint on the maximal amount of leisure. This however omits the issues related to indivisibility and irreversibility, which lead to fundamentally different implications for the resulting portfolios as we show. In sum, the fact that labor supply flexibility is modeled in a more realistic way allows a closer mapping of the results to real world institutions than is allowed by a model exhibiting continuous choice between labor and leisure.

The model is also related to a strand of the literature that studies retirement decisions. A partial listing would include Stock and Wise (1990), Rust (1994), Laezar (1986), Rust and Phelan (1997), and Diamond and Hausman (1984). Most of these models are structural estimations that are solved numerically. Here our goal is different. We do not include all the realistic ramifications that are present in actual retirement systems. Instead, we isolate and very closely analyze the new issues introduced by indivisibility and irreversibility of the labor supply / retirement decision on savings and portfolio choice. Naturally, there is a tradeoff between adding realistic condiderations and the level of theoretical analysis that can be accomplished with a more complicated model.

Sundaresan and Zapatero (1997) study optimal retirement, but in a framework without disutility of labor.

Some results of this paper share some similarities with results obtained in the literature on consumption and savings in incomplete markets. A highly partial listing would include Viceira (2001), Chan and Viceira (2000), Campbell et al. (2004), Kogan and Uppal (2001), Duffie et al. (1997), Duffie and Zariphopoulou (1993), Koo (1998), and Caroll and Kimball (1996) on the role of incomplete markets and He and Pages (1993) and El Karoui and Jeanblanc Pique (1998) on issues related to the inability of individuals to borrow against the NPV of their future income. This literature produces some insights on why consumption (as a function of wealth) should be concave, and also has some implications on life cycle portfolio choice. However, the intuitions are quite different from the ones we obtain here. In this paper the results are driven by an option component in agent's choices that is related to their ability to adjust their time of retirement. In the incomplete markets literature results are driven by agents' inability to effectively smooth their consumption due to missing markets.[5]

Throughout the paper we maintain the assumption that agents receive a constant wage. This is done not only for simplicity, but *because it makes the results most surprising*: It is well understood in the literaure[6] that allowing for a (positive) correlation between wages and the stock market can produce upward sloping portfolio holdings over the life cycle. What we show is that optimal retirement choice can induce similar effects *even when labor income is perfectly riskless*. Since the argument and the intuition for this fact is orthogonal to existing models we prefer to keep the simplest possible setup in every other dimension, so as to *isolate* the effects of optimal early retirement.

The role of labor supply flexibility in a general equilibrium model with continuous labor/leisure choice is considered in Basak (1999). It is very likely that the results we present in this paper could form the basis for a general equilibrium extension. It is well known in the macroeconomics literature that allowing indivisible labor is quite important if one is to explain the volatility of employment relative to wages. See for example Hansen (1985) and Rogerson (1988).

Technically, we extend methods proposed by Karatzas and Wang (2000) (who do not allow for income) to solve optimal consumption problems with discretionary stopping. The extension that we consider in section 5 uses some ideas proposed in Barone-Adesi and Whaley (1987), while section 6 extends the framework in He and Pages (1993) to allow for early retirement.

Finally, three papers that present parallel and independent work on similar issues are Lachance (2003), Choi and Shim (2004), and Dybvig and Liu (2005). Lachance (2003) and Choi and Sim (2004) study a model with a utility function that is separable in leisure and consumption, but without a deadline for retirement and / or borrowing constraints. Moreover, separable utility does not allow consumption drops upon retirement as the ones observed in the data. Technically, these papers use dynamic programming and not convex duality methods in order to solve the problem, which cannot be easily extended to models with deadlines, borrowing constraints, etc. Our approach overcomes these difficulties. Dybvig and Liu (2005) study a very similar model to that in section 7 of this paper with similar techniques. However they do not consider retirement prior to a deadline as we do. A deadline makes the problem considerably harder (since the critical wealth thresholds become time dependent) and we are able to provide a fairly accurate approximate closed form solution for this problem in Section 5. One can actually perform simple exercises[7] which demonstrate that in the absence of a retirement deadline the distribution of retirement times implied by the model becomes implausible. Most importantly, compared to the papers above we believe that the present paper goes into significantly greater detail in terms of the economic analysis and implications of the results. In particular we provide applications (like the analysis of portfolios of agents saving for early retirement in the late nineties) which demonstrate most clearly the real world implications of optimal portfolio choice in the presence of early retirement.

The structure of the paper is as follows: Section 2 contains the model setup. Section 3 presents the solution. In section 4 we describe the analytical results if one places no retirement deadline. Section 5 contains an extension to the case where retirement cannot take place past a deadline. Section 6 contains some calibration exercises. Section 7 extends the model by imposing borrowing constraints and section 8 concludes. The technical details and all proofs can be found in the Appendix.

2 Model Setup

2.1 Investment Opportunity Set

The consumer can invest in the money market, where she receives a fixed strictly positive interest rate $r > 0$. We place no limits on the positions that can be taken in the money market. In addition the consumer can invest in a risky security with a price per share that evolves as

$$\frac{dP_t}{P_t} = \mu dt + \sigma dB_t$$

where $\mu > r$ and $\sigma > 0$ are given constants and B_t is a one-dimensional Brownian motion on a complete probability space (Ω, F, P).[8] We finally define the state price density process (or stochastic discount factor) as

$$H(t) = \gamma(t) Z^*(t), \qquad H(0) = 1$$

where $\gamma(t)$ and $Z^*(t)$ are defined as

$$\begin{aligned}\gamma(t) &= e^{-rt} \\ Z^*(t) &= \exp\left\{-\int_0^t \kappa dB_s - \frac{1}{2}\kappa^2 t\right\}, \quad Z^*(0) = 1\end{aligned}$$

and κ is the Sharpe ratio

$$\kappa = \frac{\mu - r}{\sigma}$$

It is a standard result, that these assumptions imply a dynamically complete market (Karatzas and Shreve [1998] Chapter 1).

2.2 Portfolio and Wealth Processes

An agent chooses a portfolio process π_t and a consumption process $c_t > 0$, which are progressively measurable and satisfy the standard integrability conditions given in Karatzas and Shreve (1998) Chapters 1 and 3. She also receives a constant income stream y_0 as long as she works and no income stream once in retirement. Retirement is an irreversible decision. Until section 5 we will assume that an agent can retire at any time that she chooses.

The agent is endowed with an amount of financial wealth $W_0 \geq -\frac{y_0}{r}$. The process of stock-holdings π_t is the *dollar amount* invested in the risky asset (the "stock market") at time t. The rest, $W_t - \pi_t$, is invested in the money market. Short selling and borrowing are both allowed. We will place no extra restrictions on the (financial) wealth process W_t until section 7 of the paper. Additionally in that section we will impose the restriction $W_t \geq 0$. As long as the agent is working, the wealth process evolves as

$$dW_t = \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt - (c_t - y_0) dt \tag{1}$$

Applying Ito's Lemma to the product of $H(t)$ and $W(t)$, integrating and taking expectations we get for any stochastic time τ that is finite almost surely

$$E\left(H(\tau)W(\tau) + \int_0^\tau H(s)\left[c(s) - y_0\right] ds\right) \leq W_0 \tag{2}$$

This is the well known result that in dynamically complete markets one can reduce a dynamic budget constraint of the type (1) to a single intertemporal budget constraint of the type (2). If the agent is retired the above two equations continue to hold with $y_0 = 0$.

2.3 Leisure, Income and the Optimization Problem

To obtain closed form solutions, we assume that the consumer has a utility function of the form

$$U(l_t, c_t) = \frac{1}{\alpha} \frac{\left(l_t^{1-\alpha} c_t^{\alpha}\right)^{1-\gamma^*}}{1-\gamma^*}, \quad \gamma^* > 0 \tag{3}$$

where c_t is per period consumption, l_t is leisure and $0 < \alpha < 1$. We assume that the consumer is endowed with $\bar{l}$ units of leisure. l_t can only take two values l_1 or $\bar{l}$. If the consumer is working, then $l_t = l_1$; when retired $l_t = \bar{l}$. We will assume that the wage rate w is constant, so that the income stream is $y_0 = w(\bar{l} - l_1) > 0$. We will normalize $l_1 = 1$. Observe also that this utility is general enough so as to allow consumption and leisure to be either complements ($\gamma^* < 1$) or substitutes ($\gamma^* > 1$). The consumer maximizes expected utility

$$\max_{c_t, \pi_t, \tau} E\left[\int_0^{\tau} e^{-\beta t} U(l_1, c_t) dt + e^{-\beta \tau} \int_{\tau}^{\infty} e^{-\beta(t-\tau)} U(\bar{l}, c_t) dt\right] \tag{4}$$

where $\beta > 0$ is the agent's discount factor.[9] The easiest way to proceed is to start backwards by solving the problem

$$U_2(W_\tau) = \max_{c_t, \pi_t} E\left[\int_{\tau}^{\infty} e^{-\beta(t-\tau)} U(\bar{l}, c_t) dt\right]$$

$U_2(W_\tau)$ is the Value function once the consumer decides to retire and W_τ is the wealth at retirement. By the principle of dynamic programming we can rewrite (4) as

$$\max_{c_t, W_\tau, \tau} E\left[\int_0^{\tau} e^{-\beta t} U(l_1, c_t) dt + e^{-\beta \tau} U_2(W_\tau)\right] \tag{5}$$

It will be convenient to define the parameter γ as

$$\gamma = 1 - \alpha(1 - \gamma^*)$$

so we can then re-express the per-period utility function as

$$U(l, c) = l^{(1-\alpha)(1-\gamma^*)} \frac{c^{1-\gamma}}{1-\gamma}$$

Since we have normalized $l = 1$ prior to retirement, the per period utility prior to retirement is given by:

$$U_1(c) = U(1, c) = \frac{c^{1-\gamma}}{1-\gamma} \tag{6}$$

Notice that $\gamma > 1$ if and only if $\gamma^* > 1$ and $\gamma < 1$ if and only if $\gamma^* < 1$. Under these assumptions, it follows from standard results (See for example Karatzas and Shreve [1998], Chapter 3), that once in retirement the Value function becomes

$$U_2(W_\tau) = \left(\bar{l}^{1-\alpha}\right)^{1-\gamma^*} \left(\frac{1}{\theta}\right)^{\gamma} \frac{W_\tau^{1-\gamma}}{1-\gamma} \tag{7}$$

where

$$\theta = \frac{\gamma-1}{\gamma}\left(r + \frac{\kappa^2}{2\gamma}\right) + \frac{\beta}{\gamma}$$

In order to guarantee that the Value function is well defined, we assume throughout that $\theta > 0$[10] and $\beta - r < \frac{\kappa^2}{2}$[11]. It will be convenient to redefine the continuation Value function as

$$U_2(W_\tau) = K\frac{W_\tau^{1-\gamma}}{1-\gamma}$$

where

$$K = \left(\bar{l}^{1-\alpha}\right)^{1-\gamma^*} \left(\frac{1}{\theta}\right)^{\gamma} \tag{8}$$

Since $\bar{l} > l_1 = 1$ we have that

$$K^{\frac{1}{\gamma}} > \frac{1}{\theta} \text{ if } \gamma < 1 \tag{9}$$

$$K^{\frac{1}{\gamma}} < \frac{1}{\theta} \text{ if } \gamma > 1 \tag{10}$$

3 Solution

We present the solution to the problem described in section 2.3 and discuss the properties of the joint retirement /consumption and portfolio choice problem, when retirement is an irreversible discrete decision.

Proposition 1 *Define the constants*

$$\gamma_2 = \frac{1 - 2\frac{\beta-r}{\kappa^2} - \sqrt{(1-2\frac{\beta-r}{\kappa^2})^2 + 8\frac{\beta}{\kappa^2}}}{2} \tag{11}$$

$$\underline{\lambda} = \left(\frac{(\gamma_2 - 1)\theta}{\left(1+\gamma_2\frac{\gamma}{1-\gamma}\right)\left(K^{\frac{1}{\gamma}}\theta - 1\right)}\frac{y_0}{r}\right)^{-\gamma} \tag{12}$$

$$C_2 = \frac{\left[\frac{\gamma}{1-\gamma}\frac{(\gamma_2-1)}{\left(1+\gamma_2\frac{\gamma}{1-\gamma}\right)} - 1\right]\frac{y_0}{r}}{\underline{\lambda}^{\gamma_2-1}}$$

and assume that[12]

$$\frac{r}{\theta}\frac{\left(\frac{1-\gamma}{\gamma}+\gamma_2\right)}{\gamma_2-1}<1$$

Finally let λ^* *be the (unique) solution of*

$$\gamma_2 C_2 (\lambda^*)^{\gamma_2-1}-\frac{1}{\theta}(\lambda^*)^{-\frac{1}{\gamma}}+\frac{y_0}{r}+W_t=0 \tag{13}$$

Then

$$C_2>0, \gamma_2<0$$

and the optimal policy is

a) If $W_t<\overline{W}=\frac{(\gamma_2-1)K^{\frac{1}{\gamma}}\theta}{\left(1+\gamma_2\frac{\gamma}{1-\gamma}\right)\left(K^{\frac{1}{\gamma}}\theta-1\right)}\frac{y_0}{r}$

consumption follows the process

$$c_s = \left(\lambda^* e^{\beta(s-t)}\frac{H(s)}{H(t)}\right)^{-\frac{1}{\gamma}} 1\{t\le s<\tau^*\} \tag{14}$$

$$c_s = \bar{l}^{(1-\alpha)\frac{(1-\gamma^*)}{\gamma}}\left(\lambda^* e^{\beta(s-t)}\frac{H(s)}{H(t)}\right)^{-\frac{1}{\gamma}} 1\{s\ge\tau^*\} \tag{15}$$

The optimal retirement time is

$$\begin{aligned}\tau^* &= \inf\{s: W_s=\overline{W}\}= \\ &= \inf\left\{s:\lambda^* e^{\beta(s-t)}\frac{H(s)}{H(t)}=\underline{\lambda}\right\}\end{aligned} \tag{16}$$

Moreover, the optimal consumption and the optimal stockholdings as a function of W_t *are given by*

$$c_t = c(W_t)=(\lambda^*(W_t))^{-\frac{1}{\gamma}} \tag{17}$$

$$\pi_t = \pi(W_t)=\frac{\kappa}{\sigma}\left(\gamma_2(\gamma_2-1)C_2\lambda^*(W_t)^{\gamma_2-1}+\frac{1}{\gamma}\frac{1}{\theta}\lambda^*(W_t)^{-\frac{1}{\gamma}}\right) \tag{18}$$

where the notation $\lambda^*(W_t)$ *is used to make the dependence of* λ^* *on* W_t *explicit.*

b) If $W_t\ge\overline{W}=\frac{(\gamma_2-1)K^{\frac{1}{\gamma}}\theta}{\left(1+\gamma_2\frac{\gamma}{1-\gamma}\right)\left(K^{\frac{1}{\gamma}}\theta-1\right)}\frac{y_0}{r}$ *the optimal solution is to enter retirement immediately* $(\tau^*=t)$ *and the optimal consumption /portfolio policy is given as in Karatzas and Shreve (1998) section 3.*

The nature of the solution is intuitive: The agent enters retirement if and only if the level of her assets exceeds $\overline{W}$. Up to that point her consumption is given by (14), whereas it jumps to (15) once retired. The jump is given by:

$$\frac{c_{\tau^+}}{c_{\tau^-}} = \bar{l}^{(1-\alpha)\frac{(1-\gamma^*)}{\gamma}} = K^{1/\gamma}\theta \qquad (19)$$

where the first equality follows by dividing (15) by (14) and the second from (8). Notice that $\gamma^* > 1$ will imply a downward and $\gamma^* < 1$ an upward jump (since $\bar{l} > 1$). A key quantity in all the solutions is λ^*. It can be shown that λ^* is the derivative of the value function and is decreasing in W_t. Formally, letting $J(W_t)$ be the value function of the problem, $J_W = \lambda^*(W_t)$. Equation (17) suggests an alternate interpretation of λ^* as the marginal utility of consumption. In other words, (17) is the standard "Euler" equation $U'(c_t) = J_W$. Moreover, an application of the implicit function theorem to (13) is sufficient to show that equation (18) leads to:

$$\pi_t = -\frac{\kappa}{\sigma}\frac{J_W}{J_{WW}} = -\frac{\mu - r}{\sigma^2}\frac{J_W}{J_{WW}}$$

This is the familiar Merton formula for the optimal portfolio. λ^* solves the simple nonlinear equation (13) and hence one can study analytically the dependence of λ^* on W_t, something that we do repeatedly in the next section.

4 Properties of the solution

In this section we explore some properties of the solution. A central theme of the analysis is the presence of an option element in the decision of the agent, that fundamentally alters the nature of the optimal consumption / portfolio decision. The benchmark model to which we compare the results is a model where there is a constant labor income stream and no retirement (the worker works forever). This is the natural benchmark *for this section*, since it keeps all else equal except for the the option to retire.

The results obtained in this section allow us to isolate intuitions related to optimal retirement, in a framework where solutions are not time dependent and therefore easier to analyze. Fortunately, all of the results obtained in this section carry over to the next, where we introduce a retirement deadline. In that section the benchmark to which we compare the optimal retirement model is more natural: Namely, retirement is mandatory at time T, but without the option to retire early.

4.1 Wealth at Retirement

Wealth at retirement is given by Proposition 1 as

$$\overline{W} = \frac{(\gamma_2 - 1)K^{\frac{1}{\gamma}}\theta}{\left(1 + \gamma_2 \frac{\gamma}{1-\gamma}\right)\left(K^{\frac{1}{\gamma}}\theta - 1\right)} \frac{y_0}{r}$$

As Proposition 1 asserts, for wealth levels higher than this, it is optimal to enter retirement. For lower wealth levels it is optimal to remain in the workforce. In the appendix we show that $\overline{W}$ is strictly positive, i.e. a consumer will never enter retirement with negative wealth since there is no more income to support post-retirement consumption.

Using (19) we can rewrite $\overline{W}$ as the product of three terms:

$$\overline{W} = \frac{(\gamma_2 - 1)}{\left(1 + \gamma_2 \frac{\gamma}{1-\gamma}\right)} \frac{\left(\bar{l}^{1-\alpha}\right)^{\frac{1-\gamma^*}{\gamma}}}{\left(\left(\bar{l}^{1-\alpha}\right)^{\frac{1-\gamma^*}{\gamma}} - 1\right)} \frac{y_0}{r} \tag{20}$$

The last term shows the linear dependence of $\overline{W}$ on y_0. This homogeneity of degree 1 shows that one can express the target wealth at retirement in terms of multiples of current income, and suggests the normalization $y_0 = 1$, which we adopt in all quantitative exercises.

The second term is related to the agent's preferences over consumption and leisure. Assuming $\gamma^* > 1$, equation (19) implies that agents who value leisure and hence experience larger drops in consumption upon entering retirement, will enter retirement earlier ($\overline{W}$ is lower). An interesting implication of (19) along with (20) is that both the second and third term are in principle observable from consumption and income data. Moreover, γ_2 (defined in [11]) only depends on parameters related to the investment opportunity set and not on agent preferences. Hence, up to knowing an agents risk aversion equation (20) suggests a straightforward way to compute $\overline{W}$ and compare it to the data.

We conclude by discussing the dependence of $\overline{W}$ on κ, the Sharpe ratio. Differentiating $\overline{W}$ w.r.t. γ_2 it is not hard to establish that $\overline{W}_{\gamma_2} > 0$ and then differentiating γ_2 w.r.t. the Sharpe ratio (κ) and applying the chain rule gives that:

$$\overline{W}_\kappa < 0 \tag{21}$$

i.e. the retirement threshold is lower in economies with a higher Sharpe ratio. This is intuitive: a high Sharpe ratio implies that the stock market can produce strong gains and sustain post-retirement consumption, hence the agent is more willing to retire. In other words the gains that can be obtained in the stock market relative to the importance of a fixed income stream are more sizeable and accordingly the agent is more willing to go into retirement.

4.2 Optimal Consumption

We concentrate on a consumer with wealth lower than $\overline{W}$, so there is an incentive to continue working. Optimal consumption prior to retirement is given by Proposition 1 as

$$c_t = (\lambda^*)^{-\frac{1}{\gamma}}$$

where λ^* solves equation (13):

$$\gamma_2 C_2 (\lambda^*)^{\gamma_2 - 1} - \frac{1}{\theta}(\lambda^*)^{-\frac{1}{\gamma}} + \frac{y_0}{r} + W_t = 0 \tag{22}$$

In the appendix we show that $\theta > 0$ implies that

$$1 - \gamma_2 > \frac{1}{\gamma} \tag{23}$$

It is now useful to rewrite (22) as

$$-\gamma_2 C_2 c_t^{\gamma(1-\gamma_2)} + \frac{1}{\theta} c_t = W_t + \frac{y_0}{r} \tag{24}$$

Up to the term $-\gamma_2 C_2 c_t^{\gamma(1-\gamma_2)}$, this equation is the standard equation that one would obtain in a Merton-type framework, with a constant income stream. Indeed, if one removed the option of retirement, optimal consumption would be given by

$$c_t = \theta\left(W_t + \frac{y_0}{r}\right)$$

The difference here is that the individual wants to retire and hence has an added incentive to save for a given level of wealth (since $\gamma_2 < 0$ and $C_2 > 0$). Even though we cannot provide an explicit solution to this equation we can still calculate the marginal propensity to consume out of wealth and its derivative by using the implicit function theorem. We first define the marginal propensity to consume as

$$mpc = \frac{\partial c_t}{\partial W_t}$$

We differentiate both sides of equation (24) w.r.t. W_t, to get

$$\left(-\gamma\gamma_2(1-\gamma_2)C_2 c_t^{\gamma(1-\gamma_2)-1} + \frac{1}{\theta}\right) mpc = 1 \tag{25}$$

One can first observe from this equation, that mpc is strictly below θ since $\gamma_2 < 0$, $C_2 > 0$. Compared to the infinite horizon problem (where one stays in the workforce forever) the marginal propensity to consume out of wealth is strictly lower due to the option value embodied in (22). One can also study the dependence of the mpc on wealth. Differentiating once more and using equation (23) gives

$$mpc' = -mpc^3 \left(-\gamma\gamma_2(1-\gamma_2)(\gamma(1-\gamma_2)-1)C_2 c_t^{\gamma(1-\gamma_2)-2}\right) < 0$$

In other words, prior to retirement, the marginal propensity to consume out of wealth is a decreasing function of wealth and accordingly consumption is a concave function of wealth. To understand this, note that the consumer adopts a "threshold" policy for her retirement. If wealth is high, the time to retirement is "close" and thus an increase in W_t is counterbalanced by a decrease in the net present value of remaining income. Hence a consumer reacts less to a change in wealth, the closer her wealth is to the threshold level of retirement.

Reversing signs in the above argument, it is also true that the effect on consumption of a drop in pre-retirement wealth will be mitigated by an increase in the net present value of remaining income. Alternatively speaking, a negative shock to wealth will only partially affect consumption. A component of the drop will just postpone plans for retirement and this will in turn increase the net present value of income to be received in the future.

Of course once in retirement the problem becomes a standard Merton type problem and the usual affine relationship between consumption and wealth prevails.

It is important to note that the key to these results is not the presence of labor supply flexibility per se, but the "real option" inherent in the retirement decision. To substantiate this claim, assume that the agent never retires, and her leisure choice is determined optimally on a continuum at each point in time, so that $l_t + h_t = \bar{l}$[13] where h_t are the hours devoted to work and the instantaneous income is wh_t, with w defined as in section 2.3. The solution that one obtains for optimal consumption in such a framework with perfect labor supply flexibility is

$$c_t = \overline{C_1}\left(W_t + \frac{y_0}{r}\overline{C_2}\right)$$

for two appropriate constants $\overline{C_1}, \overline{C_2}$. Notice the simple affine relationship between wealth and consumption. These results show an important direction in which the present model sheds some new insights, beyond existing frameworks, into the relationship between retirement, consumption and portfolio choice. With endogenous retirement, wealth has a dual role. First, as in all consumption and portfolio problems, it controls the amount of resources that are available for future consumption. Second, it controls the distance to the threshold at which retirement is optimal. It is this second channel that is behind the behavior of the mpc analyzed above.[14]

The concavity of the consumption function is also a common result in models combining non-spanned income and /or borrowing constraints of the form $W_t \geq 0$ (e.g. Caroll and Kimball [1996]). A quite important difference between these models and the one considered here, is that in the present model the effects of concavity are most noticeable for high levels of wealth and not for low wealth levels. In our model the mpc asymptotes to θ as $W_t \to -\frac{y_0}{r}$,[15] and declines from there to the point where $W_t = \overline{W}$. After that it jumps back up to θ, reflecting the loss of the real option associated with remaining in the workforce. By contrast in models like Koo (1998) or Duffie et al (1997) the mpc is above θ for low levels of wealth and asymptotes to θ as $W_t \to \infty$. We discuss this issue further in section 7, when we introduce borrowing constraints.

4.3 Optimal Portfolio

By Proposition 1, the optimal holdings of stock are given by

$$\pi_t = \frac{\kappa}{\sigma}\left(\gamma_2(\gamma_2 - 1)C_2\left(\lambda^*\right)^{\gamma_2 - 1} + \frac{1}{\gamma}\frac{1}{\theta}\left(\lambda^*\right)^{-\frac{1}{\gamma}}\right)$$

From (13), we have that

$$\gamma_2 C_2\left(\lambda^*\right)^{\gamma_2 - 1} + \frac{y_0}{r} + W_t = \frac{1}{\theta}\left(\lambda^*\right)^{-\frac{1}{\gamma}}$$

therefore

$$\pi_t = \frac{\kappa}{\sigma}\frac{1}{\gamma}\left(W_t + \frac{y_0}{r}\right) + \frac{\kappa}{\sigma}\gamma_2 C_2\left(\lambda^*\right)^{\gamma_2 - 1}\left((\gamma_2 - 1) + \frac{1}{\gamma}\right)$$

The first term is equal to a standard Merton type stockholdings formula for an infinite horizon investor with constant income but no option to retire. The second term is positive. To see this, notice that

$$(\gamma_2 - 1) + \frac{1}{\gamma} < 0$$

by equation (23)and $\gamma_2 < 0, C_2 > 0$. Moreover, by the definitions of C_2, λ^* one can derive that

$$C_2 (\lambda^*)^{\gamma_2 - 1} = \frac{y_0}{r} \left[\frac{\gamma}{1-\gamma} \frac{(\gamma_2 - 1)}{\left(1 + \gamma_2 \frac{\gamma}{1-\gamma}\right)} - 1 \right] \left(\frac{\lambda^*}{\underline{\lambda}} \right)^{\gamma_2 - 1} \tag{26}$$

Thus, as $\lambda^* \to \infty$ the importance of this term disappears, whereas as $\lambda^* \to \underline{\lambda}$ this term approaches its maximal value. It is easiest to interpret this result by observing that a) λ^* is a decreasing function of wealth (W_t) and b) by (16) $\underline{\lambda}$ is the lowest value that λ^* can attain before the agent goes into retirement[16]. In words, when an agent is very poor, the relevance of early retirement is small and thus the stockholdings chosen resemble those of a simple Merton type setup. By contrast as wealth increases, so does the likelihood of early retirement and this term becomes increasingly important.

In summary, the option value of work *increases* the incentive to take risk compared to the benchmark of an infinite horizon Merton setup with constant income but no retirement. Moreover, as wealth increases and comes close to the retirement threshold, this incentive is maximized.

The intuition for this result is straightforward. As wealth increases, it becomes more likely that the "real" option to retire will be exercised. The most effective way to affect that likelihood is by investing in stocks. The agent is more willing to take risks in the stock market, because she can postpone her retirement instead of reducing consumption in the event of a negative shock.

It is interesting to relate these results to BMS. To do that, we start by normalizing the nominal stock holdings by W_t. This gives the ratio $\phi_t = \frac{\pi_t}{W_t}$, or using (26)

$$\begin{aligned} \phi &= \frac{\kappa}{\sigma}\frac{1}{\gamma}\left(1 + \frac{y_0}{W_t}\frac{1}{r}\right) + \\ &+ \frac{\kappa}{\sigma}\frac{y_0}{W_t}\frac{1}{r}\gamma_2 \left[\left(\frac{\lambda^*}{\underline{\lambda}}\right)^{\gamma_2 - 1} \left((\gamma_2 - 1) + \frac{1}{\gamma} \right) \right] \left[\frac{\gamma}{1-\gamma} \frac{(\gamma_2 - 1)}{\left(1 + \gamma_2 \frac{\gamma}{1-\gamma}\right)} - 1 \right] \end{aligned}$$

The first term in the equation for ϕ corresponds to the term one would obtain in the absence of retirement (i.e. the Merton framework where a worker never retires). The second term is the effect of the real option to retire. It is interesting to note the dependence of these terms on W_t. By fixing y_0 and increasing W_t, one can observe that the first term actually decreases. This is the standard BMS effect. In other words, ignoring the real option to retire (which is captured in the second term only) one would arrive at the conclusion that an increase in wealth should be associated with a decline in the portfolio share allocated to risky assets. This conclusion is not necessarily true if one considers the option to retire. To see why, compute ϕ_W and evaluate it around $\overline{W}$, in order to obtain after some simplifications:

$$\phi_W(\overline{W}) = -\frac{1}{\overline{\overline{W}}}\left(\phi(\overline{W}) - \frac{\kappa}{\sigma}\frac{1}{\gamma}\right) + \frac{1}{\overline{W}}\frac{1}{\phi(\overline{W})}\left(\frac{\kappa}{\sigma}\right)^2 \frac{\left(K^{\frac{1}{\gamma}}\theta - 1\right)}{K^{\frac{1}{\gamma}}\theta}\left((\gamma_2 - 1) + \frac{1}{\gamma}\right)\frac{\gamma_2}{1-\gamma}$$

The first term is clearly negative, and captures the increase in the denominator of $\phi = \frac{\pi}{W}$. The second term though is positive and potentially larger than the first term, depending on parameters.

This result is driven by the real option of work, not labor supply flexibility per se. Indeed one can show (using the methods in BMS) that allowing an agent to choose labor and leisure freely on a continuum would result in:

$$\pi_t = \frac{1}{\gamma^*}\frac{\kappa}{\sigma}\left(W_t + \frac{y_0}{r}\frac{\bar{l}}{(\bar{l} - l_1)}\right)$$

This implies that ϕ would have to be decreasing in W_t. The reason for these differences is that in BMS, the amount allocated to stocks as a fraction of total resources (financial wealth + human capital) is a constant. In our framework this fraction depends on wealth. Wealth controls both the resources available for future consumption and the likelihood of "exercising" the real option of retirement.

In summary, not only does the possibility of early retirement increase the incentive to save more, it also increases the incentive of the agent to invest in the stock market because this is the most effective way to obtain this goal. Furthermore, this incentive is strengthened as an individuals' wealth approaches the target wealth level that triggers retirement.

4.4 The Correlation between Consumption and the Stock Market

As we showed in section 4.2 the marginal propensity to consume out of wealth (and hence the sensitivity of changes to consumption w.r.t. changes in the wealth) is decreased by the possibility of early retirement. One might also wonder whether this also implies a decreased correlation between consumption and the stock market (compared to the standard Merton model). If this result were true, it could then be hoped that the model can shed some light into the equity premium puzzle, because the same fixed level of μ would be compatible with a lower correlation between consumption and the stock market than the standard Merton model.[17] The answer is unfortunately, that it does not. The reason is quite simple and can be seen by examining formula (16) in Basak (1999) which continues to be true in our setup (for agents prior to retirement)

$$\mu - r = -\frac{cU_{cc}}{U_c} cov\left(\frac{dP_t}{P_t}, \frac{dc}{c}\right) + \frac{U_{ch}}{U_c} cov\left(\frac{dP_t}{P_t}, dl\right) \tag{27}$$

U_{ch} is the cross partial of U w.r.t the hours worked and dl is the variation in leisure. In our setup $dl = 0$ prior to retirement and $\mu, r, \frac{cU_{cc}}{U_c}$ are constants in our framework. Accordingly, $cov\left(\frac{dP_t}{P_t}, \frac{dc}{c}\right)$ is constant as well. It is important to note that this result was obtained solely by the fact that $dl = 0$ along with the assumption of a constant investment opportunity set and CRRA utilities. In other words -prior to retirement- the consumption CAPM holds in this framework

This may seem surprising in light of the results we obtained for the marginal propensity to consume out of wealth. One might expect that a declining mpc would be sufficient to produce a low correlation between the stock market and consumption. The resolution of this puzzle is that a decrease in mpc in this model is accompanied by an equivalent increase in the exposure to the stock market through a portfolio that is more heavily tilted towards stocks. In other words, even though consumption becomes less responsive to shocks in the wealth process, at the same time the shocks to the wealth process become larger because of a riskier portfolio.[18] The two effects exactly balance out.

An important caveat is that the above discussion relies heavily on partial equilibrium. To see if labor supply flexibility can indeed explain the observed smoothness of aggregate consumption and accordingly a large equity premium one would have to study a general equilibrium version of this model (as Basak [1999] does for continuous choice of labor/leisure). In that case a fraction of the population would be entering retirement at each instant and would experience consumption changes due to the increase in leisure. Hence, at the aggregate the simple consumption CAPM no longer holds.

$$\mu - r = -\frac{cU_{cc}}{U_c} cov\left(\frac{dP_t}{P_t}, \frac{dc}{c}\right)$$

It can be reasonably conjectured that in this framework the behavior of the interest rate and the equity premium would be very different than in Basak (1999). Even in the base case of CRRA utilities and multiplicative technology shocks, the equity premium and the interest rate would exhibit interesting dynamics. However, these issues are beyond the scope of this paper.

5 Retirement before a Deadline

None of the claims made so far relied on restricting the time of retirement to lie in a particular interval. The exposition was facilitated by the infinite horizon setup because it allowed for explicit solutions to the associated optimal stopping problem. However, the tradeoff is that in the infinite horizon case, there is no notion of "life" cycle, since time plays no explicit role in the solution. Moreover, the "natural" theoretical benchmark for the model of the previous section is one without retirement at all. In this section we are able to extend all the intuitions of the previous section by comparing the early retirement model to a benchmark model with mandatory retirement at time T, which is more natural.[19]

Allowing for a retirement deadline introduces a new state variable (time to retirement), which considerably complicates the analysis. However, we are able to use approximation methods to solve the associated "finite horizon"[20] optimal stopping problem, that are simple to analyze, compute and seem to work very well in practice.

Formally, the only modification that we introduce compared to section 2 is that equation (5) becomes

$$\max_{c_t, W_\tau, \tau} E\left[\int_t^{\tau\wedge T} e^{-\beta(s-t)}U(l_1, c_s)ds + e^{-\beta(\tau\wedge T - t)}U_2(W_{\tau\wedge T})\right] \tag{28}$$

where T is the retirement deadline.

Proposition 2 *Define*

$$\widetilde{V}^E(\lambda, T-t) = \frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}}\frac{1}{\theta}\left[\left(K^{\frac{1}{\gamma}}\theta - 1\right)e^{-\theta(T-t)} + 1\right] + \lambda y_0 \frac{1-e^{-r(T-t)}}{r}$$

Let $\widetilde{V}(\lambda; T-t)$ *be given by:*

$$\begin{array}{lcl} C_{2(T-t)}\lambda^{\gamma_{2(T-t)}} + \widetilde{V}^E(\lambda, T-t), & \textit{if } \lambda > & \underline{\lambda}_{(T-t)} \\ \left(\frac{\gamma}{1-\gamma}K^{\frac{1}{\gamma}}(\lambda)^{\frac{\gamma-1}{\gamma}}\right) & \textit{if } \lambda \leq & \underline{\lambda}_{(T-t)} \end{array}$$

where

$$\underline{\lambda}_{(T-t)} = \left(\frac{(\gamma_{2(T-t)}-1)\theta_{(T-t)}}{\left(1+\gamma_{2(T-t)}\frac{\gamma}{1-\gamma}\right)\left(K^{\frac{1}{\gamma}}\theta_{(T-t)} - 1\right)}\frac{y_0(1-e^{-r(T-t)})}{r}\right)^{-\gamma} \tag{29}$$

$\theta_{(T-t)}$ *is given by*

$$\theta_{(T-t)} = \frac{\theta}{\left[\left(K^{\frac{1}{\gamma}}\theta - 1\right)e^{-\theta(T-t)} + 1\right]}$$

$C_{2(T-t)}$ is given by

$$C_{2(T-t)} = \frac{\left[\frac{\gamma}{1-\gamma}\frac{(\gamma_{2(T-t)}-1)}{\left(1+\gamma_{2(T-t)}\frac{\gamma}{1-\gamma}\right)} - 1\right]\frac{y_0(1-e^{-r(T-t)})}{r}}{\underline{\lambda}_{(T-t)}^{\gamma_{2(T-t)}-1}} \tag{30}$$

and $\gamma_{2(T-t)}$ is given by

$$\gamma_{2(T-t)} = \frac{1 - 2\frac{\beta-r}{\kappa^2} - \sqrt{(1-2\frac{\beta-r}{\kappa^2})^2 + 8\frac{\beta}{(1-e^{-\beta(T-t)})\kappa^2}}}{2}$$

$\widetilde{V}(\lambda, T-t)$ is continuously differentiable everywhere and $\widetilde{V}_\lambda(\lambda, T-t)$ maps $(0,\infty)$ into the interval $(-\infty, \frac{y_0}{r}\left(1-e^{-r(T-t)}\right))$. Finally compute the (unique) solution of

$$\gamma_{2(T-t)}C_{2(T-t)}\left(\lambda^*\right)^{\gamma_{2(T-t)}-1} - \frac{1}{\theta_{(T-t)}}\left(\lambda^*\right)^{-\frac{1}{\gamma}} + \frac{y_0(1-e^{-r(T-t)})}{r} + W_t = 0 \tag{31}$$

Then an approximate solution to (28) is given by

a) If $W_t < \overline{W}_{(T-t)} = K^{\frac{1}{\gamma}}\underline{\lambda}_{(T-t)}^{-\frac{1}{\gamma}}$

consumption follows the process

$$\begin{aligned} c_s &= \left(\lambda^* e^{\beta(s-t)}\frac{H(s)}{H(t)}\right)^{-\frac{1}{\gamma}} 1\{t \le s < \tau^*\} \\ c_s &= \bar{l}^{(1-\alpha)\frac{(1-\gamma^*)}{\gamma}}\left(\lambda^* e^{\beta(s-t)}\frac{H(s)}{H(t)}\right)^{-\frac{1}{\gamma}} 1\{s \ge \tau^*\} \end{aligned}$$

and the optimal retirement time is

$$\begin{aligned} \tau^* &= \inf\{s : W_s = \overline{W}_{(T-s)}\} \\ &= \inf\{s : \lambda^* e^{\beta(s-t)}\frac{H(s)}{H(t)} = \underline{\lambda}_{(T-s)}\} \end{aligned}$$

The optimal consumption and the optimal portfolio as a function of W_t are given by

$$\begin{aligned} c_t &= c(W_t) = (\lambda^*(W_t))^{-\frac{1}{\gamma}} \\ \pi_t &= \pi(W_t) = \frac{\kappa}{\sigma}\left(\gamma_{2(T-t)}(\gamma_{2(T-t)}-1)C_{2(T-t)}\lambda^*(W_t)^{\gamma_2-1} + \frac{1}{\gamma}\frac{1}{\theta_{(T-t)}}\lambda^*(W_t)^{-\frac{1}{\gamma}}\right) \end{aligned}$$

where the notation $\lambda^*(W_t)$ is used to make the dependence of λ^* on W_t explicit.

b) If $W_t \ge \overline{W}_{(T-t)}$ the optimal solution is to enter retirement immediately ($\tau^* = t$) and the optimal consumption /portfolio policy is given as in Karatzas and Shreve (1998) section 3.

The appendix discusses the nature of the approximation and examines its performance against consistent numerical methods to solve the problem. The basic idea behind the approximation is to reduce the problem to a standard optimal stopping problem and use the same approximation technique as Barone-Adesi and Whaley (1987). The most important advantage of this approximation, is that it leads to very tractable solutions for all quantities involved. This can be seen most easily by observing that equation (31) is practically identical to equation (13). One can also check easily that the formulas for optimal consumption, portfolio etc. are identical to the respective formulas of proposition 1 (the sole exception being that the constants are modified by terms that depend on $T-t$). As a result, all of the analysis in section 4 carries through to this section. This is particularly true for the dependence of consumption, portfolio etc. on wealth. Economically, the only new dimension introduced is that all constants depend explicitly on the distance to mandatory retirement, and thus enable the study of interaction effects between wealth and the an investor's age. Here we will focus only on the implications of the model for *portfolio choice over the life cycle.* The results for life-cycle consumption are similar.

To provide a benchmark against which to compare the solutions, we consider the portfolio problem of an agent with mandatory retirement in $T-t$ periods. For simplicity we also assume there is no labor supply flexibility, i.e. the agent is endowed with an income stream of y_0. By standard derivations, the portfolio of an agent in this case is

$$\pi_t^{mand.} = \frac{1}{\sigma}\frac{\kappa}{\gamma}\left(W_t + y_0\frac{1-e^{-r(T-t)}}{r}\right)$$

A constant fraction $\left(\frac{1}{\sigma}\frac{\kappa}{\gamma}\right)$ of the net present value of resources available to the individual $\left(W_0 + y_0\frac{1-e^{-rT}}{r}\right)$ is invested in the stock market irrespective of her age. Since we only observe W_t and π_t in the data, and not the net present value of future income, it is interesting to divide total holdings of stock by financial wealth, which gives:

$$\frac{\pi_t^{mand.}}{W_t} = \frac{1}{\sigma}\frac{\kappa}{\gamma}\left(1 + \frac{y_0}{W_t}\frac{1-e^{-r(T-t)}}{r}\right) \tag{32}$$

This expression captures a number of well understood intuitions. First, the allocation towards stocks as a fraction of financial wealth declines with age, *for a fixed level of* W_t. Second, in expectation W_t will increase over time, reinforcing the first effect. Therefore, (in expectation) the allocation to stocks should be downward sloping over time.

Allowing for early retirement considerably alters some of these conclusions. To demonstrate this effect, we proceed (as in section 4.3) to arrive at the optimal holdings of stock π_t in the presence of optimal early retirement

$$\begin{aligned}\pi_t &= \frac{\kappa}{\sigma}\frac{1}{\gamma}\left(W_t + y_0\frac{1-e^{-r(T-t)}}{r}\right) \\ &+\frac{\kappa}{\sigma}\frac{y_0\left(1-e^{-r(T-t)}\right)}{r}\left(\frac{\lambda^*}{\underline{\lambda}_{(T-t)}}\right)^{\gamma_{2T}-1}\gamma_{2(T-t)}\left(\frac{1}{\gamma}+\left(\gamma_{2(T-t)}-1\right)\right)\left[\frac{\gamma}{1-\gamma}\frac{\left(\gamma_{2(T-t)}-1\right)}{\left(1+\gamma_{2(T-t)}\frac{\gamma}{1-\gamma}\right)}-1\right]\end{aligned} \tag{33}$$

where λ^* is given by (31). As in section 4.3 the second term captures the "real" option of early retirement and is strictly positive. As with most options, its relevance is larger a) the more likely it is that it will be exercised (in/out of the money) and b) the more time is left until its expiration. Accordingly, the importance of the second term in (33) should be expected to decrease when $T-t$ is small and/or the ratio of W_t to the target wealth $\overline{W}_{(T-t)}$ is small.

This now opens up the possibility of rich interactions between "pure" horizon effects (variations in $T-t$, keeping W_t constant) and wealth effects, beyond the ones already present in (32). As an agent ages *but is not yet retired*, the "pure" horizon effects will tend to work in the same direction as in equation (32). However, in expectation wealth increases as well and thus the option to retire early becomes more and more relevant, counteracting the first effect.

Another property of the optimal portfolio that is implied by the present framework is a downward jump in stockholdings, immediately after the agent enters retirement. In that case stockholdings take the standard Merton form

$$\pi_t = \frac{\kappa}{\sigma}\frac{1}{\gamma}W_t$$

These effects are quantitatively illustrated in the next section.

6 Quantitative Implications

To quantitatively assess the magnitude of the effects described in section 5 we proceed as follows. First, we fix the values of the variables related to the investment opportunity set to: $r = 0.03, \mu = 0.1, \sigma = 0.2$. For β we choose 0.07 in order to account for both discounting and a constant probability of death. For γ we consider a range of values (typically $2, 3, 4$). This leaves one more parameter to be determined, namely K. K controls the shift in the marginal utility of consumption upon entering retirement. It is a well documented empirical fact that consumption drops considerably upon entering retirement. As such, the most natural way to determine the value of K is to match the agent's declining consumption upon entering retirement. Aguiar and Hurst (2004) report expenditure drops of 17%, whereas Banks et al. (1998) report changes in log consumption expenditures of almost 0.3 in the five years prior to retirement and thereafter. Since these drops are mainly calculated for food expenditures, which are likely to be inelastic, we also calibrate the model to somewhat larger drops in consumption than that.[21]

In light of (19)

$$\frac{c_{\tau^+}}{c_{\tau^-}} = K^{1/\gamma}\theta$$

where c_{τ^-} is consumption immediately prior to retirement and c_{τ^+} is the consumption immediately thereafter. We therefore determine K so that $K^{1/\gamma}\theta = \{0.5, 0.6, 0.7\}$. We fix the mandatory retirement age to be $T = 65$ throughout, and normalize y_0 to be 1. The abbreviation "*ret*" indicates the solution implied by a model with optimal early retirement (up to time T) and "*BMS*" denotes the solution of a model with mandatory retirement at time T, with no option to retire earlier or later.

Figure 1 plots the "target" wealth that is implied by the model, i.e. the level of wealth required to enter retirement. This figure demonstrates two patterns. First, "threshold" wealth declines as an agent nears mandatory retirement. This is intuitive. The option to work is more valuable the longer its "maturity". As a (working) agent ages, the incentive to keep the option "alive" is reduced and hence the wealth threshold declines. Second, the critical wealth implied by this model varies with the assumptions made about risk aversion, and the disutility of work as implied by a lower $K^{1/\gamma}\theta$. Risk aversion tends to shift the threshold upwards, whereas lower levels of $K^{1/\gamma}\theta$ (implying more disutility of labor) bring the threshold down. These are intuitive predictions. An agent who is risk averse wants to avoid the risk of losing the option to work, whereas an agent who cares a lot about leisure will want to enter retirement earlier.

Figure 2 addresses the importance of the real option to retire for portfolio choice. The figure plots the second term in equation (33) as a fraction of total stockholdings π_t. In other words, it plots the relative importance of stockholdings due to the real "option" component as a percent of total stockholdings. This percentage is plotted as a function of two variables: a) age and b) wealth. Age varies between 45 and 64. Wealth varies between 0 and x, where x corresponds to the level of wealth that would make an agent retire (voluntarily) at 64. We normalize wealth levels by x so that the (normalized) wealth levels vary between 0 and 1. We then plot a panel of figures for different levels of $\gamma, K^{1/\gamma}\theta$. Figure 2 demonstrates the joint presence of "time to maturity" and "moneyness" effects in the real option to retire. Keeping wealth fixed and varying the time to maturity (i.e. increasing age) the relative importance of the real option to retire declines. Similarly, increasing wealth makes the real option component more relevant, because the real option is more "in the money". It is interesting to note that the "real option" component is large, taking values as large as 40% for some parameter combinations.

In Figure 3 we consider the implications of the model for life cycle portfolio choice. We fix a path of returns corresponding to the realized returns on the CRSP value weighted index between 1990 and 2000. We then plot the portfolio holdings (defined as total stockholdings normalized by *financial* wealth) over time for an individual whose wealth in 1990 was just enough to allow her to retire in the beginning of 2000 at the age of 59. We repeat the same exercise assuming various combinations of $K^{1/\gamma}\theta$ and γ. In order to be able to compare the results, we also plot the portfolio that would be implied if the individual had no option of retiring early. We label this later case as "BMS". What figure 3 shows, is that the portfolio of the agent is initially declining and then flat or even increasing over time after 1995. This is in contrast to what would be predicted by ignoring the option to retire early (the "BMS" case). This fundamentally different behavior of the portfolio of the agent over time is due to the extraordinary returns during the latter half of the 1990's, that makes wealth grow faster, and hence the real option to retire very important towards the end of the sample. By contrast, if one assumed away the possibility of early retirement, the natural conclusion would be that a run-up in prices would change the composition of total resources (financial wealth + human capital) of the agent towards financial wealth. For a constant income stream this would mean accordingly a decrease in the portfolio chosen.

Figure 4 demonstrates the above effect more clearly. In this figure we normalize total stockholdings by total resources (human capital + financial wealth). As already shown, for the BMS case we get a constant equal to $\left(\frac{\kappa}{\sigma}\frac{1}{\gamma}\right)$. By allowing for an early retirement option we observe that the fraction of total resources invested in stock, exhibits a stark increase towards the latter half of the 1990's, because the option of early retirement becomes more relevant. The increase in this fraction is small for the first half of the 1990's and large for the latter part of the decade.

Figure 4 is useful in understanding the behavior of the portfolio holdings in Figure 3. In the first half of the sample the standard BMS intuition applies. The fraction of total resources invested in the stock market is roughly constant even after taking the option of early retirement into account. Hence by the standard intuition behind the BMS results the portfolio of the agent (total stockholdings normalized by financial wealth) declines over time. However, in the latter half of the sample, the increase in the real option to retire is strong enough to counteract the decline in the portfolio implied by standard BMS intuitions.

Figures 5 and 6 repeat the same exercise as figures 3 and 4, only now for an agent who "came close" to retirement in 2000. However, we assume that her wealth at that point was slightly less than enough in order to actually retire. To achieve this we just assume that in 1990 she started with slightly less initial wealth than necessary to retire by 2000. It is interesting to note what happens post 2000. Now, the option of early retirement starts to become irrelevant and the agent's portfolio declines. The effect of a disappearing option magnifies the drop in the portfolio. By contrast in the BMS case the abrupt drop in the stock market (and hence wealth) would be counterbalanced by a change in the composition between financial wealth and human capital towards human capital. This effect tends to somehow counteract the effects of aging, and produces a much more moderate drop in portfolio holdings.

These figures are meant to demonstrate the fundamentally different economic implications that can result once one takes into account the real option to retire. As such they should be seen as merely an illustrative application. Note however, that a stronger result can be shown in the context of this exercise. For wealth levels close to the retirement threshold, the portfolio would increase with age *in expectation,* and not just for the sample path that we consider. What is special about the path that we consider is that the strong gains in the stock market drive investor wealth close to the optimal retirement threshold. This increase of the portfolio with age (*in expectation*) would be impossible in the absence of an early retirement option.

The present paper is theoretical in nature, and we don't claim to have modeled even a small fraction of all the issues that influence real life retirement, consumption, and portfolio decisions (like liquidity constraints, shorting and leverage constraints, transaction costs, undiversifiable income and health shocks etc.). However, note that the model does produce "sensible" portfolios (for the combination $\gamma = 4$, $K^{1/\gamma}\theta = 0.7$) as well as variations in portfolio shares between 1995 and 2003. In the bottom right plots of figure 5 for instance the portfolio of the agent grows from 0.58 to 0.62 between 1995 and 1999 and then declines to roughly 0.5 by the beginning of 2003. By comparison, the EBRI[22] reports that the average equity share in a sample of 401(k)'s grew steadily from 0.46 to 0.53 between 1995 and 1999 only to fall to 0.4 by the beginning of 2003. The reason why the model does not perform poorly is that we are considering an agent close to retirement, at a time when the remaining NPV of her income is not a big component of her total wealth in the first place. For early stages in the life cycle the model (unsurprisingly) has similar problems matching the data as BMS, which is to be expected.

We conclude with the following caveat. These results cannot be taken so far as to suggest a rational explanation for agents, who increased their stockholdings during the latter half of the 1990's but then chose to sharply reduce holdings once the market was in decline. The individuals in our model do not take into account any predictability in returns. In light of equation (21) a decrease in the Sharpe ratio (as was suggested by rising P/E ratios during the latter half of the 1990's) would have to be accompanied by an increase in the wealth threshold. In simple terms, an agent who understood that expected returns would be low (potentially because of irrational exuberance) would have been more careful about retiring compared to what is suggested by our model. Extending the model to allow for predictability would be a fascinating extension, that is beyond the scope of the present paper.

With this caveat in mind, it is true that the present model shows a number of differences that can result from the real option to retire. As the quote from Business Week at the beginning of this paper suggests, for many agents the ability to retire earlier is an important consideration. The present framework seems to be a natural starting point to examine the implications of early retirement for portfolio choice and savings over the life cycle. Again, the examples in this section are meant merely to illustrate how the presence of the real option to retire can potentially modify some of the commonly held views on portfolio choice.

7 Borrowing constraint

We have thus far assumed that the agent was able to borrow against the value of her future labor income. In this section we impose an additional restriction: It is impossible for the agent to borrow against the value of future income. Formally, we add the requirement that $W_t \geq 0$, for all $t > 0$. To preserve tractability, we assume in this section that the agent is able to go into retirement at any time that she chooses without a deadline. This makes the problem stationary and as a result the optimal consumption and portfolio policies will be given by functions of W_t alone.

The borrowing constraint is never binding post-retirement because the agent receives no income and has constant relative risk aversion. This implies that once the agent is retired, her consumption, her portfolio, and her value function are the same with or without borrowing constraints. In particular, if she enters retirement at time τ with wealth W_τ, her expected utility is still $U_2(W_\tau)$.

The problem the agent now faces is

$$\max_{c_t, W_\tau, \tau} E\left[\int_0^\tau e^{-\beta t} U(l_1, c_t)dt + e^{-\beta\tau} U_2(W_\tau)\right] \tag{34}$$

subject to the borrowing constraint

$$W_t \geq 0, \forall t \geq 0, \tag{35}$$

and the budget constraint

$$dW_t = \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} rdt - (c_t - y_0 1\{t < \tau\})\, dt. \tag{36}$$

By arguments similar to the ones in section 2 we can rewrite these two constraints as

$$E\left[\int_0^\tau H_s c_s ds + H_\tau W_\tau\right] \leq E\left[\int_0^\tau H_s y_0 ds\right] + W_0,$$

$$\frac{E\left[\int_t^\tau H_s c_s ds + H_\tau W_\tau\right]}{H_t} \geq \frac{E\left[\int_t^\tau H_s y_0 ds\right]}{H_t}, \forall t \geq 0.$$

We present the solution in Proposition 3. The post retirement solution is the standard Merton solution and is completely determined by the critical wealth $\overline{\overline{W}}$ required to enter retirement.

Proposition 3 *Under technical conditions (71) and (72) in the Appendix, there exist appropriate constants* $C_1, C_2, Z_L, Z_H, \gamma_1, \gamma_2$ *(also given in the Appendix) and a positive decreasing process* X_s^* *with* $X_t^* = 1$ *so that the optimal policy triple* $< \widehat{c}_s, \widehat{W}_{\widehat{\tau}}, \widehat{\tau} >$ *is*

a) If $W_t < \overline{\overline{W}} = K^{\frac{1}{\gamma}} Z_L^{-\frac{1}{\gamma}}$

$$\begin{aligned}
\widehat{c}_s &= \left(\lambda^* e^{\beta(s-t)} X_s^* \frac{H(s)}{H(t)}\right)^{-\frac{1}{\gamma}} 1\{s < \widehat{\tau}\} \\
\widehat{W}_{\widehat{\tau}} &= \overline{\overline{W}} \\
\widehat{\tau} &= \inf\{s : W_s = \overline{\overline{W}}\} = \\
&= \inf\{s : \lambda^* e^{\beta(s-t)} X_s^* \frac{H(s)}{H(t)} = Z_L\}
\end{aligned}$$

and λ^* *is given by*

$$\gamma_1 C_1 (\lambda^*)^{\gamma_1 - 1} + \gamma_2 C_2 (\lambda^*)^{\gamma_2 - 1} - \frac{1}{\theta} (\lambda^*)^{-\frac{1}{\gamma}} + \frac{y_0}{r} + W_t = 0 \qquad (37)$$

Using the notation $\lambda^*(W_t)$ *to make the dependence of* λ^* *on* W_t *explicit, the optimal consumption and portfolio policy is given by*

$$\begin{aligned}
c_t &= c(W_t) = (\lambda^*(W_t))^{-\frac{1}{\gamma}} \\
\pi_t &= \pi(W_t) = -\frac{\kappa}{\sigma} \frac{\lambda^*(W_t)}{\lambda^*_{W_t}(W_t)}
\end{aligned}$$

where $\lambda^*_{W_t}(W_t)$ *denotes the first derivative of* $\lambda^*(W_t)$ *with respect to* W_t.

b) If $W_t \geq \overline{\overline{W}} = K^{\frac{1}{\gamma}} Z_L^{-\frac{1}{\gamma}}$ *the optimal solution is to enter retirement immediately* ($\widehat{\tau} = t$) *and the optimal consumption policy is given as in the standard Merton (1971) infinite horizon problem.*

We devote the remainder of this section to a comparison of results obtained in section 4 with the resultant optimal policies obtained in Proposition 3.

A simple intuitive argument shows that (compared with section 3) wealth at retirement is smaller with borrowing constraints than without: $\overline{\overline{W}} < \overline{W}$. ($\overline{W}$ is the threshold at which an individual facing no borrowing constraints goes into retirement). The reasoning is the following: Given a level of wealth $\overline{W}$, the agent will achieve the same utility $U_2(\overline{W})$ if she goes into retirement at t irregardless of any borrowing constraints. But the expected utility the agent will achieve by postponing her retirement decision to $t + dt$ is strictly lower if she faces borrowing constraints (the inequality is strict because there is a non-zero probability that the constraint will bind between t

and $t+dt$). As a result, the value of postponement is strictly lower with borrowing constraints than without, i.e. $\overline{\overline{W}} < \overline{W}$.

The total stockholdings can be computed by steps similar to section 4. The implicit function theorem gives

$$\frac{\lambda^*}{\lambda^*_{W_t}} = -\left(\gamma_1(\gamma_1-1)C_1\left(\lambda^*\right)^{\gamma_1-1} + \gamma_2(\gamma_2-1)C_2\left(\lambda^*\right)^{\gamma_2-1} + \frac{1}{\gamma}\frac{1}{\theta}\left(\lambda^*\right)^{-\frac{1}{\gamma}}\right) \tag{38}$$

and by steps similar to section 4 we obtain the optimal stockholdings as

$$\pi_t = \frac{\kappa}{\sigma}\frac{1}{\gamma}\left(W_t + \frac{y_0}{r}\right) + \frac{\kappa}{\sigma}\gamma_1 C_1\left(\lambda^*\right)^{\gamma_1-1}\left((\gamma_1-1)+\frac{1}{\gamma}\right) + \frac{\kappa}{\sigma}\gamma_2 C_2\left(\lambda^*\right)^{\gamma_2-1}\left((\gamma_2-1)+\frac{1}{\gamma}\right).$$

The first term is equal to a standard Merton type portfolio for an infinite horizon problem. One can show that the second term is negative and decreasing in λ^* (increasing in wealth), while the third term is positive and decreasing in λ^* (increasing in wealth). π_t is therefore decreasing in λ^* (increasing in wealth) as in section 4.

The presence of borrowing constraints has two effects: it moderates holdings of stock, and decreasingly so as the wealth of the agent increases (and hence by (37) λ^* decreases). In economic terms, considerations related to optimal retirement are relevant for large levels of wealth, whereas considerations related to borrowing constraints are relevant for low levels of wealth. As a result, the economic intuitions that apply to the two cases can be analyzed separately.

Figure 7 compares optimal portfolios for four cases. with and without the early retirement option, and with and without the imposition of borrowing constraints. For the cases where we allow retirement, we take wealth levels close to retirement but lower than the threshold that would imply retirement. The figure demonstrates that for levels of wealth close to retirement there are only (minor) quantitative differences between agents with borrowing constrains and agents without. The qualitative properties are the same. Holdings of stock increase with wealth (more than linearly). One can observe that the optimal stockholdings in the presence of early retirement are more tilted towards stocks whether we impose borrowing constraints or not. Similarly, the optimal holdings of stock are smaller when one imposes borrowing constraints (whether one assumes early retirement or not).

We conclude by summarizing the key insights of this section: Borrowing constraints are relevant for levels of wealth where optimal retirement is not an issue. Similarly, the effects of optimal retirement are relevant for levels of wealth where borrowing constraints are highly unlikely to bind in the future. Hence, as long as one examines the effects of the option to retire close to the threshold levels of wealth, borrowing constraints can be safely ignored. However, it is important to note that borrowing constraints can fundamentally affect quantities related to e.g. the expected time to retirement for a person who starts with wealth close to 0 because they will typically imply lower levels of stockholdings and hence a more prolonged time (in expectation) to reach the retirement threshold.

8 Conclusion

In this paper we proposed a simple partial equilibrium model of consumer behavior which allows for the joint determination of optimal consumption, portfolio and the retirement time of a consumer. Essentially closed form solutions were obtained for virtually all quantities of interest. The results can be summarized as follows: The ability to time one's retirement introduces an option type character to the optimal retirement decision. This option is most relevant for individuals with a high likelihood of early retirement, which are individuals with high wealth levels. This option in turn affects both an agent's incentive to consume out of current wealth and her investment decisions. In general, the presence of the option value to retire will lead to portfolios that are more exposed to stock market risk. The marginal propensity to consume out of wealth will be lower as one approaches early retirement, reflecting the increased incentives to reinvest gains in the stock market in order to bring retirement "closer". In turn, the likelihood of attaining early retirement is more relevant for individuals who are young and/or wealthy.

An important practical implication of our model is that the relationship between stockholdings and age is likely to be more complicated than is suggested in BMS. This can be most easily seen by dividing the stockholdings by total wealth (financial wealth + the net present value of future income) in order to control for the effects discussed in BMS. In our model the resulting fraction is not constant (as in BMS), but has clear option pricing properties and depends on both wealth and the distance to mandatory retirement, if such a deadline is imposed. Even though the fraction of total wealth invested in the stock market decreases as an individual ages *for a given level of (financial) wealth,* this fraction increases as wealth increases *for a given time* to retirement.

The model makes some quite intuitive predictions. We single out some of the predictions that seem to be particularly interesting: First, during stock market booms, there should be an increase in the numbers of people opting for retirement as a larger percent of the population hits the retirement threshold. (Some evidence for this may be found in Gustman and Steinmeir [2002] and references therein). Second, it is possible that portfolios over the life cycle could exhibit increasing holdings of stock over time, even when there isn't variation in the investment opportunity set and the income stream exhibits no correlation with the stock market (or any risk whatsoever). This is interesting in light of evidence in Ameriks and Zeldes (2001) that portfolios tend to be increasing or hump-shaped with age for the datasets that they consider. Third, according to the model there should be a discontinuity in the holdings of stock and in consumption upon entering retirement. There is ample empirical evidence on the latter. (see e.g. Aguiar and Hurst (2004) and references therein). The former seems to have been less tested an hypothesis. Fourth, all else equal, switching to a more flexible retirement system should imply increased stock market participation. It is an empirical fact that stock market participation increased in the U.S. during the last years, while at the same time 401(k)'s were gaining popularity. Fifth, increasing levels of stock holdings during a stock market runup and liquidations during a stock market fall might not be due to irrational herding. Instead they might be due to the behavior of the real option to retire that emerges during the runup and becomes irrelevant after the fall.

In this paper we have tried to outline the basic new intuitions that are introduced by the timing of the retirement decision. By no means do we claim that we have addressed all the issues that are likely to be relevant for actual retirement decisions (unfairly priced health insurance, unspanned income etc.). We view the theory developed in this paper as a complement to our understanding of richer -typically numerically solved- models of retirement. There are however many interesting extensions to this model that should be relatively tractable.

A first important extension would be to include features that are realistically present in actual 401(k) type plans such as tax deferral, employee matching contributions and tax provisions related to withdrawals. By so doing, the solutions developed in this model could be used to determine the optimal saving, retirement and portfolio decisions of consumers that are contemplating retirement and taking into account tax considerations.

A second extension would allow the agent to reenter the workforce (at a lower income rate) once retired. We doubt this would alter the qualitative features of the model, but it is very likely that it would alter the quantitative predictions. It can be reasonably conjectured that the wealth thresholds would be significantly lower in that case, and the portfolios even stronger tilted towards stocks, because of the added flexibility.

A third extension of the model would be to introduce predictability and more elaborate preferences. If one were to introduce predictability, while keeping the market complete (like Wachter [2002]) the methods of this paper can be easily extended. It is also very likely that the model does not loose its tractability if one uses Epstein-Zin type utilities, in conjunction with the methods recently developed by Schroder and Skiadas (1999).

A fourth extension of the model that we currently pursue is to study its general equilibrium implications. This is of particular interest as it would enable one to make some predictions about how the properties of returns are likely to change as worldwide retirement systems begin to offer more freedom to agents in making investment and retirement decisions.

9 Appendix

9.1 Proofs for section 3

The goal of this section is the proof of Proposition 1. Throughout this section we fix $t = 0$ without loss of generality. We start with some useful definitions that are standard in the convex duality approach[23]

For a concave, strictly increasing and cont. differentiable function $U : (0, \infty) \to R$ satisfying

$$U'(0^+) = \lim_{x \downarrow 0} U'(x) = \infty \text{ and } U'(\infty) = \lim_{x \to \infty} U'(x) = 0 \tag{39}$$

we can define the inverse $I()$ of $U'()$. $I()$ maps $(0, \infty) \to (0, \infty)$ and satisfies

$$I(0^+) = \infty, I(\infty) = 0$$

A very convenient concept is that of a Legendre Fenchel transform ($\widetilde{U}$) of a concave function $U : (0, \infty) \to$ R

$$\widetilde{U}(y) = \max_{x>0} [U(x) - xy] = U(I(y)) - yI(y), \quad 0 < y < \infty \tag{40}$$

It is easy to verify that $\widetilde{U}()$ is strictly decreasing and convex and satisfies

$$\begin{aligned} \widetilde{U}'(y) &= -I(y), \quad 0 < y < \infty \qquad (41) \\ U(x) &= \min_{y>0} \left[\widetilde{U}(y) + xy \right] = \widetilde{U}(U'(x)) + xU'(x), \quad 0 < x < \infty \end{aligned}$$

The inequality

$$U(I(y)) \geq U(x) + y [I(y) - x]$$

follows from (40).

With these definitions we can proceed to extend the duality approach proposed by Karatzas and Wang (2000) to address portfolio problems with discretionary stopping to a setting with income.

We start by fixing a stopping time τ and defining

$$V_\tau(W_0) = \max_{c_t, \pi_t} E \left[\int_0^\tau e^{-\beta t} U_1(c_t) dt + e^{-\beta \tau} U_2(W_\tau) \right] \tag{42}$$

where U_1 and U_2 were defined in (6),(7). The following result is a generalization of the equivalent result in Karatzas and Wang (2000)[24] to allow for income.

Lemma 1 *Let:*

$$\widetilde{J}(\lambda;\tau) = E\left[\int_0^\tau \left[e^{-\beta t}\widetilde{U_1}(\lambda e^{\beta t}H(t)) + \lambda H(t)y_0\right]dt + e^{-\beta\tau}\widetilde{U}_2(\lambda e^{\beta\tau}H(\tau))\right]$$

For any τ that is finite almost surely, there exists λ^* such that

$$V_\tau(W_0) = \inf_{\lambda>0}\left[\widetilde{J}(\lambda;\tau) + \lambda W_0\right] = \widetilde{J}(\lambda^*;\tau) + \lambda^* W_0$$

and the optimal solution to (42) entails

$$W_\tau = I_2(\lambda^* e^{\beta\tau}H_\tau) \tag{43}$$

$$c_t = I_1(\lambda^* e^{\beta t}H(t))1\{t<\tau\} \tag{44}$$

with I_1, I_2 defined in a similar way to equation (41). Moreover the value function $V(W_0)$ of the problem outlined in section 2 satisfies

$$V(W_0) = \sup_\tau V_\tau(W_0) = \sup_\tau \inf_{\lambda>0}\left[\widetilde{J}(\lambda;\tau) + \lambda W_0\right] = \sup_\tau\left[\widetilde{J}(\lambda^*;\tau) + \lambda^* W_0\right]$$

This result shows that one can proceed in two steps to solve the entire problem. First fix a stopping time. Conditional on that one can determine the optimal consumption and portfolio policies. Then determine the resulting value function and maximize over stopping times. This approach is unfortunately somewhat unfruitful. Karatzas and Wang (2000) show that instead one can reduce the entire joint portfolio-consumption-stopping problem into a pure optimal stopping problem by investigating cases in which the following inequality

$$V(W_0) = \sup_\tau \inf_{\lambda>0}\left[\widetilde{J}(\lambda;\tau) + \lambda W_0\right] \le \inf_{\lambda>0}\sup_\tau\left[\widetilde{J}(\lambda;\tau) + \lambda W_0\right] = \inf_\lambda\left[\widetilde{V}(\lambda) + \lambda W_0\right] \tag{45}$$

becomes an equality, with $\widetilde{V}(\lambda)$ is defined as

$$\widetilde{V}(\lambda) = \sup_\tau \widetilde{J}(\lambda;\tau) = \sup_\tau E\left[\int_0^\tau \left[e^{-\beta t}\widetilde{U_1}(\lambda e^{\beta t}H(t)) + \lambda H(t)y_0\right]dt + e^{-\beta\tau}\widetilde{U}_2(\lambda e^{\beta\tau}H(\tau))\right] \tag{46}$$

The inequality (45) follows from a standard result in convex duality (See e.g. Rockafellar (1997)).

The interesting fact about (46) is that it is a standard optimal stopping problem, for which one can apply well known results. In particular, the parametric assumptions that we made in section 2.3 allow us to solve this optimal stopping problem explicitly. We do this in the following Lemma.

Lemma 2 *Assume that*

$$\frac{r}{\theta}\frac{\left(\frac{1-\gamma}{\gamma}+\gamma_2\right)}{\gamma_2-1}<1 \tag{47}$$

and let

$$\gamma_2=\frac{1-2\frac{\beta-r}{\kappa^2}-\sqrt{(1-2\frac{\beta-r}{\kappa^2})^2+8\frac{\beta}{\kappa^2}}}{2}<0$$

$$\underline{\lambda}=\left(\frac{(\gamma_2-1)\theta}{\left(1+\gamma_2\frac{\gamma}{1-\gamma}\right)\left(K^{\frac{1}{\gamma}}\theta-1\right)}\frac{y_0}{r}\right)^{-\gamma}$$

$$C_2=\frac{\left[\frac{\gamma}{1-\gamma}\frac{(\gamma_2-1)}{\left(1+\gamma_2\frac{\gamma}{1-\gamma}\right)}-1\right]\frac{y_0}{r}}{\underline{\lambda}^{\gamma_2-1}}$$

Finally let

$$Z_t=\lambda e^{\beta t}H_t$$

Then a) The function $\widetilde{V}(\lambda)$ is strictly convex. b) The value function $\widetilde{V}(\lambda)$ of the optimal stopping problem of (46) is given by

$$\begin{array}{ll} C_2\lambda^{\gamma_2}-\frac{\gamma}{\gamma-1}\frac{1}{\theta}\lambda^{\frac{\gamma-1}{\gamma}}+\frac{y_0}{r}\lambda, & \text{if } \lambda > \underline{\lambda} \\ \left(\frac{\gamma}{1-\gamma}K^{\frac{1}{\gamma}}(\lambda)^{\frac{\gamma-1}{\gamma}}\right) & \text{if } \lambda \leq \underline{\lambda} \end{array}$$

The optimal stopping strategy is to stop the first time the process Z_t reaches $\underline{\lambda}$. $\widetilde{V}(\lambda)$ is continuously differentiable everywhere and $\widetilde{V}'(\lambda)$ maps $(0,\infty)$ into $(-\infty,\frac{y_0}{r})$.

Proof. *(Lemma 2).* The proof of convexity is fairly standard (available upon request). We proceed by calculating the solution to the optimal stopping problem. It is easy to show that

$$\begin{aligned}\widetilde{U}_1(\lambda) &= \max_c \frac{c^{1-\gamma}}{1-\gamma}-\lambda c= \\ &= \frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}}\end{aligned}$$

and

$$\begin{aligned}\widetilde{U}_2(\lambda) &= \max_X\left(K\frac{X^{1-\gamma}}{1-\gamma}-\lambda X\right)= \\ &= K^{\frac{1}{\gamma}}\frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}}\end{aligned}$$

so that the expression (46) becomes

$$\sup_{\tau\in S} E\left[\int_0^\tau e^{-\beta t}\frac{\gamma}{1-\gamma}\left(\lambda e^{\beta t}H_t\right)^{\frac{\gamma-1}{\gamma}}dt + e^{-\beta\tau}\left(\frac{\gamma}{1-\gamma}K^{\frac{1}{\gamma}}\left(\lambda e^{\beta\tau}H_\tau\right)^{\frac{\gamma-1}{\gamma}}\right)+\lambda y_0\int_0^\tau H_t dt\right]$$

Consider the process

$$Z_t = \lambda e^{\beta t}H_t = \lambda H_0 e^{\left(\beta - r - \frac{1}{2}\kappa^2\right)t - \kappa B_t}$$

so that

$$\frac{dZ_t}{Z_t} = (\beta - r)\,dt - \kappa dB_t, \quad Z_0 = \lambda$$

With this new notation we can rewrite the above optimal stopping problem as

$$\sup_{\tau\in S} E\left[\int_0^\tau e^{-\beta t}\left(\frac{\gamma}{1-\gamma}\left(Z_t\right)^{\frac{\gamma-1}{\gamma}} + y_0 Z_t\right)dt + e^{-\beta\tau}\left(\frac{\gamma}{1-\gamma}K^{\frac{1}{\gamma}}\left(Z_\tau\right)^{\frac{\gamma-1}{\gamma}}\right)\right]$$

To solve this pure optimal stopping problem we proceed as in Oksendal (1998) p. 213. It is easy to show that the continuation region will have the form[25]

$$\underline{Z} \le Z_t < \infty$$

To determine $\underline{Z}$ we apply the standard methodology of smooth pasting, i.e. we search for $\phi(Z_t)$ satisfying the following properties

$$-\beta\phi + (\beta - r)\,Z\frac{\partial\phi}{\partial Z} + \frac{1}{2}\frac{\partial^2\phi}{\partial Z^2}Z^2\kappa^2 + \left(\frac{\gamma}{1-\gamma}Z^{\frac{\gamma-1}{\gamma}} + y_0 Z\right) = 0 \text{ on } U \tag{48}$$

$$\phi(Z_t) \ge \left(\frac{\gamma}{1-\gamma}K^{\frac{1}{\gamma}}\left(Z_t\right)^{\frac{\gamma-1}{\gamma}}\right) \text{ everywhere} \tag{49}$$

$$-\beta\phi + (\beta - r)\,Z\frac{\partial\phi}{\partial Z} + \frac{1}{2}\frac{\partial^2\phi}{\partial Z^2}Z^2\kappa^2 + \left(\frac{\gamma}{1-\gamma}Z^{\frac{\gamma-1}{\gamma}} + y_0 Z\right) \le 0 \text{ on } R\backslash U \tag{50}$$

$$\phi(Z_t) \text{ is } C^2 \text{ a.e. and } C^1 \tag{51}$$

where U is the continuation region and D is the exercise boundary. The general solution to (48) is given by

$$\phi(Z) = C_1 Z^{\gamma_1} + C_2 Z^{\gamma_2} - \frac{\gamma}{\gamma - 1}Z^{\frac{\gamma-1}{\gamma}}\frac{1}{\theta} + \frac{y_0}{r}Z$$

where

$$\gamma_{1,2} = \frac{1 - 2\frac{\beta - r}{\kappa^2} \pm \sqrt{(1 - 2\frac{\beta - r}{\kappa^2})^2 + 8\frac{\beta}{\kappa^2}}}{2}$$

It is straightforward to verify that

$$\gamma_1 > 0, \gamma_2 < 0,\ \gamma_1 + \gamma_2 = 1 - 2\frac{\beta - r}{\kappa^2}$$

Since the continuation region is of the form $\underline{Z} < Z < \infty$ we require

$$C_1 = 0$$

and thus we are left with determining the optimal exercise point and the constant C_2 and $\underline{Z}$. We can do that by invoking (51) to get the set of conditions

$$C_2\underline{Z}^{\gamma_2} - \frac{\gamma}{\gamma-1}\frac{1}{\theta}\underline{Z}^{\frac{\gamma-1}{\gamma}} + \frac{y_0}{r}\underline{Z} = \frac{\gamma}{1-\gamma}K^{\frac{1}{\gamma}}\left(\underline{Z}\right)^{\frac{\gamma-1}{\gamma}} \quad (52)$$

$$\gamma_2 C_2\underline{Z}^{\gamma_2-1} - \frac{1}{\theta}\underline{Z}^{-\frac{1}{\gamma}} + \frac{y_0}{r} = -K^{\frac{1}{\gamma}}\underline{Z}^{-\frac{1}{\gamma}} \quad (53)$$

Notice that we can rewrite the above as

$$C_2\underline{Z}^{\gamma_2-1} - \frac{\gamma}{\gamma-1}\frac{1}{\theta}\underline{Z}^{-\frac{1}{\gamma}} + \frac{y_0}{r} = \frac{\gamma}{1-\gamma}K^{\frac{1}{\gamma}}\left(\underline{Z}\right)^{-\frac{1}{\gamma}}$$

$$\gamma_2 C_2\underline{Z}^{\gamma_2-1} - \frac{1}{\theta}\underline{Z}^{-\frac{1}{\gamma}} + \frac{y_0}{r} = -K^{\frac{1}{\gamma}}\underline{Z}^{-\frac{1}{\gamma}}$$

Solving for $\underline{Z}$ leads to

$$\underline{Z}^{-\frac{1}{\gamma}} = \frac{(\gamma_2-1)\theta}{\left(1+\gamma_2\frac{\gamma}{1-\gamma}\right)\left(K^{\frac{1}{\gamma}}\theta-1\right)}\frac{y_0}{r}$$

One can show that

$$1+\gamma_2\frac{\gamma}{1-\gamma} < 0 \text{ if } \gamma < 1$$

$$1+\gamma_2\frac{\gamma}{1-\gamma} > 0 \text{ if } \gamma > 1$$

and hence $\underline{Z}^{-\frac{1}{\gamma}} > 0$, independent of γ.We also note that the above two inequalities also imply:

$$(\gamma_2-1) < -\frac{1}{\gamma} \quad (54)$$

independent of γ, an inequality that we use throughout in the text. It is also the case that:

$$C_2 = \frac{\left[\frac{\gamma}{1-\gamma}\frac{(\gamma_2-1)}{\left(1+\gamma_2\frac{\gamma}{1-\gamma}\right)}-1\right]\frac{y_0}{r}}{\underline{Z}^{\gamma_2-1}} > 0$$

since

$$\frac{\gamma}{1-\gamma}\frac{(\gamma_2-1)}{\left(1+\gamma_2\frac{\gamma}{1-\gamma}\right)} - 1 = \frac{-\frac{1}{1-\gamma}}{\left(1+\gamma_2\frac{\gamma}{1-\gamma}\right)} > 0$$

The previous considerations allow us to guess that the solution to the optimal stopping problem under consideration is given by

$$C_2 Z^{\gamma_2} - \frac{\gamma}{\gamma-1}\frac{1}{\theta}Z^{\frac{\gamma-1}{\gamma}} + \frac{y_0}{r}Z, \quad \text{if } Z > \underline{Z} \quad (55)$$

$$\left(\frac{\gamma}{1-\gamma}K^{\frac{1}{\gamma}}\left(Z\right)^{\frac{\gamma-1}{\gamma}}\right) \quad \text{if } Z \leq \underline{Z} \quad (56)$$

We proceed to verify that this is indeed the optimal stopping time by considering the rest of the conditions (namely (50) and (49)). To verify (49) we need to show that

$$C_2 Z^{\gamma_2} - \frac{\gamma}{\gamma - 1}\frac{1}{\theta} Z^{\frac{\gamma-1}{\gamma}} + \frac{y_0}{r} Z \geq \left(\frac{\gamma}{1-\gamma} K^{\frac{1}{\gamma}} Z^{\frac{\gamma-1}{\gamma}} \right)$$

for $Z \geq \underline{Z}$. We do this by considering the difference

$$T(Z) = C_2 Z^{\gamma_2} + \frac{\gamma}{\gamma - 1}\left(K^{\frac{1}{\gamma}} - \frac{1}{\theta} \right) Z^{\frac{\gamma-1}{\gamma}} + \frac{y_0}{r} Z$$

It is clear that $T(Z)$ satisfies: $T(\underline{Z}) = 0$ and $T'(\underline{Z}) = 0$ by construction. The claim that $T(Z) \geq 0$ for $Z > \underline{Z}$ will be proved if we can show that $T'(Z) \geq 0$ for $Z > \underline{Z}$. $T'(Z)$ is given by

$$T'(Z) = C_2 \gamma_2 Z^{\gamma_2 - 1} + Z^{-\frac{1}{\gamma}}\left(K^{\frac{1}{\gamma}} - \frac{1}{\theta} \right) + \frac{y_0}{r}$$

or

$$T'(Z) = C_2 \gamma_2 \left(\frac{Z}{\underline{Z}} \right)^{\gamma_2 - 1} \underline{Z}^{\gamma_2 - 1} + \left(\frac{Z}{\underline{Z}} \right)^{-\frac{1}{\gamma}} \underline{Z}^{-\frac{1}{\gamma}} \left(K^{\frac{1}{\gamma}} - \frac{1}{\theta} \right) + \frac{y_0}{r}$$

Observe that for $Z > \underline{Z}$ we have that

$$\left(\frac{Z}{\underline{Z}} \right)^{\gamma_2 - 1} < \left(\frac{Z}{\underline{Z}} \right)^{-\frac{1}{\gamma}}$$

and accordingly

$$C_2 \gamma_2 \left(\frac{Z}{\underline{Z}} \right)^{\gamma_2 - 1} \underline{Z}^{\gamma_2 - 1} > C_2 \gamma_2 \left(\frac{Z}{\underline{Z}} \right)^{-\frac{1}{\gamma}} \underline{Z}^{\gamma_2 - 1}$$

since $\gamma_2 < 0$ and $(\gamma_2 - 1) < -\frac{1}{\gamma}$:

$$T'(Z) \geq \left(\frac{Z}{\underline{Z}} \right)^{-\frac{1}{\gamma}} \left[C_2 \gamma_2 \underline{Z}^{\gamma_2 - 1} + \underline{Z}^{-\frac{1}{\gamma}} \left(K^{\frac{1}{\gamma}} - \frac{1}{\theta} \right) \right] + \frac{y_0}{r} \tag{57}$$

But

$$C_2 \gamma_2 \underline{Z}^{\gamma_2 - 1} + \underline{Z}^{-\frac{1}{\gamma}} \left(K^{\frac{1}{\gamma}} - \frac{1}{\theta} \right) = -\frac{y_0}{r}$$

by the fact that $T'(\underline{Z}) = 0$. Accordingly (57) becomes

$$T'(Z) \geq \left[1 - \left(\frac{Z}{\underline{Z}} \right)^{-\frac{1}{\gamma}} \right] \frac{y_0}{r} > 0$$

for $Z > \underline{Z}$. This verifies that $T(Z) > 0$ for $Z > \underline{Z}$.

We are left with checking that (50) holds. Observe first, that for $Z < \underline{Z}$, the function under consideration becomes: $\frac{\gamma}{1-\gamma} K^{\frac{1}{\gamma}} (Z_t)^{\frac{\gamma-1}{\gamma}}$. Letting $\phi(Z) = \frac{\gamma}{1-\gamma} K^{\frac{1}{\gamma}} (Z_t)^{\frac{\gamma-1}{\gamma}}$ and computing

$$A(\phi) = -\beta\phi + \phi' Z(\beta - r) + \frac{\phi''}{2} Z^2 \kappa^2 + \left(\frac{\gamma}{1-\gamma} Z^{\frac{\gamma-1}{\gamma}} + y_0 Z \right)$$

it is easy to check that $A(\phi) \leq 0$ whenever $Z < \left(\frac{\frac{1-\gamma}{\gamma}y_0}{K^{\frac{1}{\gamma}}\theta-1}\right)^{-\gamma}$. Hence in order to prove the claim it suffices to show that

$$\underline{Z} < \left(\frac{\frac{1-\gamma}{\gamma}y_0}{K^{\frac{1}{\gamma}}\theta-1}\right)^{-\gamma} \tag{58}$$

To check (58) we need to show that

$$\underline{Z}^{-\frac{1}{\gamma}} = \frac{(\gamma_2-1)\theta}{\left(1+\gamma_2\frac{\gamma}{1-\gamma}\right)\left(K^{\frac{1}{\gamma}}\theta-1\right)}\frac{y_0}{r} > \frac{\frac{1-\gamma}{\gamma}y_0}{K^{\frac{1}{\gamma}}\theta-1}$$

We need to distinguish cases. For $\gamma < 1$, $\left(K^{\frac{1}{\gamma}}\theta-1\right) > 0$ and thus the above inequality can be rewritten as

$$\frac{(\gamma_2-1)\theta}{\left(1+\gamma_2\frac{\gamma}{1-\gamma}\right)r} > \frac{1-\gamma}{\gamma}$$

or equivalently

$$\frac{r}{\theta}\frac{\left(\frac{1-\gamma}{\gamma}+\gamma_2\right)}{\gamma_2-1} < 1 \tag{59}$$

Observe that we arrive at the same inequality for $\gamma > 1$ too with identical steps. This is exactly assumption (47). The previous reasoning allows us to obtain the function $\widetilde{V}(\lambda)$ as

$$\begin{aligned}
\widetilde{V}(\lambda) &= C_2\lambda^{\gamma_2} - \frac{\gamma}{\gamma-1}\frac{1}{\theta}\lambda^{\frac{\gamma-1}{\gamma}} + \frac{y_0}{r}\lambda, \quad \text{if } \lambda > \underline{\lambda} \\
\widetilde{V}(\lambda) &= \left(\frac{\gamma}{1-\gamma}K^{\frac{1}{\gamma}}(\lambda)^{\frac{\gamma-1}{\gamma}}\right) \quad \text{if } \lambda \leq \underline{\lambda}
\end{aligned}$$

where

$$\underline{\lambda} = \underline{Z} = \left(\frac{(\gamma_2-1)\theta}{\left(1+\gamma_2\frac{\gamma}{1-\gamma}\right)\left(K^{\frac{1}{\gamma}}\theta-1\right)}\frac{y_0}{r}\right)^{-\gamma}$$

The function $\widetilde{V}(\lambda)$ is continuously differentiable everywhere and convex. Accordingly, we can calculate the derivative

$$\begin{aligned}
\widetilde{V}'(\lambda) &= \gamma_2C_2\lambda^{\gamma_2-1} - \frac{1}{\theta}\lambda^{-\frac{1}{\gamma}} + \frac{y_0}{r}, \quad \text{if } \lambda > \underline{\lambda} \\
\widetilde{V}'(\lambda) &= \left(-K^{\frac{1}{\gamma}}\lambda^{-\frac{1}{\gamma}}\right) \quad \text{if } 0 < \lambda \leq \underline{\lambda}
\end{aligned}$$

The range of $\widetilde{V}'(\lambda)$ for positive λ is $(-\infty, \frac{y_0}{r})$ implying that the equation

$$\widetilde{V}'(\lambda) = -W_0$$

will always have a solution as long as $W_0 \in (-\frac{y_0}{r}, \infty)$, since $\widetilde{V}'(\lambda)$ is an increasing continuous function. ■

Remark 1 *Notice that in contrast to Karatzas and Wang (2000) the function $\widetilde{V}(\lambda)$ does not have to be decreasing due to the presence of income. As a matter of fact we show that $\widetilde{V}'(\lambda)$ takes values in $(-\infty, \frac{y_0}{r})$.*

Remark 2 *Assumption 47 can be shown to be always satisfied as long as* $\theta > 0$ *in two special cases: i) if* $\gamma > 1$ *and* $\beta \geq r$ *or ii) if* $\beta = r$ *. We conjecture that* $\theta > 0$ *is sufficient for assumption 47 more generally but we haven't been able to prove it algebraically. However, in all numerical exercises that we consider we satisfy the condition i) of this remark.*

Proof. *Remark (2)* To see this, observe that we can rewrite (47) as

$$\begin{aligned} r + (\theta - r)\gamma &> (\theta - r)\gamma_2\gamma, \\ r &> (r - \theta)\gamma(1 - \gamma_2). \end{aligned}$$

This inequality is clearly satisfied if $\theta \geq r$, that is if

$$\frac{\gamma - 1}{\gamma}\frac{\kappa^2}{2\gamma} + \frac{\beta - r}{\gamma} \geq 0.$$

Observe that this last equation is verified if $\gamma > 1$ and $\beta \geq r$. To show ii) assume now that $\beta = r$ and that γ is arbitrary as long as $\theta > 0$. Multiplying both numerator and denominator of (47) by γ_2 and using the fact that$\gamma_2(\gamma_2 - 1) = \frac{2r}{\kappa^2}$ we reduce to showing that

$$\frac{\kappa^2}{2\theta}\left[\frac{\gamma_2}{\gamma} + \frac{2r}{\kappa^2}\right] < 1$$

Now using the definition of $\theta = r - \frac{1}{2}\frac{1-\gamma}{\gamma}\frac{\kappa^2}{\gamma}$ and the fact $\gamma_1 + \gamma_2 = 1$ we reduce the above problem to checking that the ratio

$$\frac{r - \frac{1}{2}\frac{\kappa^2}{\gamma}(\gamma_1 - 1)}{r - \frac{1}{2}\frac{1-\gamma}{\gamma}\frac{\kappa^2}{\gamma}} < 1$$

which will be trivially the case if $\gamma > 1$. For $\gamma < 1$ this will be true if

$$\gamma_1 - 1 > \frac{1 - \gamma}{\gamma}$$

but this is immediate since

$$\gamma_1 > \frac{1}{\gamma}$$

■

The following result is a straightforward extension of a result in Karatzas and Wang (2000) and is given without proof.

Lemma 3 *Let* $\widehat{\tau_\lambda}$ *be the optimal stopping rule associated with* λ *and given in Lemma 2. Then*

$$\widetilde{V}'(\lambda) = -E\left[\int_0^{\widehat{\tau_\lambda}} H(t)\left(I_1(\lambda e^{\beta t}H(t)) - y_0\right)dt + H(\widehat{\tau_\lambda})I_2(\lambda e^{\beta\widehat{\tau_\lambda}}H_{\widehat{\tau_\lambda}})\right],\ \lambda \in (0, \infty)$$

This Lemma shows that the derivative of $\widetilde{V}(\lambda)$ informs us of the amount of initial wealth that would be needed in order to sustain a stream of consumption and retirement wealth of

$$\begin{aligned} \widehat{c}_t &= I_1(\lambda e^{\beta t} H(t)) 1\{t < \widehat{\tau_\lambda}\} \\ \widehat{W}_{\widehat{\tau_\lambda}} &= I_2(\lambda e^{\beta \widehat{\tau_\lambda}} H(\widehat{\tau_\lambda})) \end{aligned}$$

To prove Proposition 1 we use this observation in order to replace the inequality in (45) with an equality sign and thus compute the Value function of the problem of interest.

Proof. *(Proposition 1)* We will verify that the triplet

$$\begin{aligned} \widehat{c}_t &= (\lambda^* e^{\beta t} H(t))^{-\frac{1}{\gamma}} 1\{0 \leq t < \widehat{\tau}\} \\ \widehat{\tau} &= \inf\{t : \lambda^* e^{\beta t} H(t) = \underline{\lambda}\} \\ \widehat{W}_{\widehat{\tau}} &= I_2(\lambda^* e^{\beta \widehat{\tau_\lambda}} H(\widehat{\tau_\lambda})) = I_2(\underline{\lambda}) = \overline{W} \end{aligned}$$

where λ^* is given by (13) is an optimal policy. We start by showing that this policy is feasible. To see this, consider the function $\widetilde{V}(\lambda)$ as obtained in Lemma 2. Since $\widetilde{V}(\lambda)$ is strictly convex, and $\widetilde{V'}(\lambda)$ maps $(0, \infty)$ to $(-\frac{y_0}{r}, \infty)$ we know that there exists a unique $\lambda^* > 0$ s.t.

$$\widetilde{V}(\lambda^*) + \lambda^* W_0 = \inf_{\lambda > 0} \left[\widetilde{V}(\lambda) + \lambda W_0 \right]$$

which can be rewritten as

$$\widetilde{V}(\lambda) + \lambda W_0 \geq \widetilde{V}(\lambda^*) + \lambda^* W_0 \quad \forall \lambda > 0$$

Moreover, λ^* as obtained in equation (13) minimizes $\left[\widetilde{V}(\lambda) + \lambda W_0\right]$ over all $\lambda > 0$, since

$$\widetilde{V'}(\lambda^*) = -W_0$$

By Lemma 3

$$W_0 = -\widetilde{V'}(\lambda^*) = E\left[\int_0^{\widehat{\tau}_{\lambda^*}} \left[H(t)(\lambda^* e^{\beta t} H(t))^{-\frac{1}{\gamma}}) - H(t) y_0\right] dt + H(\widehat{\tau}_{\lambda^*}) \overline{W}\right]$$

so that we can create portfolios that can finance the consumption stream

$$\widehat{c}_t = I_1(\lambda^* e^{\beta t} H(t)) 1\{t < \widehat{\tau}_{\lambda^*}\}$$

and the retirement Wealth

$$\widehat{W}_{\widehat{\tau}_{\lambda^*}} = I_2(\lambda^* e^{\beta \widehat{\tau}_{\lambda^*}} H(\widehat{\tau}_{\lambda^*})) = \overline{W}$$

We now verify optimality of this policy as follows

$$\begin{aligned}
V(W_0) &\geq E\left[\int_0^{\widehat{\tau}_{\lambda^*}} e^{-\beta t}U_1(I_1(\lambda^* e^{\beta t}H(t)))dt + e^{-\beta\widehat{\tau}_{\lambda^*}}U_2\left(I_2(\lambda^* e^{\beta\widehat{\tau}_{\lambda^*}}H(\widehat{\tau}_{\lambda^*}))\right)\right] = \\
&= E\left[\int_0^{\widehat{\tau}_{\lambda^*}} e^{-\beta t}\widetilde{U}_1(I_1(\lambda^* e^{\beta t}H(t)))dt + e^{-\beta\widehat{\tau}_{\lambda^*}}\widetilde{U}_2\left(I_2(\lambda^* e^{\beta\widehat{\tau}_{\lambda^*}}H(\widehat{\tau}_{\lambda^*}))\right)\right] + \\
&\quad \lambda^* E\left[H(\widehat{\tau}_{\lambda^*})\widehat{W}_{\widehat{\tau}_{\lambda^*}} + \int_0^{\widehat{\tau}_{\lambda^*}} H(t)\widehat{c}_t dt\right] \\
&= E\left[\int_0^{\widehat{\tau}_{\lambda^*}} \left[e^{-\beta t}\widetilde{U}_1(I_1(\lambda^* e^{\beta t}H(t))) + \lambda^* H(t)y_0\right]dt + e^{-\beta\widehat{\tau}_{\lambda^*}}\widetilde{U}_2\left(I_2(\lambda^* e^{\beta\widehat{\tau}_{\lambda^*}}H(\widehat{\tau}_{\lambda^*}))\right)\right] + \lambda^* W_0 = \\
&= \widetilde{V}(\lambda^*) + \lambda^* W_0 = \inf_{\lambda>0}\left[\widetilde{V}(\lambda) + \lambda W_0\right]
\end{aligned}$$

The first equality follows from the definitions of $\widetilde{U}_1, I_1, \widetilde{U}_2, I_2$. The second equality follows from the intertemporal budget constraint and the last from the definition of $\widetilde{V}(\lambda)$. The fact that $V(W_0) \geq \inf_{\lambda>0}\left[\widetilde{V}(\lambda) + \lambda W_0\right]$ along with (45) delivers the result that

$$V(W_0) = \inf_{\lambda>0}\left[\widetilde{V}(\lambda) + \lambda W_0\right]$$

In particular the optimal policies are given by $< \widehat{c}_t, \widehat{W}_{\widehat{\tau}_{\lambda^*}}, \widehat{\tau}_{\lambda^*} >$. The final claim of the proposition concerns the optimal portfolio. To actually compute it in feedback form we make use of formula (3.8.24) in Karatzas and Shreve (1998)

$$\pi_0 = -\frac{\kappa}{\sigma}\frac{\lambda^*(W_0)}{\lambda^*_{W_0}(W_0)}$$

where $\lambda^*(W_0)$ solves equation (13). The implicit function theorem gives

$$\frac{\lambda^*}{\lambda^*_{W_0}} = -\left(\gamma_2(\gamma_2 - 1)C_2\lambda^{*\gamma_2 - 1} + \frac{1}{\gamma}\frac{1}{\theta}\lambda^{*-\frac{1}{\gamma}}\right) \tag{60}$$

■

9.2 Proofs for section 5

Proposition 2 can be established by virtually identical steps as Proposition 1. Once again we set $t = 0$ without loss of generality. The only substantial difference to section 3 is that now $\widetilde{V}(\lambda, T)$ solves the optimal stopping problem

$$\widetilde{V}(\lambda, T) = \sup_{\tau\leq T} E\left[\int_0^{\tau}\left[e^{-\beta t}\widetilde{\widetilde{U_1}}(\lambda e^{\beta t}H(t)) + \lambda H(t)y_0\right]dt + e^{-\beta\tau}\widetilde{U}_2(\lambda e^{\beta\tau}H(\tau))\right] \tag{61}$$

We proceed with the proof of Proposition 2

Proof. *(Proposition 2)* Only a sketch is given. The idea behind the approximation is to define

$$\widetilde{V}^E(\lambda;T)\widehat{=}E\left[\int_0^T\left[e^{-\beta t}\widetilde{\widetilde{U_1}}(\lambda e^{\beta t}H(t))+\lambda H(t)y_0\right]dt+e^{-\beta\tau}\widetilde{U}_2(\lambda e^{\beta T}H(T))\right]$$

and compute

$$\widetilde{V}^E(\lambda;T)=\frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}}\frac{1-e^{-\theta T}}{\theta}+\lambda y_0\frac{1-e^{-rT}}{r}+\frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}}K^{\frac{1}{\gamma}}e^{-\theta T}$$

which can be shown by elementary methods. The next step is to study the difference between

$$P(\lambda;T)=\widetilde{V}(\lambda;T)-\widetilde{V}^E(\lambda;T)$$

which we will refer to as the early exercise premium. One can then show that inside the continuation region the "early exercise premium" $P(\lambda;T)$ solves the PDE

$$-\beta P+P_Z Z(\beta-r)+\frac{1}{2}P_{ZZ}Z^2\kappa^2-P_T=0$$

By the same approximation idea as in Barone-Adesi and Whaley (1987) we will postulate a solution of the form $P=Y(T)f(Z,Y(T))$, take $Y(T)=1-e^{-\beta T}$ and ignore P_Y. This allows to reduce the problem to the determination of solutions of the equation

$$f_Z Z(\beta-r)+\frac{1}{2}Z^2\kappa^2 f_{ZZ}-\frac{\beta}{Y(T)}f=0$$

which is a simple linear ODE. The solution is given just as in the infinite horizon case by

$$f(Z)=C_{2T}Z^{\gamma_{2T}}$$

where

$$\gamma_{2T}=\frac{1-2\frac{\beta-r}{\kappa^2}-\sqrt{(1-2\frac{\beta-r}{\kappa^2})^2+8\frac{\beta}{Y(T)\kappa^2}}}{2}$$

To determine the complete solution we require continuity and smooth pasting of $\widetilde{V}(\lambda;T)$ to $\frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}}K^{\frac{1}{\gamma}}$. Then by arguments identical to the infinite horizon case we get (29) and (30). The rest of the results follow easily by the arguments used in the case where there is no deadline. ∎

An important concern is the accuracy of the approximation. The most straightforward way to see whether the solution is accurate or not, is to compare it with a consistent numerical scheme. A particularly attractive numerical scheme in our single-dimensional framework is a binomial tree of the Cox-Ross-Rubinstein kind. For a fixed λ and T we can treat (61) as a standard optimal stopping problem and solve it numerically. For the numbers that we present we chose $dt=1/200$. We fixed the parameters of the investment opportunity set to the levels described in section 6 and experimented with various levels of $K^{1/\gamma}\theta$ and γ. We report results for $\gamma=3$. For other levels of γ the results are similar.

Figure 8 depicts the value function of problem (61) for $T - t = 5$. We choose the initial value of λ so that it coincides with $\underline{\lambda}_{(T-t)}$.We then choose the range of λ so that the largest value of λ is given by $\underline{\lambda}_{(T-t)}e^{10\kappa}$. In simple terms we look at a range that starts from the critical barrier $\underline{\lambda}_{(T-t)}$ and extends from there by 10 standard deviations.[26]. The top panel in figure 8 gives the value function of problem (61) obtained through the binomial tree and the analytical approximation. Since these curves are practically indistinguishable, we find it more informative to focus on the results in the lower panel which depicts the relative numerical error. This is defined as the absolute value of the difference between the two value functions over the absolute value of the level of the value function obtained numerically. As can be seen the relative error is typically indistinguishable from 0, except on a small strip (very close to $\underline{\lambda}_{(T-t)}$) where it is in the second digit range. There are two remarks on this: a) the amount of time that the state variable is likely to spend in the neighborhood of these areas is very small. The relative error seems to be in the second digit range for a range of $\lambda's$ with mass 0.05. The standard deviation of the state variable is 7 times larger than that. b) It is a well known fact that binomial trees tend to be "jagged" close to the critical region and this might also be accounting for some of the difference between the two value functions.

Figure 8 shows that except on a negligible set, the value functions of the two problems practically coincide. This is important since all other quantities (critical wealth levels, consumption and portfolios) are derived from the solution to (61).

Figure 9 examines the practical implications of using the analytical approximation instead to the binomial tree. We focus on critical wealth levels, because they demonstrate how closely the analytical solution approximates the numerical one. We plot the critical wealth level as a function of the distance to retirement. Once again, the two solutions coincide as might be expected in light of the previous figure. Except for times very close to the retirement deadline, the analytical solution is within the "error" band of the binomial tree. Even when it isn't the relative error seems to be negligible. (typically less than 5%). Similar results hold unsurprisignly for optimal consumption and portfolios, which are not reported here.

In conclusion, we find that the numerical accuracy of the analytical solution that we propose is very good. Nothing economically meaningful seems to rest on whether we use a consistent numerical scheme, or the considerably more tractable analytical approximation.

9.3 Proofs for section 7

In what follows we sketch how to obtain the solution to this problem and prove proposition 3. The basic modification of the approach used so far is that $\widetilde{V}(\lambda)$ needs to be minimized over a set of decreasing processes in a manner analogous to He and Pages (1993). The reader is referred to that paper for a number of technical details. Once again we set $t = 0$ without loss of generality.

We start by fixing a stopping time τ and defining

$$J(W;\pi_s,c_s,\tau)=E\left[\int_0^\tau e^{-\beta t}U_1(c_t)dt+e^{-\beta\tau}U_2(W_\tau)\right]$$

for any admissible pair (c_s,π_s) satisfying (35) and (36). Let λX_t be a non-increasing process starting at $X_0=1$ with $\lambda>0$. We obtain the following set of inequalities for any admissible pair (c_s,π_s)

$$\begin{aligned}J(W;\pi_s,c_s,\tau) &= E\left[\int_0^\tau e^{-\beta t}U_1(c_t)dt+e^{-\beta\tau}U_2(W_\tau)\right] &(62)\\ &\leq E\left[\int_0^\tau e^{-\beta t}\widetilde{U_1}(\lambda X_t e^{\beta t}H(t))dt+e^{-\beta\tau}\widetilde{U}_2(\lambda X_\tau e^{\beta\tau}H(\tau))\right] &(63)\\ &\quad+\lambda E\left[X_\tau H(\tau)W_\tau+\int_0^\tau X_t H(t)c(t)dt\right]\end{aligned}$$

Integrating by parts and using the fact that $X_0=1$, the second term of the right hand side can be rewritten as

$$\begin{aligned}E\left[\int_0^\tau X_tH_tc_tdt+X_\tau H_\tau W_\tau\right] &= E\left[\int_0^\tau X_tH_t(c_t-y_0)dt+X_\tau H_\tau W_\tau+\int_0^\tau X_tH_ty_0dt\right]=\\ &\quad E\left[\int_0^\tau X_tH_ty_0dt+H_\tau W_\tau+\int_0^\tau H_t(c_t-y_0)dt\right]\\ &\quad+E\left[\int_0^\tau H_t\frac{E_t\left[\int_t^\tau H_s(c_s-y_0)ds+H_\tau W_\tau\right]}{H_t}dX_t\right]\end{aligned}$$

so that we have

$$\begin{aligned}J(W;\pi_s,c_s,\tau) &\leq E\left[\int_0^\tau e^{-\beta t}\widetilde{U_1}(\lambda X_te^{\beta t}H(t))dt+e^{-\beta\tau}\widetilde{U}_2(\lambda X_\tau e^{\beta\tau}H(\tau))\right]\\ &\quad+\lambda E\left[\int_0^\tau X_tH_ty_0dt+H_\tau W_\tau+\int_0^\tau H_t(c_t-y_0)dt\right]\\ &\quad+\lambda E\left[\int_0^\tau H_t\frac{E_t\left[\int_t^\tau H_s(c_s-y_0)ds+H_\tau W_\tau|F_t\right]}{H_t}dX_t\right]\\ &\leq E\left[\int_0^\tau e^{-\beta t}\widetilde{U_1}(X_te^{\beta t}H(t))dt+e^{-\beta\tau}\widetilde{U}_2(X_\tau e^{\beta\tau}H(\tau))\right]\\ &\quad+\lambda\left(W_0+E\left[\int_0^\tau X_tH_ty_0dt\right]\right)\end{aligned}$$

where the last inequality comes from the fact that

$$\frac{E_t\left[\int_t^\tau H_s(c_s - y_0)ds + H_\tau W_\tau\right]}{H_t} dX_t \leq 0, \text{ and}$$

since $dX_t \leq 0, W_t \geq 0$ and

$$E\left[\int_0^\tau H_t(c_t - y_0)dt + H_\tau W_\tau\right] \leq W_0.$$

by the budget constraint.The equality occurs if and only if

$$W_\tau = I_2\left(e^{\beta\tau} X_\tau H_\tau\right) \text{ and } c(t) = I_1\left(e^{\beta t} X_t H_t\right), \quad \text{for all } 0 \leq t \leq \tau \tag{64}$$

and

$$E\left[H(\tau)W_\tau + \int_0^\tau H(t)c(t)dt\right] = W_0 + E\left[\int_0^\tau H(t)y_0 dt\right]$$

and

$$\frac{E_t\left[\int_t^\tau H_s(c_s - y_0)ds + H_\tau W_\tau\right]}{H_t} dX_t = 0.$$

Evaluating the above set of inequalities at the optimal stopping time τ, and observing that it holds for all X_t decreasing, we have that

$$V(W_0) \leq \sup_\tau \inf_{\{\lambda, X_t\}} \left[\widetilde{J}(\{X_t, \lambda\}; \tau) + \lambda W_0\right] \tag{65}$$

where $V(W_0)$ is the value function of the original problem and $\widetilde{J}(X_t; \tau)$ is given by

$$\widetilde{J}(\{X_t\}; \tau, \lambda) = E\left[\int_0^\tau \left[e^{-\beta t}\widetilde{U_1}(\lambda X_t e^{\beta t} H(t)) + \lambda X_t H(t) y_0\right] dt + e^{-\beta\tau}\widetilde{U}_2(\lambda X_\tau e^{\beta\tau} H(\tau))\right]$$

Let

$$\begin{aligned} \widetilde{V}(\{X_t\}, \lambda) &= \sup_\tau \widetilde{J}(\{X_t\}; \tau, \lambda) & (66) \\ &= \sup_\tau E\left[\int_0^\tau \left[e^{-\beta t}\widetilde{U_1}(\lambda X_t e^{\beta t} H(t)) + \lambda X_t H(t) y_0\right] dt + e^{-\beta\tau}\widetilde{U}_2(\lambda X_\tau e^{\beta\tau} H(\tau))\right], & (67) \end{aligned}$$

and

$$\widetilde{V}(\lambda) = \inf_{\{X_t\}} \widetilde{V}(\{X_t\}, \lambda) \tag{68}$$

and define the process Z_t:

$$Z_t = \lambda e^{\beta t} X_t H_t$$

We now proceed by analogy to the case without borrowing constraints. For our constant parameters case, it can be shown that[27]

$$V(W_0) = \sup_{\tau} \inf_{\{\lambda, X_t\}} \left[\widetilde{J}(\{X_t\}; \tau, \lambda) + \lambda W_0\right] = \inf_{\{\lambda, X_t\}} \sup_{\tau} \left[\widetilde{J}(\{X_t\}; \tau, \lambda) + \lambda W_0\right] = \inf_{\lambda} \left[\widetilde{V}(\lambda) + \lambda W_0\right]. \tag{69}$$

The optimal policy functions are given by

$$W_\tau = I_2\left(\lambda^* e^{\beta\tau} X_\tau^* H_\tau\right) \text{ and } c(t) = I_1\left(\lambda^* e^{\beta t} X_t^* H_t\right), \quad \text{for all } 0 \le t \le \tau \tag{70}$$

where λ^*, τ^*, X_t^* solve (69). To solve the infimization over the space of decreasing processes one can proceed in a fashion analogous to He Pages (1993) to construct the Value of the min-max game of equation (69). The following generalization of Lemma 2 is required

Lemma 4 *For appropriate constants C_1, C_2, Z_L, Z_H (given in the proof) define the function $\widetilde{V}(\lambda)$ as*

$$\begin{aligned}
\widetilde{V}(\lambda) &= C_1\lambda^{\gamma_1} + C_2\lambda^{\gamma_2} - \frac{\gamma}{\gamma-1}\lambda^{\frac{\gamma-1}{\gamma}}\frac{1}{\theta} + \frac{y_0\lambda}{r} \;\; \textit{if } Z_L \le \lambda \le Z_H, \\
\widetilde{V}(\lambda) &= \left(\frac{\gamma}{1-\gamma}K^{\frac{1}{\gamma}}(\lambda)^{\frac{\gamma-1}{\gamma}}\right) \;\; \textit{if } \lambda < Z_L, \\
\widetilde{V}(\lambda) &= C_1 Z_H^{\gamma_1} + C_2 Z_H^{\gamma_2} - \frac{\gamma}{\gamma-1} Z_H^{\frac{\gamma-1}{\gamma}}\frac{1}{\theta} + \frac{y_0 Z_H}{r} \;\; \textit{if } \lambda > Z_H,
\end{aligned}$$

Assume moreover that

$$-\beta\widetilde{V} + (\beta - r)\lambda\frac{\partial\widetilde{V}}{\partial\lambda} + \frac{1}{2}\frac{\partial^2\widetilde{V}}{\partial\lambda^2}\lambda^2\kappa^2 + \left(\frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}} + y_0\lambda\right) \le 0 \;\; \textit{for } \lambda < Z_L \tag{71}$$

and

$$\widetilde{V} \ge \left(\frac{\gamma}{1-\gamma}K^{\frac{1}{\gamma}}(\lambda)^{\frac{\gamma-1}{\gamma}}\right) \quad \textit{everywhere} \tag{72}$$

Then $\widetilde{V}(\lambda)$ provides the value to the game

$$\widetilde{V}(\lambda) = \sup_{\tau} \inf_{\{X_t\}} \left[\widetilde{J}(\{X_t\}, \tau) \right] = \inf_{\{X_t\}} \sup_{\tau} \left[\widetilde{J}(\{X_t\}, \tau) \right]$$

Define also the process

$$Z_t = \lambda e^{\beta t} X_t H_t$$

The optimal stopping policy is to stop once Z_t crosses Z_L whereas the optimal X_t decreases once $Z_t = Z_H$.

Proof. *(Lemma 4)* We give a sketch[28]. To keep the notation consistent with section 3 we let $t = 0$ without loss of generality and use the fact that at time 0, $Z_0 = \lambda$. We will denote $Z = Z_0$ for convenience. The purpose is to determine the value $\phi(Z)$ of the game

$$\phi(Z) = \sup_{\tau} \inf_{\{X_t\}} \left[\widetilde{J}(\{X_t\}, \tau) \right] = \inf_{\{X_t\}} \sup_{\tau} \left[\widetilde{J}(\{X_t\}, \tau) \right]$$

i.e. to fix a given initial value of the multiplier $\lambda = Z_0$ and determine a decreasing process X_t^* and a stopping time τ^* so that X_t^* minimizes $\widetilde{J}$ conditional on λ and τ and τ^* maximizes $\widetilde{J}$ conditional on X_t^* and λ. In this context it is not difficult to establish a verification theorem, asserting that $\phi(Z)$ is the value of the game, as long as we can find a function $\phi(Z)$ and two barriers Z_L and Z_H with $Z_L < Z_H$ satisfying

$$-\beta\phi + (\beta - r) Z \frac{\partial \phi}{\partial Z} + \frac{1}{2} \frac{\partial^2 \phi}{\partial Z^2} Z^2 \kappa^2 + \left(\frac{\gamma}{1-\gamma} Z^{\frac{\gamma-1}{\gamma}} + y_0 Z \right) = 0 \text{ for } Z \in (Z_L, Z_H) \quad (73)$$

$$\phi(Z) \geq \left(\frac{\gamma}{1-\gamma} K^{\frac{1}{\gamma}} (Z_t)^{\frac{\gamma-1}{\gamma}} \right) \text{ everywhere} \quad (74)$$

$$-\beta\phi + (\beta - r) Z \frac{\partial \phi}{\partial Z} + \frac{1}{2} \frac{\partial^2 \phi}{\partial Z^2} Z^2 \kappa^2 + \left(\frac{\gamma}{1-\gamma} Z^{\frac{\gamma-1}{\gamma}} + y_0 Z \right) \leq 0 \text{ for } Z < Z_L \quad (75)$$

$$\phi_z < \infty \text{ in } (Z_L, Z_H),\ \phi(Z) \text{ is } C^2 \text{ a.e. and } \quad (76)$$

$$\frac{\partial \phi}{\partial Z} \leq 0 \text{ everywhere} \quad (77)$$

$$\frac{\partial \phi}{\partial Z} = 0 \text{ for } Z \in (Z_H, \infty) \quad (78)$$

A proof of this verification Theorem can be given along the lines of Theorem 3 in He and Pages (1993) and standard arguments for optimal stopping problems along the lines of Oksendal (1998). We now proceed to construct a function and two barriers that satisfy these equations. The general solution to

$$-\beta\phi + (\beta - r) Z \frac{\partial \phi}{\partial Z} + \frac{1}{2} \frac{\partial^2 \phi}{\partial Z^2} Z^2 \kappa^2 + \left(\frac{\gamma}{1-\gamma} Z^{\frac{\gamma-1}{\gamma}} + y_0 Z \right) = 0$$

is given by

$$\phi(Z) = C_1 Z^{\gamma_1} + C_2 Z^{\gamma_2} - \frac{\gamma}{\gamma-1} Z^{\frac{\gamma-1}{\gamma}} \frac{1}{\theta} + \frac{y_0 Z}{r}$$

where γ_1 and γ_2 are given by

$$\gamma_1 = \frac{1 - 2\frac{\beta-r}{\kappa^2} + \sqrt{(1-2\frac{\beta-r}{\kappa^2})^2 + 8\frac{\beta}{\kappa^2}}}{2}$$

and

$$\gamma_2 = \frac{1 - 2\frac{\beta-r}{\kappa^2} - \sqrt{(1-2\frac{\beta-r}{\kappa^2})^2 + 8\frac{\beta}{\kappa^2}}}{2}$$

It is straightforward to verify that

$$\gamma_1 > 0, \gamma_2 < 0.$$

To enforce the condition (76) we will search for Z_L, Z_H and C_1, C_2 so that

$$\begin{aligned}
C_1 Z_L^{\gamma_1} + C_2 Z_L^{\gamma_2} - \frac{\gamma}{\gamma-1} Z_L^{\frac{\gamma-1}{\gamma}} \frac{1}{\theta} + \frac{y_0 Z_L}{r} &= \left(\frac{\gamma}{1-\gamma} K^{\frac{1}{\gamma}} (Z_L)^{\frac{\gamma-1}{\gamma}} \right) \\
\gamma_1 C_1 Z_L^{\gamma_1 - 1} + \gamma_2 C_2 Z_L^{\gamma_2 - 1} - Z_L^{-\frac{1}{\gamma}} \frac{1}{\theta} + \frac{y_0}{r} &= \left(-K^{\frac{1}{\gamma}} (Z_L)^{-\frac{1}{\gamma}} \right) \\
\gamma_1 C_1 Z_H^{\gamma_1 - 1} + \gamma_2 C_2 Z_H^{\gamma_2 - 1} - Z_H^{-\frac{1}{\gamma}} \frac{1}{\theta} + \frac{y_0}{r} &= 0 \\
-\beta \left(C_1 Z_H^{\gamma_1} + C_2 Z_H^{\gamma_2} - \frac{\gamma}{\gamma-1} Z_H^{\frac{\gamma-1}{\gamma}} \frac{1}{\theta} + \frac{y_0 Z_H}{r} \right) + \left(\frac{\gamma}{1-\gamma} Z_H^{\frac{\gamma-1}{\gamma}} + y_0 Z_H \right) &= 0
\end{aligned}$$

For notational simplicity, we will define

$$\begin{aligned}
A_1 &= C_1 Z_L^{\gamma_1 - 1}, \\
A_2 &= C_2 Z_L^{\gamma_2 - 1}, \\
B &= (Z_L)^{-\frac{1}{\gamma}} \\
C &= \frac{Z_H}{Z_L}
\end{aligned}$$

With this new notation the above 4x4 system becomes

$$\begin{aligned}
A_1 + A_2 &= -\frac{y_0}{r} - \frac{\gamma}{\gamma-1} \left(K^{\frac{1}{\gamma}} - \frac{1}{\theta} \right) B \\
\gamma_1 A_1 + \gamma_2 A_2 &= -\frac{y_0}{r} - \left(K^{\frac{1}{\gamma}} - \frac{1}{\theta} \right) B \\
\gamma_1 A_1 C^{\gamma_1 - 1} + \gamma_2 A_2 C^{\gamma_2 - 1} &= B \frac{C^{-\frac{1}{\gamma}}}{\theta} - \frac{y_0}{r} \\
\beta \left(A_1 C^{\gamma_1 - 1} + A_2 C^{\gamma_2 - 1} \right) &= -y_0 \left(\frac{\beta}{r} - 1 \right) + \frac{\gamma}{1-\gamma} \left[1 - \frac{\beta}{\theta} \right] B C^{-\frac{1}{\gamma}}
\end{aligned}$$

The first two equations allow us to solve for A_1 and A_2 as functions of B

$$
\begin{aligned}
A_1 &= \frac{\frac{y_0}{r}(1-\gamma_2) + \left(K^{\frac{1}{\gamma}} - \frac{1}{\theta}\right) B \left(1 - \gamma_2 \frac{\gamma}{\gamma-1}\right)}{\gamma_2 - \gamma_1}, \\
A_2 &= \frac{-\frac{y_0}{r}(1-\gamma_1) - \left(K^{\frac{1}{\gamma}} - \frac{1}{\theta}\right) B \left(1 - \gamma_1 \frac{\gamma}{\gamma-1}\right)}{\gamma_2 - \gamma_1}.
\end{aligned}
$$

The last two equations also allow us to solve for A_1 and A_2 as functions of B and C

$$
\begin{aligned}
A_1 &= \frac{\frac{y_0}{r}\left(1 - \gamma_2 + \frac{r\gamma_2}{\beta}\right) + \left[\gamma_2 \frac{\gamma}{\gamma-1}(\frac{1}{\theta} - \frac{1}{\beta})C^{-\frac{1}{\gamma}} - \frac{C^{-\frac{1}{\gamma}}}{\theta}\right] B}{C^{\gamma_1 - 1}(\gamma_2 - \gamma_1)}, \\
A_2 &= \frac{-\frac{y_0}{r}\left(1 - \gamma_1 + \frac{r\gamma_1}{\beta}\right) - \left[\gamma_1 \frac{\gamma}{\gamma-1}(\frac{1}{\theta} - \frac{1}{\beta})C^{-\frac{1}{\gamma}} - \frac{C^{-\frac{1}{\gamma}}}{\theta}\right] B}{C^{\gamma_2 - 1}(\gamma_2 - \gamma_1)}.
\end{aligned}
$$

By equating the A_1 and A_2 obtained from the two subsystems, we get

$$
\begin{aligned}
\frac{\frac{y_0}{r}\left(1 - \gamma_2 + \frac{r\gamma_2}{\beta}\right) + \left[\gamma_2 \frac{\gamma}{\gamma-1}(\frac{1}{\theta} - \frac{1}{\beta})C^{-\frac{1}{\gamma}} - \frac{C^{-\frac{1}{\gamma}}}{\theta}\right] B}{C^{\gamma_1 - 1}(\gamma_2 - \gamma_1)} - \frac{\frac{y_0}{r}(1-\gamma_2) + \left(K^{\frac{1}{\gamma}} - \frac{1}{\theta}\right) B \left(1 - \gamma_2 \frac{\gamma}{\gamma-1}\right)}{\gamma_2 - \gamma_1} &= 0, \\
\frac{\frac{y_0}{r}\left(1 - \gamma_1 + \frac{r\gamma_1}{\beta}\right) + \left[\gamma_1 \frac{\gamma}{\gamma-1}(\frac{1}{\theta} - \frac{1}{\beta})C^{-\frac{1}{\gamma}} - \frac{C^{-\frac{1}{\gamma}}}{\theta}\right] B}{C^{\gamma_2 - 1}(\gamma_2 - \gamma_1)} - \frac{\frac{y_0}{r}(1-\gamma_1) + \left(K^{\frac{1}{\gamma}} - \frac{1}{\theta}\right) B \left(1 - \gamma_1 \frac{\gamma}{\gamma-1}\right)}{\gamma_2 - \gamma_1} &= 0
\end{aligned}
$$

.We can rewrite these two equations as

$$
\begin{aligned}
B &= \frac{\frac{y_0}{r}\left[(1-\gamma_2)C^{\gamma_1 - 1} - \left(1 - \gamma_2 + \frac{r\gamma_2}{\beta}\right)\right]}{\gamma_2 \frac{\gamma}{\gamma-1}(\frac{1}{\theta} - \frac{1}{\beta})C^{-\frac{1}{\gamma}} - \frac{C^{-\frac{1}{\gamma}}}{\theta} - C^{\gamma_1 - 1}\left(K^{\frac{1}{\gamma}} - \frac{1}{\theta}\right)\left(1 - \gamma_2 \frac{\gamma}{\gamma-1}\right)}, \\
B &= \frac{\frac{y_0}{r}\left[(1-\gamma_1)C^{\gamma_2 - 1} - \left(1 - \gamma_1 + \frac{r\gamma_1}{\beta}\right)\right]}{\gamma_1 \frac{\gamma}{\gamma-1}(\frac{1}{\theta} - \frac{1}{\beta})C^{-\frac{1}{\gamma}} - \frac{C^{-\frac{1}{\gamma}}}{\theta} - C^{\gamma_2 - 1}\left(K^{\frac{1}{\gamma}} - \frac{1}{\theta}\right)\left(1 - \gamma_1 \frac{\gamma}{\gamma-1}\right)}.
\end{aligned}
$$

so we finally get the following non-linear equation for C

$$
\frac{\frac{y_0}{r}\left[(1-\gamma_2)C^{\gamma_1 - 1} - \left(1 - \gamma_2 + \frac{r\gamma_2}{\beta}\right)\right]}{\gamma_2 \frac{\gamma}{\gamma-1}(\frac{1}{\theta} - \frac{1}{\beta})C^{-\frac{1}{\gamma}} - \frac{C^{-\frac{1}{\gamma}}}{\theta} - C^{\gamma_1 - 1}\left(K^{\frac{1}{\gamma}} - \frac{1}{\theta}\right)\left(1 - \gamma_2 \frac{\gamma}{\gamma-1}\right)} =
$$

$$
\frac{\frac{y_0}{r}\left[(1-\gamma_1)C^{\gamma_2 - 1} - \left(1 - \gamma_1 + \frac{r\gamma_1}{\beta}\right)\right]}{\gamma_1 \frac{\gamma}{\gamma-1}(\frac{1}{\theta} - \frac{1}{\beta})C^{-\frac{1}{\gamma}} - \frac{C^{-\frac{1}{\gamma}}}{\theta} - C^{\gamma_2 - 1}\left(K^{\frac{1}{\gamma}} - \frac{1}{\theta}\right)\left(1 - \gamma_1 \frac{\gamma}{\gamma-1}\right)}
$$

Thus we are left with determining C from this equation and then, substituting above to obtain B, A_1 and A_2.Given A_1, A_2, B, C , we can recover Z_L, Z_H and C_1, C_2. Conditions (74) and (75) are stated as part of the assumptions of the Lemma (compare equations (71) and (72)). Finally, condition (77) can be shown by elementary methods. ■

The solution proposed has the same form as the one obtained in section 3. An agent should enter retirement when her wealth is sufficiently high. This will occur when Z_t is sufficiently low, which in turn is going to be the case when W_t is high. Similarly, the borrowing constraints will bind once Z_t is high, which will typically be associated with a period of low performance in the stock market. The consumption process will exhibit a similar behavior to the one described in He and Pages (1993).

The rest of the proposition follows steps similar to section 3.

References

Aguiar, M., and E. Hurst (2004): "Consumption vs. Expenditure," NBER Working Paper, w.10307.

Ameriks, J., and S. P. Zeldes (2001): "How Do Household Portfolio Shares Vary With Age?," Working Paper, Columbia GSB.

Banks, J., R. Blundell, and S. Tanner (1998): "Is There a Retirement-Savings Puzzle?," *American Economic Review*, 88(4), 769–788.

Barone-Adesi, G., and R. E. Whaley (1987): "Efficient Analytic Approximation of American Option Values," *Journal of Finance*, 42(2), 301–320.

Basak, S. (1999): "On the Fluctuations in Consumption and Market Returns in the Presence of Labor and Human Capital: An Equilibrium Analysis," *Journal of Economic Dynamics and Control*, 23(7), 1029–64.

Bodie, Z., J. B. Detemple, S. Otruba, and S. Walter (2004): "Optimal consumption-portfolio choices and retirement planning," *Journal of Economic Dynamics and Control*, 28(3), 1115–1148.

Bodie, Z., R. C. Merton, and W. F. Samuelson (1992): "Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model," *Journal of Economic Dynamics and Control*, 16(3-4), 427–49.

Campbell, J. Y., J. F. Cocco, F. J. Gomes, and P. J. Maenhout (2004): "Investing Retirement Wealth: A Life-Cycle Model," *Review of Financial Studies - forthcoming*.

Campbell, J. Y., and L. M. Viceira (2002): *Strategic Asset Allocation*, Clarendon Lectures in Economics. Oxford University Press.

Carroll, C. D., and M. S. Kimball (1996): "On the Concavity of the Consumption Function," *Econometrica*, 64(4), 981–92.

Chan, Y. L., and L. Viceira (2000): "Asset Allocation with Endogenous Labor Income: The Case of Incomplete Markets," Working Paper, Harvard University.

Choi, K. J., and G. Shim (2004): "Disutility, Optimal Retirement, and Portfolio Selection," mimeo. Graduate School of Management, Korea Advanced Institute of Science and Technology.

Cox, J. C., and C.-f. Huang (1989): "Optimal consumption and portfolio policies when asset prices follow a diffusion process," *Journal of Economic Theory*, 49(1), 33–83.

Cox, J. C., S. A. Ross, and M. Rubinstein (1979): "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7(3), 229–263.

Diamond, P. A., and J. A. Hausman (1984): "Individual Retirement and Savings Behavior," *Journal of Public Economics*, 23(1-2), 81–114.

Duffie, D. (2001): *Dynamic asset pricing theory*. Princeton University Press, Princeton and Oxford.

Duffie, D., W. Fleming, M. Soner, and T. Zariphopoulou (1997): "Hedging in Incomplete Markets with HARA Utility," *Journal of Economic Dynamics and Control*, 21, 753–782.

Duffie, D., and T. Zariphopoulou (1993): "Optimal Investment with Undiversifiable Income Risk," *Mathematical Finance*, 3, 135–148.

Dybvig, P. H., and H. Liu (2005): "Lifetime Consumption and Investment: Retirement and Constrained Borrowing," mineo. Washington University in St. Louis.

El Karoui, N., and M. Jeanblanc-Picque (1998): "Optimization of Consumption with Labor Income," *Finance and Stochastics*, 2(4), 409–440.

Gustman, A. L., and T. L. Steinmeier (1986): "A Structural Retirement Model," *Econometrica*, 54(3), 555–84.

——— (2002): "Retirement and the Stock Market Bubble," *NBER Working Paper*, 9404.

Hansen, G. (1985): "Indivisible labor and the business cycle," *Journal of Monetary Economics*, 16(3), 309–327.

Harrison, J. M. (1985): *Brownian motion and stochastic flow systems*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York.

He, H., and H. F. Pages (1993): "Labor Income, Borrowing Constraints, and Equilibrium Asset Prices," *Economic Theory*, 3(4), 663–696.

Karatzas, I., J. P. Lehoczky, and S. E. Shreve (1987): "Optimal portfolio and consumption decisions for a "small investor" on a finite horizon," *SIAM Journal on Control and Optimization*, 25(6), 1557–1586.

Karatzas, I., and S. E. Shreve (1991): *Brownian motion and stochastic calculus*, vol. 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edn.

——— (1998): *Methods of mathematical finance*, Applications of Mathematics. Springer-Verlag, New York.

Karatzas, I., and H. Wang (2000): "Utility maximization with discretionary stopping," *SIAM Journal on Control and Optimization*, 39(1), 306–329.

Kocherlakota, N., and R. Jagannathan (1996): "Why Should Older People Invest Less in Stocks than Younger People?," *Federal Reserve Bank of Minneapolis Quarterly Review*, pp. 11–23.

Kogan, L., and R. Uppal (2001): "Risk Aversion and Optimal Portfolio Policies in Partial and General Equilibrium Economies," NBER Working Papers: 8609, 2001.

Koo, H. K. (1998): "Consumption and Portfolio Selection with Labor Income: A Continuous Time Approach," *Mathematical Finance*, 8(1), 49–65.

Lachance, M.-E. (2003): "Optimal Investment Behavior as Retirement Looms," Working Paper, Wharton.

Lazear, E. P. (1986): *Retirement from the Labor Force*pp. 305–55, Handbook of Labor Economics, vol. I eds. O. Ashenfelter and R. Layard. Elsveier.

Liu, J., and E. Neis (2002): "Endogenous Retirement and Portfolio Choice," Working Paper, UCLA.

Merton, R. C. (1971): "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory*, 3(4), 373–413.

Musiela, M., and M. Rutkowski (1998): *Martingale Methods in Financial Modelling*, vol. 36 of *Applications of Mathematics*. Springer-Verlag, New York, fffirst edn.

Oksendal, B. (1998): *Stochastic differential equations: An Introduction with Applications - Fifth Edition*, Universitext. Springer-Verlag, Berlin.

Poterba, J. M., S. F. Venti, and D. A. Wise (1998): "401(k) Plans and Future Patterns of Retirement Saving," *American Economic Review*, 88(2), 179–184.

——— (2000): "Saver Behavior and 401(k) Retirement Wealth," *American Economic Review*, 90(2), 297–302.

Rockafellar, R. T. (1997): *Convex analysis*, Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ.

Rogerson, R. (1988): "Indivisible labor, lotteries and equilibrium," *Journal of Monetary Economics*, 21(1), 3–16.

Rust, J. (1994): *Structural Estimation of Markov Decision Processes*pp. 3081–3143, Handbook of Econometrics, vol. 4 eds. Engle R.F and McFadden D.L,. North-Holland, Amsterdam.

Rust, J., and C. Phelan (1997): "How Social Security and Medicare Affect Retirement Behavior in a World of Incomplete Markets," *Econometrica*, 65(4), 781–831.

Schroder, M., and C. Skiadas (1999): "Optimal Consumption and Portfolio Selection with Stochastic Differential Utility," *Journal of Economic Theory*, 89, 68–126.

Stock, J. H., and D. A. Wise (1990): "Pensions, the Option Value of Work, and Retirement," *Econometrica*, 58(5), 1151–80.

Sundaresan, S., and F. Zapatero (1997): "Valuation, Asset Allocation and Incentive Retirements of Pension Plans," *Review of Financial Studies*, 10, 631–660.

VICEIRA, L. (2001): "Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income," *Journal of Finance*, 56(2), 433–470.

WACHTER, J. (2002): "Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets," *Journal of Financial and Quantitative Analysis*, 37, 63–91.

Notes

[1]This assumption can actually be easily relaxed. For instance, we could assume that retirees can return to the workforce (at a lower wage rate), without affecting any of the major predictions of the model.

[2]Some (indirect) evidence to this fact is given in the August 2004 Issue Brief of the Employee Benefit Research Institute (Figure 2 - based on the EBRI/ICI 401(k) Data)

[3]See also the very interesting discussion in Jagannathan and Kocherlakota (1996).

[4]If we impose a retirement deadline, this multiple also depends on the distance to this deadline.

[5]Chan and Viceira (2000) combines intuitions of both literatures. However, they assume labor/ leisure choices that can be adjusted continuously.

[6]See e.g. Jagannathan and Kocherlakota (1996) and BMS

[7]Available upon request.

[8]We shall denote by $F = \{F_t\}$ the P-augmentation of the filtration generated by B_t.

[9]By standard arguments the constant discount factor β could also incorporate a constant hazard rate of death λ.

[10]Observe that this is guaranteed if $\gamma > 1$

[11]As we show in the appendix, this will guarantee that retirement takes place with probability 1 in this stochastic setup.

[12]In the appendix we show that this condition is redundant in many special cases since it is implied by $\theta > 0$. In the (empirically relevant) region $\gamma > 1, \beta \geq r$ this condition is satisfied automatically as we show in the appendix.

[13]See BMS for a proof. Liu and Neis (2002) impose the constraint $h_t \geq 0$ and obtain different results. It is interesting to note that in the framework of Liu and Neiss (2002) an individual starts loosing labor supply flexibility as she approaches the constraint $h_t = 0$. Hence, she effectively becomes more risk averse. In our framework this is true only post retirement. Pre-retirement, the individual exposes herself to more risk because this is the only way in which she can accelerate retirement. This shows that taking indivisibility and irreversibility into account, the properties of the solution are fundamentally different.

[14]It is also the key factor behind the behavior of optimal portfolios that will be analyzed in a subsequent subsection

[15] This follows directly from equation (25). As $W_t \to -\frac{y_0}{r}$ we know by the budget constraint will imply consumption levels arbitrarily close to 0. Also, as we show in the appendix $\gamma(1-\gamma_2)-1>0$ and hence the term $c_t^{\gamma(1-\gamma_2)-1}$ goes to 0 leading to $\lim_{W_t \to -\frac{y_0}{r}} mpc = \theta$.

[16] Actually, it is not difficult to show by the results in the Appendix that $\underline{\lambda}$ is the solution to equation (13) if $W_t = \overline{W}$.

[17] It is indeed commonly accepted that the empirically observed low correlation between the stock market and consumption is one of the key issues behind the equity premium puzzle.

[18] This can be formally shown by applying Ito's Lemma to $c(W_t)$ together with the dynamics of the wealth process (1) and the optimal portfolio $\pi(W_t)$ to arrive at (27)

[19] Allowing for continuous adjustment of labor until T does not affect the main conclusions of this section.

[20] An important remark on terminology: The term "finite horizon" refers to the fact that the optimal stopping region becomes a function of the deadline to mandatory retirement. The individual continues to be infinitely lived.

[21] Admittedly, not all of these effects are purely due to non-separability between leisure and consumption. Home production is undoubtedly a key determinant behind these drops. It is important to note however, that our model is not incompatible with such an explanation. As long as a) the agent can leverage consumption utility with her increased leisure and b) time spent on home production is not as painful as work, the present model can be seen as a good reduced form approximation to a more complicated model that would model home production explicitly.

[22] Employee Benefits Research Institute, Issue Brief 272 (Aug 2004), especially Figure 2.

[23] This section is based on Karatzas and Wang (2000). For a more explicit presentation see also Karatzas and Shreve (1998)

[24] The proof is omitted and is available upon request.

[25] Proof available upon request.

[26] Recall that the state variable evolves as: $\frac{dZ_t}{Z_t} = (\beta - r)\,dt - \kappa dB_t, \quad Z_0 = \lambda$, so that κ is its standard deviation.

[27] This proof is available upon request. It is ommited because it effectively replicates the steps in Karatzas and Wang (2000), combined with the results in He and Pages (1993)

[28] A more detailed proof is available upon request.

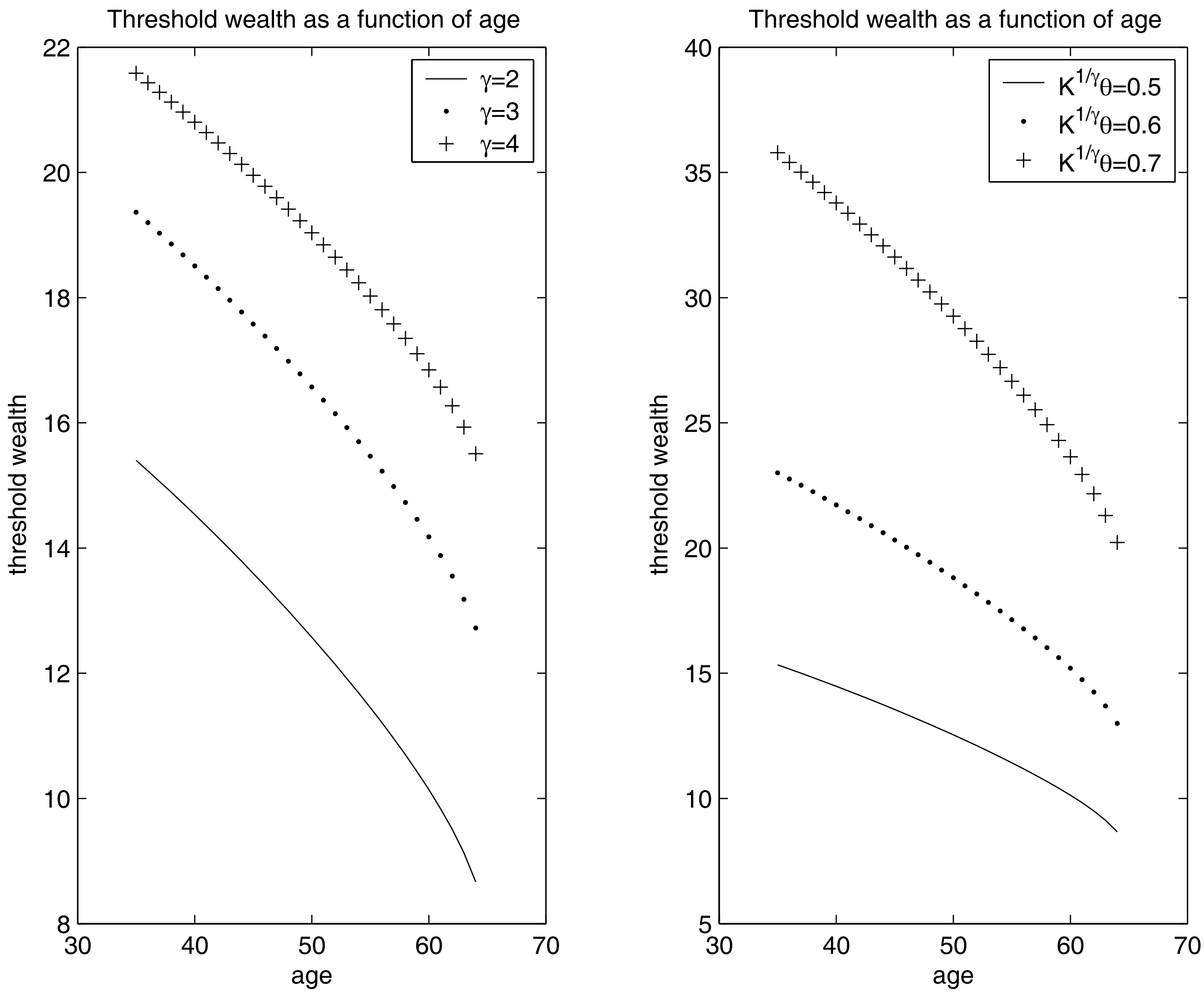

Figure 1: Wealth Thresholds as a function of age. For the left figure we use $K^{1/\gamma}\theta = 0.5$, whereas for the right figure we use $\gamma = 2$.

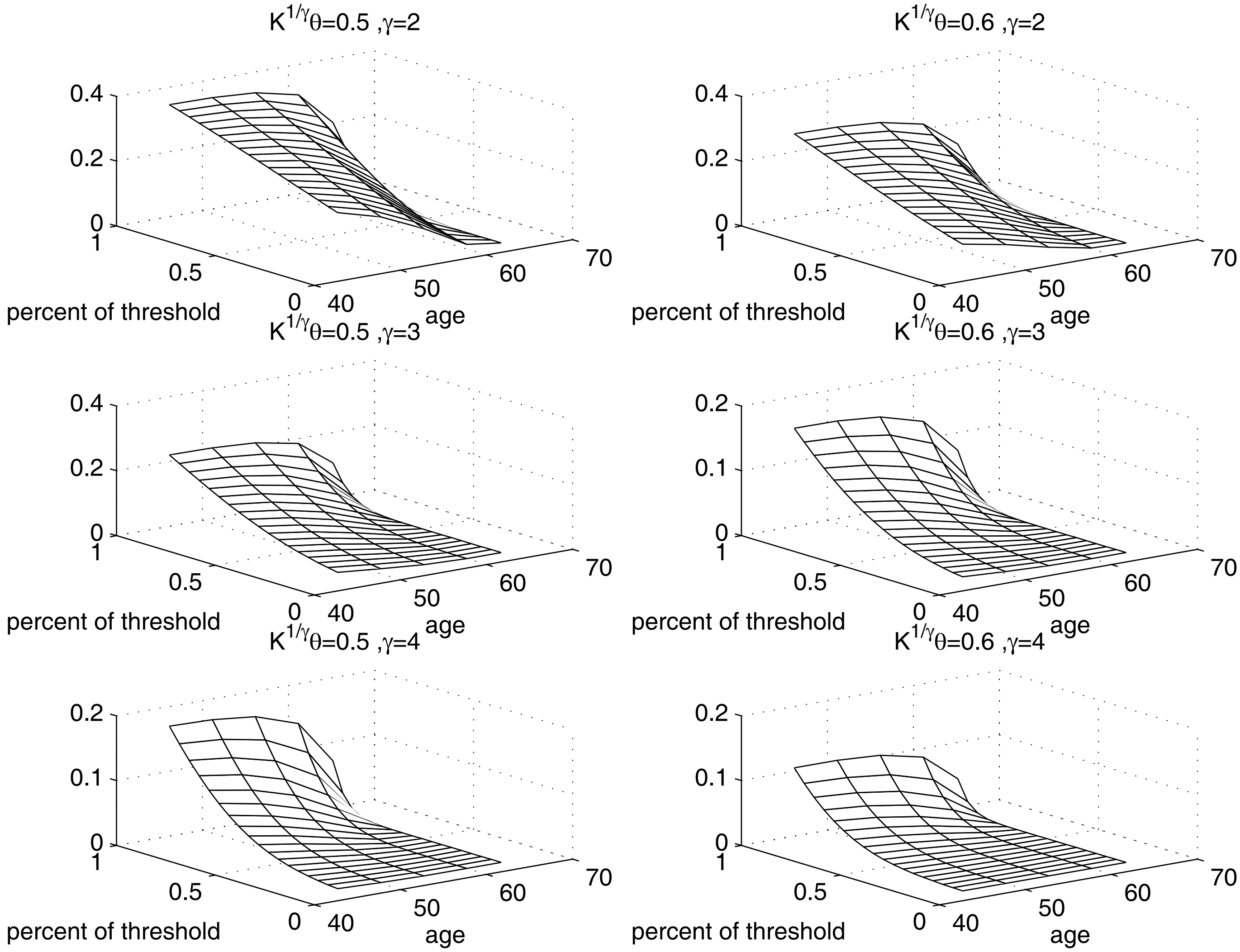

Figure 2: Relative importance of the portfolio holdings due to the real option to retire. Percent of threshold refers to the level of wealth normalized by the threshold wealth that would imply retirement at age 64.

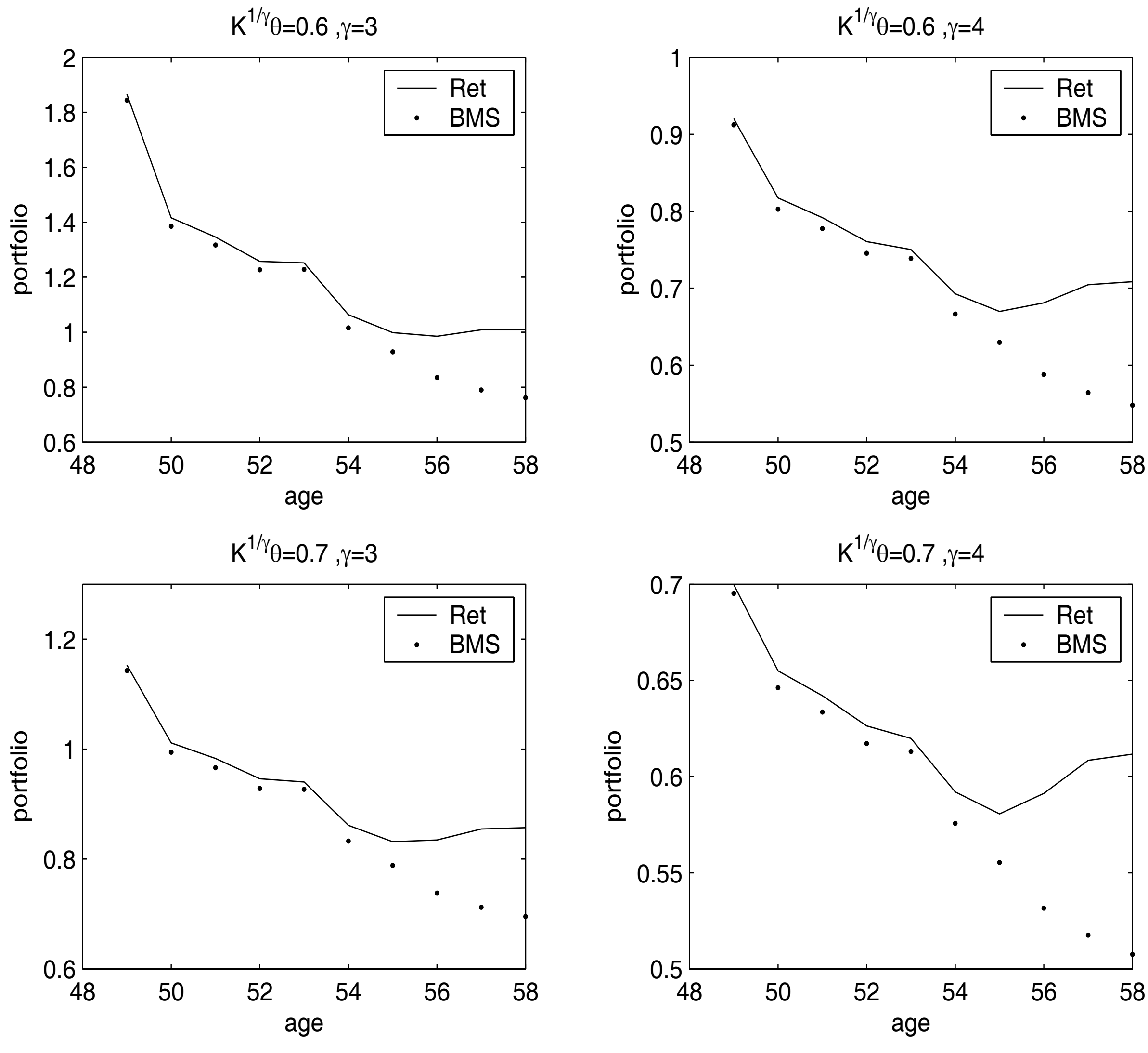

Figure 3: Portfolio holdings for an individual who retired in 1999 at age 58. Portfolio refers to total stockholdings divided by financial wealth.

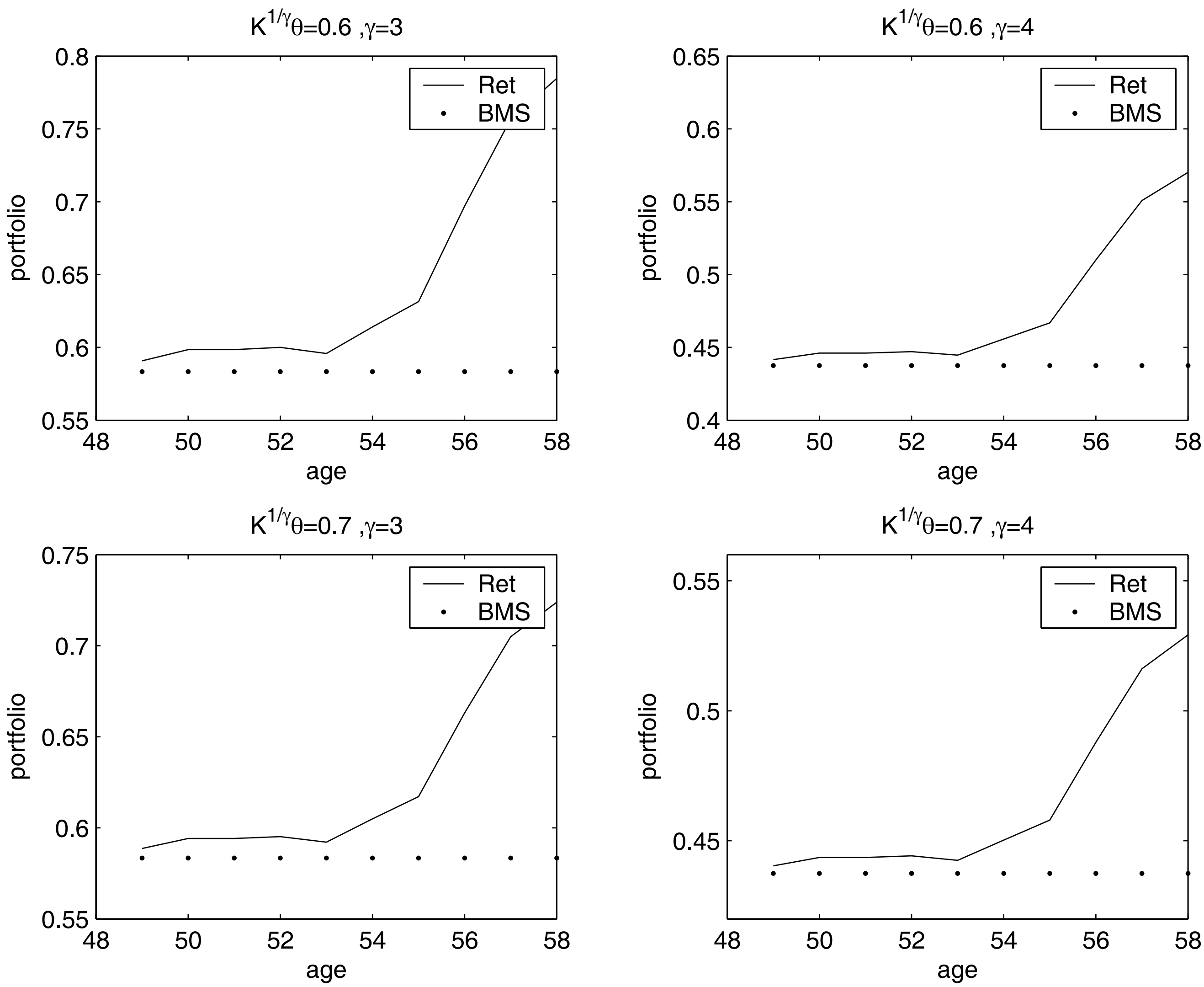

Figure 4: Portfolio holdings for an individual who retired in 1999 at age 58. Portfolio refers to total stockholdings divided by total resources (financial wealth + human capital).

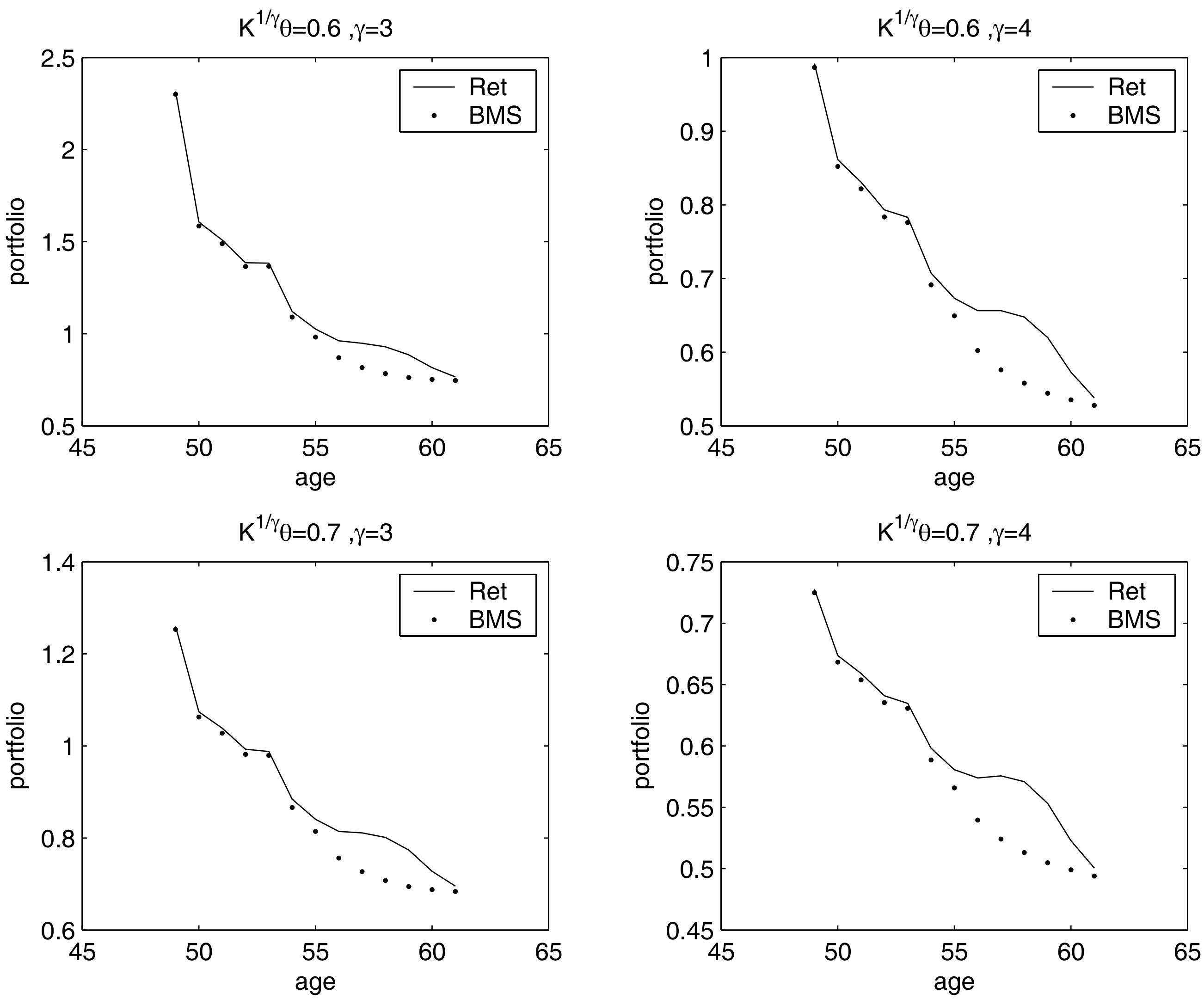

Figure 5: Portfolio holdings for an individual who was 58 years old in 1999 but did not have enough wealth to retire. Portfolio refers to total stockholdings divided by financial wealth.

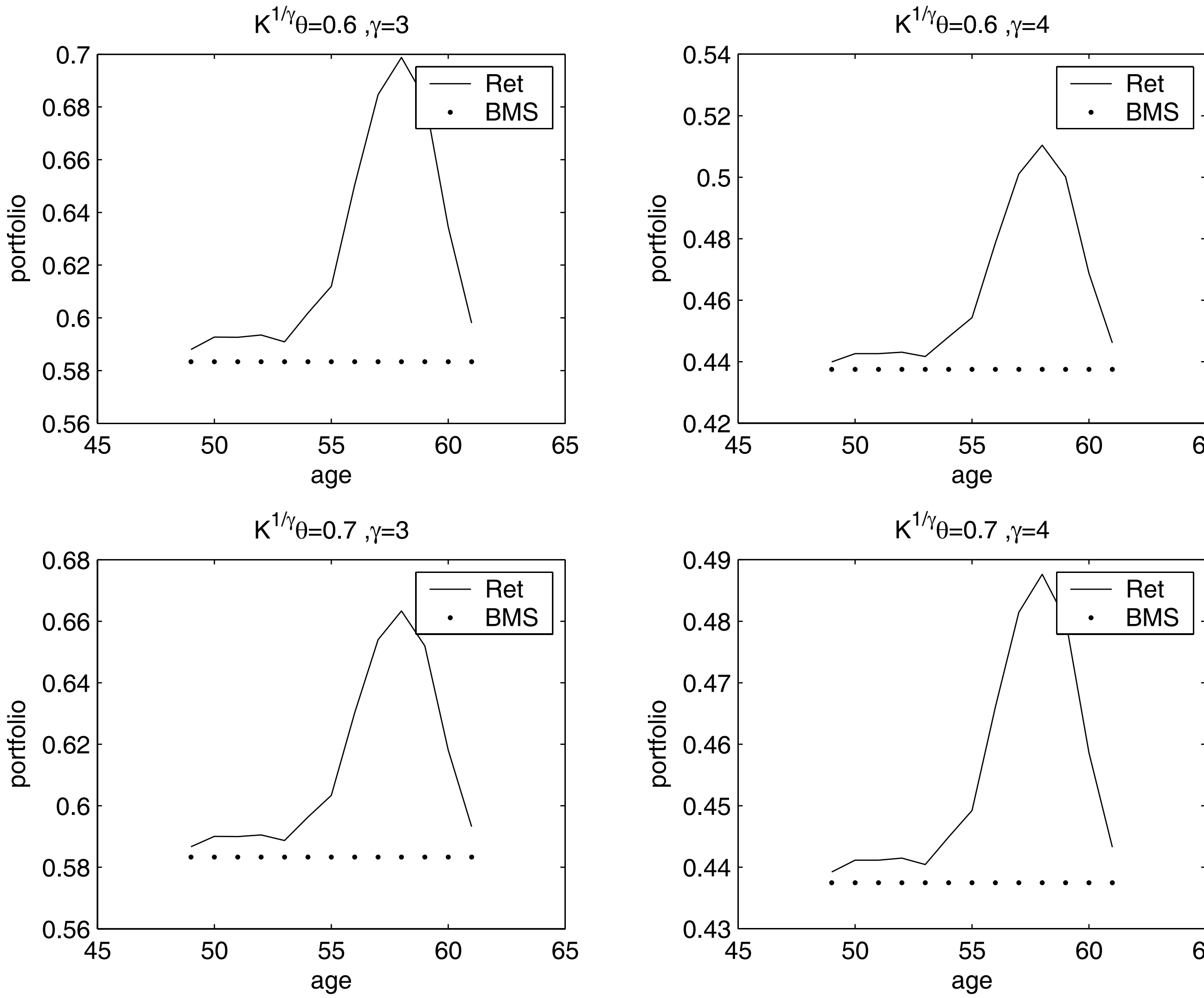

Figure 6: Portfolio holdings for an individual who was 58 years old in 1999 but did not have enough wealth to retire. Portfolio refers to total stockholdings divided by total resources (financial wealth + human capital).

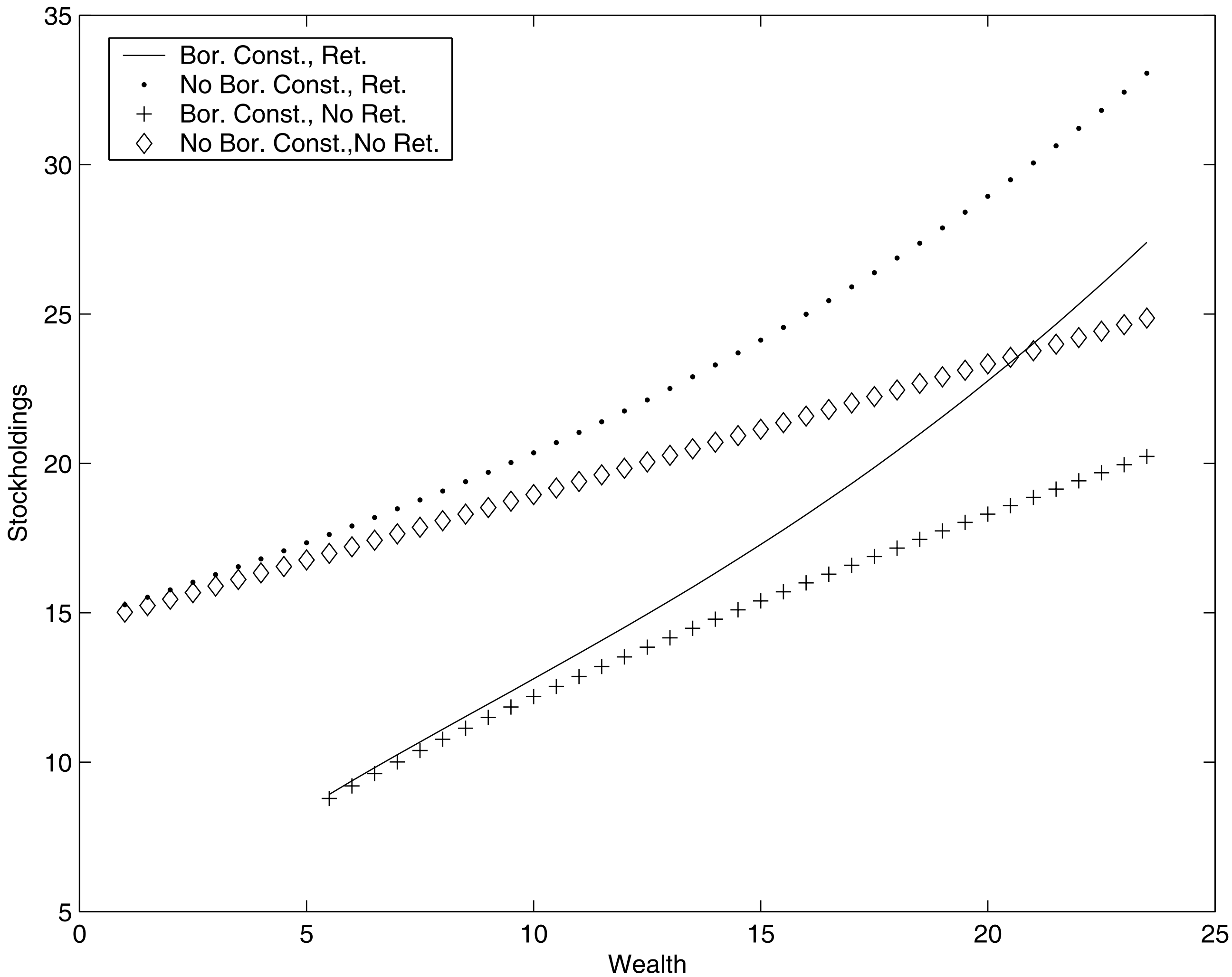

Figure 7: Total stockholdings as a function of wealth for 4 scenarios depending on whether borrowing constraints are imposed or not and whether (optimal) retirement is allowed or not. For this quantitative exercise we take $\gamma = 4, K^{1/\gamma}\theta = 0.5$

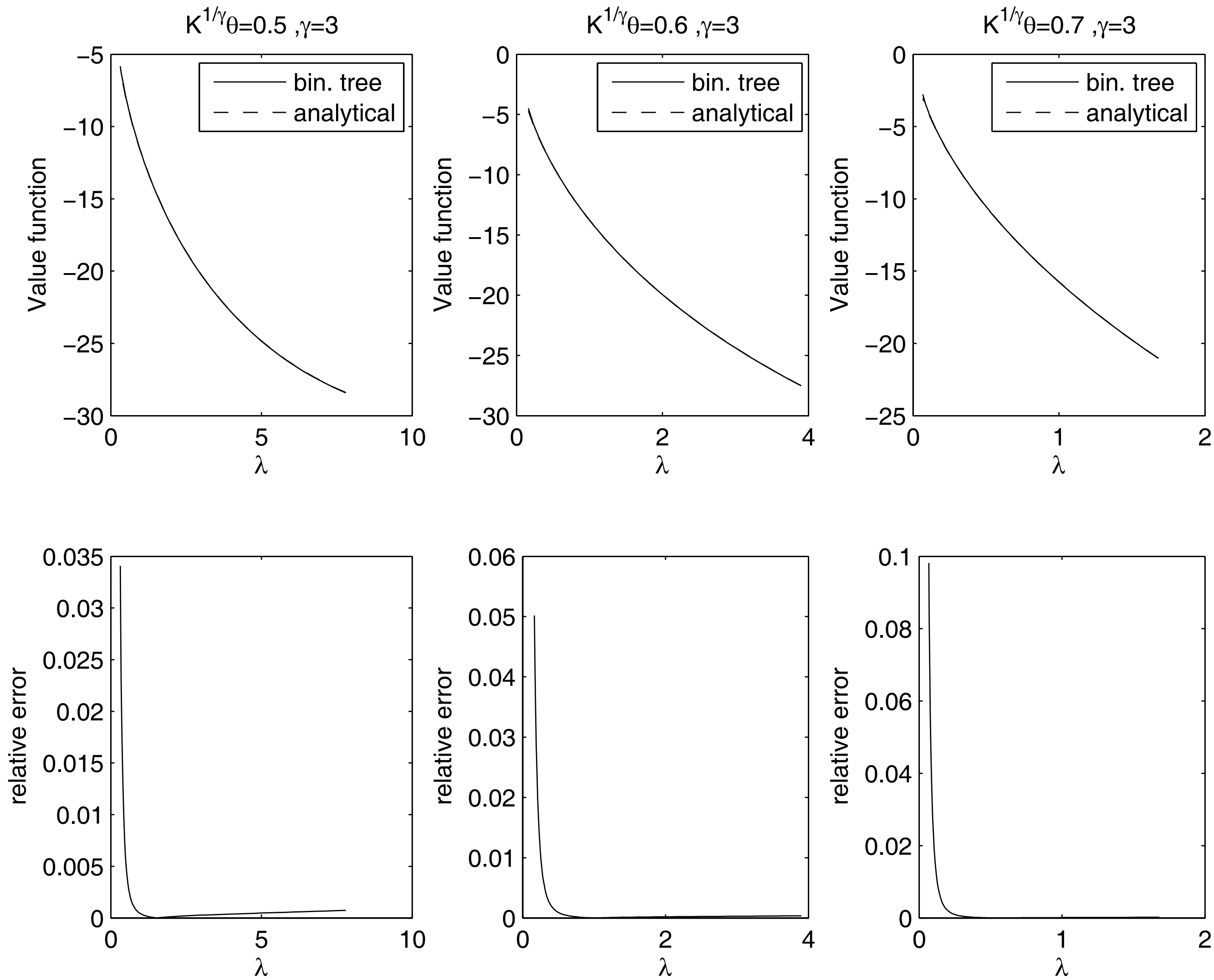

Figure 8: Value functions for the analytical approximation and the numerical solution. The top panel depicts levels, whereas the bottom panel depicts relative error.

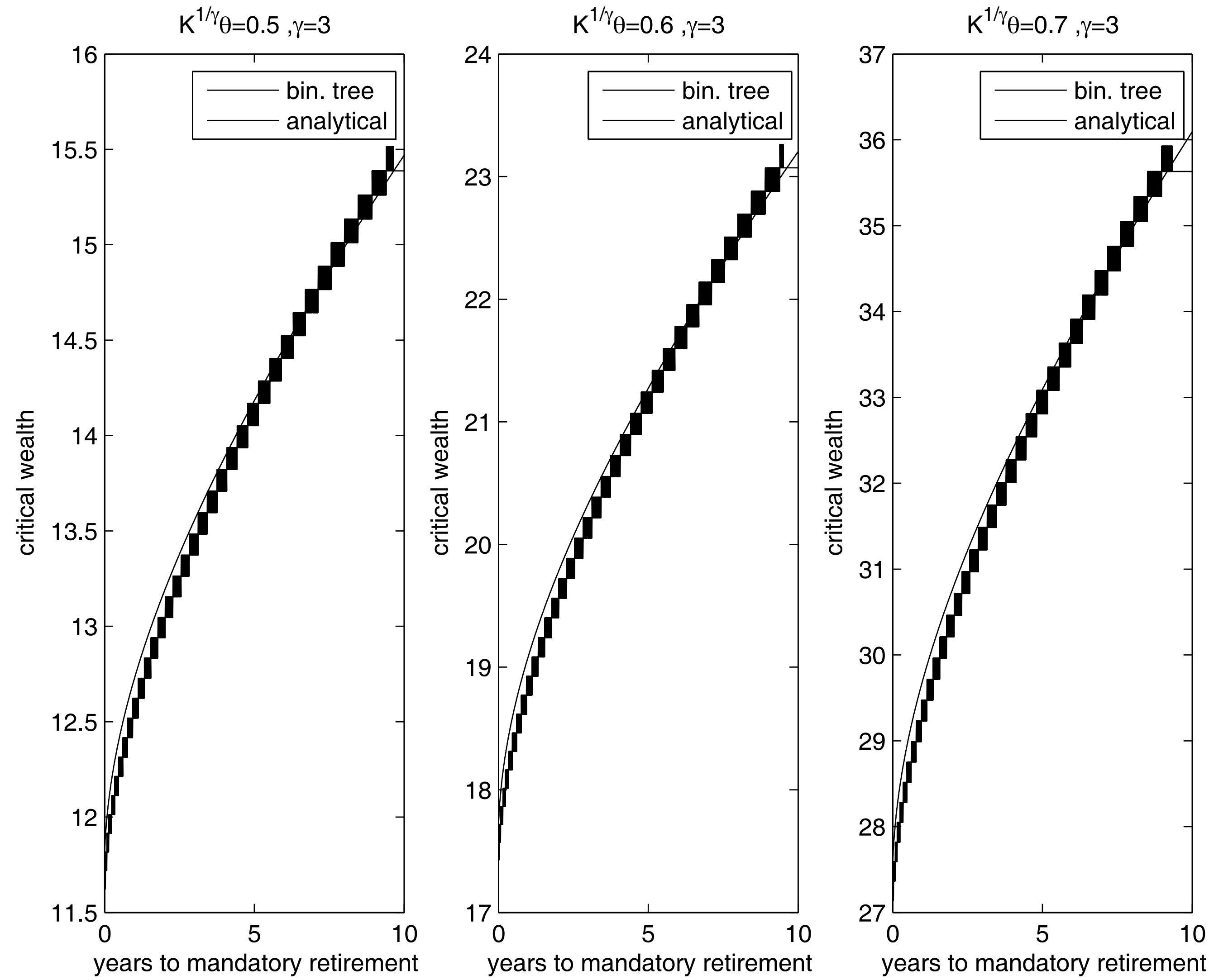

Figure 9: Critical wealth levels as a function of years to mandatory retirement for the analytical solution, and the numerical solution obtained via a binomial tree.